1 Circle maps

Several techniques used for the analysis of unimodal maps have their origin in the theory of circle maps. To avoid annoying factors of 2π, we define the circle S^1 as $\mathbb{R}\backslash\mathbb{Z}$. We shall sometimes consider the dynamics on $\mathbb{R}$ (usually for the purpose of computations) and sometimes on S^1 (for conceptual purposes). A *circle map* is a homeomorphism of the circle. Note that circle maps are invertible, in contrast with unimodal maps. The simplest example of a circle map is a *rotation*. For any angle $\alpha \in \mathbb{R}$, we define

$$R_\alpha(x) = x + \alpha \mod 1. \tag{1.1}$$

There is a fundamental difference between rational and irrational rotations. It is easy to show that if $\alpha \in \mathbb{Q}$ every orbit is periodic. If $\alpha \notin \mathbb{Q}$ then no orbit is periodic and every orbit is dense (Exercise 1). This example illustrates the typical dichotomy between the presence and absence of periodic orbits. We shall mainly focus on homeomorphisms that do not have periodic orbits, for the following reason.

Proposition 1.1. *Suppose f is a circle map that has a periodic point. Then every orbit is asymptotic to a periodic orbit.*

Proof. Suppose $f^p(x_*) = x_*$. Observe that $S^1\backslash\{x_*\}$ is an interval. Since f^p is a homeomorphism, it is a monotone map of this interval, thus every orbit f^{pk}, $k \geq 0$ asymptotes to a fixed point of f^p. □

We shall first focus on the topological theory of circle maps. This is the study of properties of a circle map that remain invariant under homeomorphisms. The fundamental topological invariant for circle maps is the *rotation number* introduced by Poincaré. The classical approach to the rotation number is introduced in the Exercises. Here we follow a longer route beginning with the idea of *first return times*. The connection with the classical theory is that the sequence of first return times constitutes the continued fraction expansion of the rotation number. This approach is well suited to the study of a similar invariant for unimodal maps.

In all that follows, we will consider only *orientation preserving* homeomorphisms. The extension to orientation reserving homeomorphisms are trivial.

1.1 Lifting circle maps to an interval

We first reduce circle maps to discontinuous maps of the unit interval. Let $g : S^1 \to S^1$ be a homeomorphism, and $\hat{g} : \mathbb{R} \to \mathbb{R}$ a lift of g. There is a unique point $c \in [0,1]$ such that $\hat{g}(c)$ is an integer. We assume that g has no fixed points, so that $c \in (0,1)$. We define the map $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} \hat{g}(x) \mod 1, & x \in [0,1]\backslash\{c\}, \\ 0, & x = c. \end{cases} \tag{1.2}$$

Note that $\lim_{x \downarrow c} f(x) = f(c)$, that is f is right-continuous. This is an arbitrary, but useful convention. Note also that $f(0) = f(1) \in (0,1)$. Figure 1.1 illustrates this class of maps. Conversely, it is easy to check that every f of this form lifts to a circle map. The point c is called the *critical point.* Note

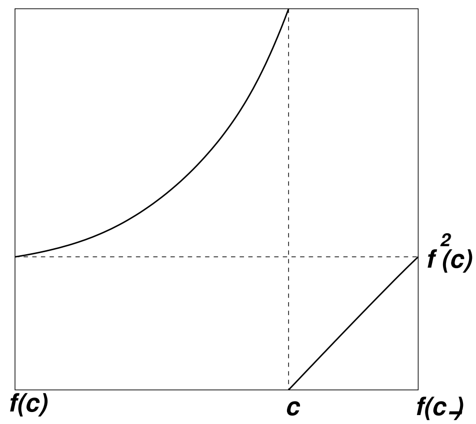

Figure 1.1: Lift of a circle map to the class $S(I)$

that with this *definition* of S^1, there is a unique c associated to every map g.

In order to define renormalization of circle maps, let us isolate the essential features of these maps in slightly greater generality. Let I denote a closed interval. The class $S(I)$ consists of maps from $I \to I$ such that

1. f has a unique point of discontinuity, denoted c, in the interior of I. We call c the *critical point* of f.

2. f is continuous and strictly increasing in each component of $I\backslash\{c\}$ and is right-continuous at c. The left limit $f(c_-)$ is the right-endpoint of I, and the right limit $f(c_+)$ is the left-endpoint of I.

3. f maps both boundary points to a single point in the interior of I.

Though c is a point of discontinuity for f, notice that it is a point of continuity for f^2. We will use this observation often.

1.2 Symbolic dynamics

Let us first introduce the space of symbol sequences.

Definition 1.2. Σ denotes the space of sequences $\{0, c, 1\}^{\mathbb{N}}$ with typical element $\underline{x} = (x_0, x_1, \ldots, x_n, \ldots)$. The shift map $\sigma : \Sigma \to \Sigma$ is defined by

$$(\sigma x)_n = x_{n+1}, \quad n \geq 0. \tag{1.3}$$

We order Σ *lexicographically* with the convention $0 < c < 1$. That is, $\underline{x} \prec \underline{y}$ if $x_k < y_k$ where k is the first index where $x_j \neq y_j$. The natural distance on a space of symbols is usually the *Hamming distance* . If $a, b \in \{0, c, 1\}$, the distance $d(a, b)$ is 0 if $a = b$ and 1 otherwise. This distance can be used to induce a distance between two elements of Σ. We set

$$d(\underline{x}, \underline{y}) = \sum_{k=0}^{\infty} \frac{d(x_k, y_k)}{2^k}. \tag{1.4}$$

The metric space (Σ, d) is compact and totally disconnected. σ is a continuous function on this metric space.

There is a natural sequence associated to any orbit. The interval I may be decomposed into

$$I = I_0 \cup \{c\} \cup I_1,$$

where I_0 and I_1 denote the subintervals to the left and right of c.

Definition 1.3. The *address* of a point $x \in I$ is the symbol 0 if $x \in I_0$, c if $x = c$, and 1 if $x \in I_1$.

Definition 1.4. Let $f \in S(I)$ and $x \in I$. The *itinerary* of x under the map f is the sequence of addresses of $(x, f(x), \ldots, f^n(x), \ldots)$ and is denoted

$$i_f(x) = (i_0(x), i_1(x), \ldots), \tag{1.5}$$

where $i_k(x)$ is the address of $f^k(x)$.

The following monotonicity lemma is fundamental. Heuristically, it tells is that the lexicographical ordering on Σ is 'correct' for circle maps. This should be contrasted with a similar lemma for unimodal maps where the lexicographical ordering has to be replaced by a different ordering.

Lemma 1.5. *Let $f \in S(I)$.*

(a) If $x < y$ then $i_f(x) \preceq i_f(y)$.

(b) If $i_f(x) \prec i_f(y)$ then $x < y$.

(c) $i_f(f^k(x)) = \sigma^k(i_f(x))$.

The assertion in (a) cannot be strengthened to strict inequality without further assumptions. If f has a periodic orbit, then we may have $i_f(x) = i_f(y)$ for x, y in an interval.

Proof. (a) Suppose $x < y$. If $i_f(x) = i_f(y)$ there is nothing to prove. Therefore, let k be the first index in which the sequences $i_f(x)$ and $i_f(y)$ differ. We must show that $i_k(x) < i_k(y)$. Since $i_j(x) = i_j(y)$, $0 \le j < k$, the points $f^j(x)$ and $f^j(y)$ lie in the same subinterval for $0 \le j < k$ (what happens if $i_j(x) = c$ for some $j < k$?). Since f is increasing on both I_0 and I_1 we have inductively $f(x) < f(y)$, $f^2(x) < f^2(y)$, $\ldots$, $f^{k-1}(x) < f^{k-1}(y)$. Since $i_k(x) \neq i_k(y)$, the monotonicity of f now implies $i_k(x) < i_k(y)$ as desired.

(b) This is almost identical to the previous argument. Let k be the first index where $i_k(x) < i_k(y)$. This implies $f^k(x) < f^k(y)$ and $f^j(x)$ and $f^j(y)$ are contained in the same subinterval for $0 \le j < k$. But now we may use the monotonicity of f to inductively deduce that $f^{k-1}(x) < f^{k-1}(y)$, $\ldots$, $f^2(x) < f^2(y)$, $f(x) < f(y)$ and $x < y$.

(c) This is left as an exercise. □

A general theme in symbolic dynamics is to deduce properties of an orbit via knowledge of $i_f(x)$. The following lemma is an example of this approach.

Lemma 1.6. *If $i_f(x)$ is periodic with period p, then f has a periodic point with period not greater than p.*

Proof. We may as well assume c does not appear in $i_f(x)$. If it did, periodicity of $i_f(x)$ implies $f^p(c) = c$ and we are done.

Let I_r, $0 \le r \le p-1$, denote the smallest closed interval containing the points $\{f^r(x), f^p(f^r(x)), f^{2p}(f^r(x)), \ldots\}$. Then I_r is contained within I_0 or I_1 depending on whether $i_r(x)$ is 0 or 1. In either case, f is monotone increasing on I_r, and maps it homeomorphically to I_{r+1}. We compose maps to find $f^p : I_0 \to I_0$ is a homeomorphism. Thus, f^p has a fixed point. □

The basic invariants we will consider are the symbol sequences

$$K^+(f) = \lim_{x \downarrow c} i_f(x), \quad K^-(f) = \lim_{x \uparrow c} i_f(x). \tag{1.6}$$

Both limits exist by Lemma 1.5. Observe also that

$$K^+(f) = (1,0) \cdot i_f(f^2(c)), \quad K^-(f) = (0,1) \cdot i_f(f^2(c)),$$

where the symbol $\cdot$ denotes concatenation of two strings. Thus, it will suffice to consider $K^+(f)$.

Definition 1.7. Two maps $f, \tilde{f} \in S(I)$ are *combinatorially equivalent* if the orbit $O_f(c)$ of c under f has the same order as the orbit of $O_{\tilde{f}}(\tilde{c})$. That is, the map $h : O_f(c) \to O_{\tilde{f}}(\tilde{c})$ is strictly order preserving.

For any n, the ordering of $c, f(c), \dots, f^{n-1}(c)$ defines a permutation of $1, 2, \dots, n$. Two maps are combinatorially equivalent if these permutations are identical for every n.

Proposition 1.8. *Suppose $f, \tilde{f} \in S(I)$ are maps without periodic orbits. Then f and $\tilde{f}$ are combinatorially equivalent if and only if $K^+(f) = K^+(\tilde{f})$.*

Proof. The forward implication is trivial. If f and $\tilde{f}$ are combinatorially equivalent, then the ordering of $O_f(c)$ and $O_{\tilde{f}}(\tilde{c})$ is identical. Therefore, $i_f(f^2(c)) = i_{\tilde{f}}(\tilde{f}^2(\tilde{c})$, and $K^+(f) = K^+(\tilde{f})$.

Let us now assume that $K^+(f) = K^+(\tilde{f})$. We must show that order is preserved, that is for every k, l, $f^k(c) < f^l(c)$ if and only if $\tilde{f}^k(\tilde{c}) < \tilde{f}^l(\tilde{c})$. The proof relies on a strengthened form of Lemma 1.5(a):

Claim: If $f^k(c) < f^l(c)$, then $i_f(f^k(c)) \prec i_f(f^l(c))$.

Proof of the claim. The point here is strict inequality. If the claim were false, we would have $i_f(f^k(c)) = i_f(f^l(c))$. But then Lemma 1.5(c) implies

$$i_f(f^l(c)) = \sigma^{l-k}\left(i_f(f^k(c))\right) = i_f(f^k(c)).$$

Thus, $i_f(f^k(c))$ is periodic with period $l - k$ and Lemma 1.6 implies the existence of a periodic orbit, contradicting our assumptions. □

We now complete the proof of the Proposition. For any $k, l \geq 2$ we have

$$\begin{aligned} f^k(l) < f^l(c) &\Rightarrow i_f(f^k(c)) \prec i_f(f^l(c)) \\ &\Rightarrow i_{\tilde{f}}(\tilde{f}^k(\tilde{c})) \prec i_{\tilde{f}}(\tilde{f}^l(\tilde{c})) \Rightarrow \tilde{f}^k(\tilde{c}) < \tilde{f}^l(\tilde{c}). \end{aligned}$$

The second implication relies on the hypothesis $K^+(f) = K^+(\tilde{f})$ and the last on Lemma 1.5(b). □

The first part of the proof also shows that $K^+(g)$ is well-defined for a circle map. Indeed, it is easy to show that any two lifts $f, \tilde{f}$ of g into the unit interval are combinatorially equivalent. Another proof of the converse is outlined in the exercises.

1.3 First return times and the first return map

We now develop a concrete example of renormalization. Suppose $J \subset I$ is a subinterval such that $f^l(x) \cup J \neq \phi$ for every $x \in J$ and some positive integer $l(x)$. We define the *first return time*

$$k(x) = \min\{l > 0 \,\Big|\, f^l(x) \in J\} \tag{1.7}$$

Maps in the class $S(I)$ satisfy either $f(I_0) \subset I_1$ or $f(I_1) \subset I_0$ (why?). If $f(I_0) = I_1$ then f simply permutes the intervals, and f^2 has a fixed point. We ignore this case. To be concrete, we shall always denote the interior of the 'smaller' interval by J_0, and the interior of the 'larger' interval by J_1 in order that $f(J_0) \subset J_1$. (This may mean that $J_0 = I_1$ as in the example below). We also set $J = I$. We sometimes abuse notation and let the same letter denote both the open and closed interval with the same endpoints.

The return time k depends on the point x. It is continuous, except at preimages of c. The following times provide uniform control.

$$a(f) = \min\{l \,\Big|\, J_0 \cap f^{l+1}(J_0) \neq \phi\} \tag{1.8}$$

Lemma 1.9. *Suppose $f \in S(J)$ has no fixed points. $a(f)$ is the smallest integer such that the closure of J_0, $f(J_0)$, …, $f^{a(f)+1}(J_0)$ covers the closed interval J.*

Proof. Let c_n denote the orbit of c and $a' = \sup\{l \,\big|\, f^k(J_0) \subset J_1 \text{ for } 1 \leq k \leq l\}$. For $k \leq a'$ the closed intervals $f^k(J_0)$ are $[c_{k+1}, c_k]$. Thus, they are ordered and have a common endpoint. Since f does not have a fixed point, $a' < \infty$. It follows that $a' = a$. □

We use $a(f)$ to define the *first return map*

$$\mathcal{F}(f)(x) = f^{k(x)}(x), \tag{1.9}$$

whose domain is the interval

$$J(f) = \text{closure } \left(J_0 \cup f^{a+1}(J_0)\right). \tag{1.10}$$

There are two cases to consider. Our interest is usually in the second case.

Lemma 1.10. *(a) If $c \in \overline{f^{a(f)}(J_0)}$ then c is a fixed point of $f^{a(f)+1}$. In this case, $f^{a(f)+1}(J_0) = J_0$, $J(f) = J_0$, and $\mathcal{F}(f)$ is $f^{a(f)+1}$.*

(b) If not, $J(f)$ strictly contains J_0, the first return map is in $S(J(f))$, and given by

$$\mathcal{F}(f)(x) = \begin{cases} f^{a(f)+1}(x), & x \in J_0, \\ f(x), & x \in J_1 \cap J(f). \end{cases} \tag{1.11}$$

Proof. This follows directly from the geometry in the proof of Lemma 1.9. □

1.4 Renormalization of rotations

To illustrate these definition, let us compute the first return map when f is a rotation R_α. We assume α is irrational, and $\alpha \in (0, 1/2)$. In this case,

$$J_0 = I_1 = (1-\alpha, 1), \; f(J_0) = (0, \alpha), \dots, \; f^k(J_0) = ((k-1)\alpha, k\alpha),$$

provided $k\alpha < 1 - \alpha$. Therefore, we obtain

$$a(f) = \left\lfloor \frac{1}{\alpha} \right\rfloor - 1, \quad \text{and} \quad J(f) = [a(f)\alpha, 1]. \tag{1.12}$$

Here $\lfloor y \rfloor = \max_{k \in \mathbb{Z}} \{k \le y\}$. The corresponding first return map is given by

$$\mathcal{F}(f)(x) = \begin{cases} x + \alpha, & a(f)\alpha \le x < 1 - \alpha, \\ x + (a(f)+1)\alpha, & 1 - \alpha \le x < 1. \end{cases} \tag{1.13}$$

If we plot the graph of $\mathcal{F}(f)$ we observe that it has the same form as the graph of R_α. To make this precise, let $h : [0,1] \to J(f)$, denote the affine map with $h(0) = a(f)\alpha$ and $h(1) = 1$. The *renormalization* of R_α is defined by $h^{-1} \circ \mathcal{F}(f) \circ h$. We use (1.12) and (1.13) to compute

$$\mathcal{R}(R_\alpha) = R_{\alpha'}, \quad \alpha' = \frac{1}{1 + G(\alpha)}, \quad \alpha \in (0, \frac{1}{2}), \tag{1.14}$$

where the *Gauss map* $G : (0,1] \to (0,1]$ is defined by

$$G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor. \tag{1.15}$$

Thus, the renormalization of an irrational rotation is another irrational rotation. When $\alpha \in (1/2, 1)$ we have

$$J_0 = I_0 = (0, 1-\alpha), \; f(J_0) = (\alpha, 1), \dots, \; f^k(J_0) = (k(\alpha-1)+1, (k-1)(\alpha-1)+1),$$

provided $k(\alpha-1)+1 > 1-\alpha$. The largest index that satisfies this inequality is $a(f)$ and after some algebra we find

$$a(f) = \left\lfloor \frac{1}{1-\alpha} \right\rfloor - 1, \quad J(f) = [0, a(f)(\alpha-1)+1].$$

The return map may be renormalized as before yielding

$$\mathcal{R}(R_\alpha) = R_{\alpha'}, \quad \alpha' = \frac{G(1-\alpha)}{1+G(1-\alpha)}, \quad \alpha \in (\frac{1}{2}, 1). \tag{1.16}$$

Further analysis of this example, and the relation to continued fractions is explored in the exercises.

1.5 The structure of $K^+(f)$

We now iterate the construction of the first return map to unravel the structure of $K^+(f)$. We inductively construct a sequence of first return times a_n, closest return intervals $J^{(n)}$ and closest return maps $\varphi_n \in S(J^{(n)})$ as follows. Suppose $n \geq 2$ and suppose a_k, $J^{(k)}$, and φ_k have been defined for $1 \leq k \leq n-1$. If $\varphi^{(n-1)}$ has no fixed point, as in Section 1.3 we define

$$a_n = a(\varphi_{n-1}), \quad J^{(n)} = J(\varphi_{n-1}), \quad \varphi_n = \mathcal{F}(\varphi_{n-1}). \tag{1.17}$$

If φ_{n-1} has a fixed point we set $a_n = \infty$ and the process terminates. The initial conditions and first iterate are fixed as follows:

$$J^{(0)} = J, \quad \varphi_0 = f. \tag{1.18}$$

For $n = 1$, if J_0 is to the right of c we set

$$a_1 = a(f) + 1, \quad J^{(1)} = J(f), \quad \varphi_1 = \mathcal{F}(f). \tag{1.19}$$

(There is a slight inconsistency between the definition of a_1 and a_n, $n > 1$. This notation proves convenient later in (1.24)).On the other hand, if J_0 is to the left of c we set

$$a_1 = 1, \quad J^{(1)} = J, \quad \varphi_1 = f. \tag{1.20}$$

Let $J_0^{(n)}$ denote the interior of the left component of $J^{(n)} \backslash \{c\}$ if n is odd, and the right component if n is even. (The normalization in the first step is chosen to ensure this condition). The role of left and right is interchanged at each step by the procedure (1.17). We have

$$J_0^{(n)} = J_1^{(n-1)} \cap J^{(n)}, \quad J_1^{(n)} = J_0^{(n-1)} \cap J^{(n)} = J_0^{(n-1)}. \tag{1.21}$$

The first return map is defined as in Lemma 1.10. We have

$$\varphi_1(x) = \begin{cases} f(x), & x \in J_0^{(1)}, \\ f^{a_1}, & x \in J_1^{(1)}. \end{cases} \tag{1.22}$$

For $n \geq 2$ we use (1.21) and Lemma 1.10 to obtain

$$\varphi_n|_{J_0^{(n)}} = \varphi_{n-1}|_{J_1^{(n-1)}}, \quad \varphi_n|_{J_1^{(n)}} = \left(\varphi_{n-1}|_{J_1^{(n-1)}}\right)^{a_n} \circ \left(\varphi_{n-1}|_{J_0^{(n-1)}}\right). \tag{1.23}$$

To get a feel for this, write out the first few terms. For example, we have

$$\varphi_2|_{J_0^{(2)}} = f^{a_1}, \quad \varphi_2|_{J_1^{(2)}} = f^{a_1 a_2 + 1}.$$

We define a sequence of *closest return times* by

$$\begin{aligned} &q_{n+1} = a_{n+1} q_n + q_{n-1}, \quad n \geq 1, \\ &q_0 = 1, \quad q_1 = a_1. \end{aligned} \tag{1.24}$$

The closest return maps φ_n, $n \geq 2$ is then given explicitly by

$$\varphi_n|_{J_0^{(n)}} = f^{q_{n-1}}, \quad \varphi_n|_{J_1^{(n)}} = f^{q_n}. \tag{1.25}$$

This process is illustrated in Figures 1.2 and Figure 1.3.

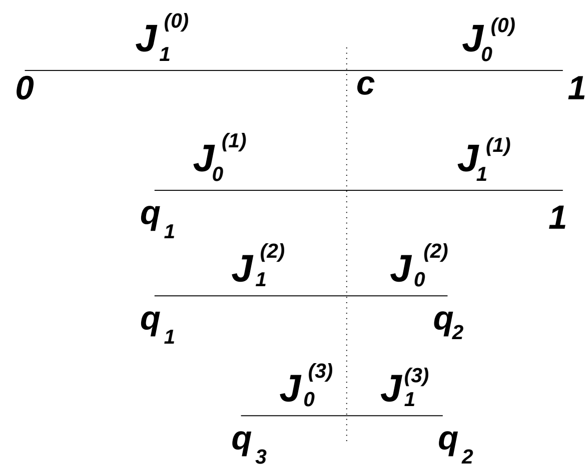

Figure 1.2: The geometry of $J^{(n)}$. The points $f^{q_k}(c)$ are denoted q_k for brevity. Observe the role of (1.21).

The terminology *closest* return time or interval is motivated by the following lemma.

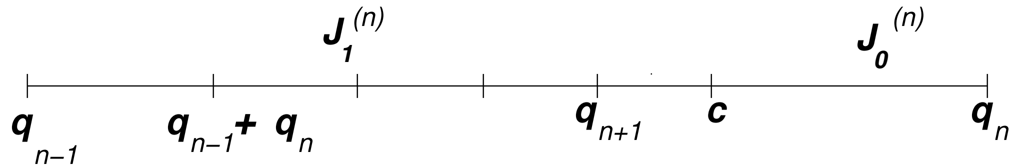

Figure 1.3: Note that $q_{n+1} = q_{n-1} + a_{n+1}q_n$. The return map is f^{q_n} on the left interval, and $f^{q_{n-1}}$ on the right.

Lemma 1.11. *Suppose j is an index, $0 \leq j \leq q_{n+1}$. Then $f^j(c) \in J^{(n)}$ if and only if $j = q_{n-1} + kq_n$ for some index k, $0 \leq k \leq a_{n+1}$.*

Proof. The proof for $n = 1$ and $n = 2$ are explained in Figure 1.5. It is clear that this generalizes. □

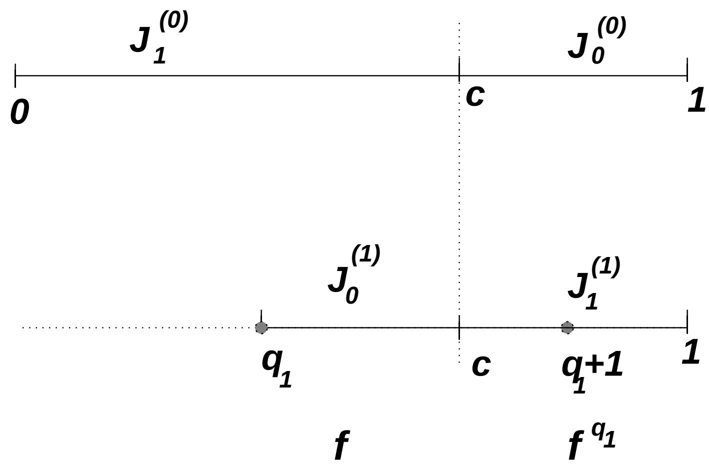

Figure 1.4: All iterates $f^j(c)$ for $1 < j < q_1$ lie to the left of q_1. Observe that $c \in (f^{q_1}(c), f^{q_1+1}(c))$ by definition. Moreover, the first return map is f on $J_0^{(1)}$, and f^{q_1} on $J_0^{(1)}$.

Lemma 1.12. *The union of $\cup_{k=0}^{q_{n-1}-1} f^k(J_0^{(n)})$ and $\cup_{k=0}^{q_n-1} f^k(J_1^{(n)})$ tiles the interval J. That is, these intervals have disjoint interior, and the closure of their union covers J.*

Proof. 1. Let us first verify that the intervals are disjoint. First consider $J_0^{(n)}$. The first return map is $f^{q_{n-1}}$ on this interval, therefore if $j \neq k$ are

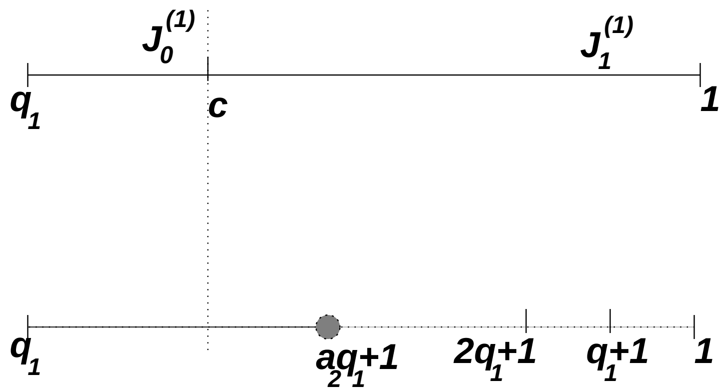

Figure 1.5: All iterates $f^{kq_1+1}(c)$ for $1 \leq k < a_2$ lie to the right of $f^{q_2}(c)$ ($q_2 = a_2 q_1 + 1$). Moreover, if the index $j > q_1 + 1$ is not of the form $kq_1 + 1$, then $f^j(c)$ does not lie in $J_1^{(1)}$ (if it did, it would contradict the definition of f^{q_1} as the first return map).

indices less than q_{n-1} we must have $f^j(J_0^{(n)}) \cap f^k(J_0^{(n)}) = \phi$. The same argument applies on $J_1^{(n)}$: we simply note that $J_1^{(n)} = J_0^{(n-1)}$.

2. Next we show that the intervals cover J. For $n = 1$,

$$\cup_{k=1}^{q_1 - 1} f^k(J_1^{(1)}) = [0, c_2] \cup [c_2, c_3] \cup \ldots [c_{q_1 - 1}, c_{q_1}].$$

Moreover,

$$J_1^{(1)} = [c, 1], \quad J_0^{(1)} = [c_{q_1}, c].$$

At the next stage, the same arguments yield that $\cup_{k=0}^{q_2} f^k(J_0^{(1)})$ covers $J^{(1)}$. Therefore, it also covers $J_0^{(0)}$, and it follows that $\cup_{k=0}^{q_1+q_2} f^k(J_0^{(1)})$ covers J. Proceed inductively: if $Q_n = q_1 + q_2 + \ldots + q_n$, then $\cup_{k=0}^{Q_n} f^k(J_0^{(n)})$ covers J.

3. We are using too many intervals at this stage. The minimal number is obtained as follows. Suppose k is the smallest positive integer such that $f^{-k}(x) \in \overline{J_0^{(n)}}$. (The previous step ensure that such k exists). Then also $f^{-k+q_{n-1}}(x) \in \overline{J_0^{(n)}}$. Thus, $k < q_{n-1}$. Similarly, if k is the minimal integer such that $f^{-k}(x) \in \overline{J_1^{(n)}}$ we find $k < q_n$. □

The sequences a_n and q_n impose severe restrictions on the invariants K^+ and K^-: an arbitrary string in Σ is not admissible as an itinerary as a consequence of the following lemma. The notation adopted is as follows. If

$\underline{x}$ is a string, $\underline{x}_m$ denotes the truncation $(x_0, \ldots, x_{m-1})$, $\underline{x}_m \cdot \underline{y}_n$ denotes the concatenation of the strings $\underline{x}_m$ and $\underline{y}_n$, and $\underline{x}_m^p$ denotes the concatenation $\underline{x}_m \cdot \underline{x}_m^{p-1}$.

Lemma 1.13. *Assume $f \in S(I)$ has no periodic orbits. Then for $n \geq 1$*

$$\begin{aligned} K^+_{q_{2n+2}} &= K^+_{q_{2n}} \cdot \left(K^+_{q_{2n+1}}\right)^{a_{2n+2}}, \\ K^+_{q_{2n+1}} &= K^+_{q_{2n-1}} \cdot \left(K^-_{q_{2n}}\right)^{a_{2n+1}}. \end{aligned} \tag{1.26}$$

Proof. We prove only the first identity. Since $q_{2n+2} = a_{2n+2}\, q_{2n+1} + q_{2n}$, it suffices to show

$$\left(i_f(c_{q_{2n}+jq_{2n+1}})\right)_{q_{2n+1}} = K^+_{q_{2n+1}}, \quad 0 \leq j \leq a_{2n+2} - 1. \tag{1.27}$$

The point c is contained in the interval $J^{(2n+1)} = [c_{q_{2n+1}}, c_{q_{2n}}]$. Since $f^{q_{2n+1}}$ is the first return map on the interval $(c, c_{q_{2n}})$, and f has no periodic orbits, if $x \in [c, c_{q_{2n}}]$ then $f^j(x)$ does not lie in $[c, c_{q_{2n}}]$ for $0 \leq j \leq q_{2n+1}$. That is, the intervals $f^j([c, c_{q_{2n}}])$ do not contain the point c, so that the itinerary of every point in this interval agrees for $j \leq q_{2n+1}$. Finally, we note that the points $c_{q_{2n}+jq_{2n+1}} \in [c, c_{q_{2n}}]$, $0 \leq j \leq a_{2n+2} - 1$ (by the definition of a_{2n+2}). This proves (1.27). □

A simple and powerful consequence of Lemma 1.13 is that the knowledge of K^+ is equivalent to knowledge of a_n.

Lemma 1.14. *Suppose $f, \tilde{f} \in S(I)$ have no periodic points. Then $K^+(f) = K^+(\tilde{f})$ if and only if $a_n(f) = a_n(\tilde{f})$ for $n \geq 1$.*

Proof. Suppose $a_n = \tilde{a}_n$. Then $q_n = \tilde{q}_n$ by (1.24) and Lemma 1.13 implies $K^+(f) = K^+(\tilde{f})$.

Conversely, suppose $K^+(f) = K^+(\tilde{f})$. Proposition 1.8 implies f and $\tilde{f}$ are combinatorially equivalent. It is then immediate that $a_n = \tilde{a_n}$. □

1.6 The rotation number and Poincaré's theorem

A finite set of positive integers $a_1, \ldots, a_n$ determines a *continued fraction*

$$[0; a_1, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots \cfrac{1}{a_n}}}. \tag{1.28}$$

Given an infinite sequence a_k, $k \geq 1$ we construct the sequence of rational numbers

$$\frac{p_n}{q_n} = [0; a_1, \ldots, a_n]. \tag{1.29}$$

We adopt the convention that rational numbers are always written in reduced form, that is p_n and q_n are relatively prime, thus uniquely determined by a_k, $1 \leq k \leq n$. The terms p_n/q_n are called the *convergents* of the continued fraction. $\lim_{n\to\infty} p_n/q_n$ is an irrational number in $(0, 1)$ (Exercise 8). The continued fraction expansion of a rational number has finite depth, and we adopt the obvious convention

$$[0; a_1, \ldots, a_n] = [0; a_1, \ldots, a_n, +\infty].$$

Definition 1.15. The *rotation number* of a circle map g, denoted $\rho(g)$ is defined to be $[0; a_1, a_2, \ldots]$ if all $a_n < \infty$, and $[0; a_1, \ldots, a_{n-1}]$ if $a_n = \infty$.

The basic properties of the rotation number are the following.

Proposition 1.16. *$\rho(g)$ is a rational number p/q if and only if g has a periodic orbit with period q.*

Proof. g has a periodic point if and only if φ_n has a fixed point for some n. In this case, $a_n < \infty$, $1 \leq k \leq n$ and $a_{n+1} = \infty$. Let $p/q = [0; a_1, \ldots, a_n]$, so that f^q has a fixed point. The period of the orbit cannot be less than q (else, φ_k would have a fixed point for some $k < n$.) □

Proposition 1.17. *The map $g \mapsto \rho(g)$ is continuous in the C^0 topology.*

Proof. For every finite N, the map from g to the first N points of the orbit $O_g(c)$ is continuous in the C^0 topology. It now follows from the definitions that the map $g \mapsto a_n(g)$ is continuous. More precisely, if $a_n(g) < \infty$, there is an $\varepsilon > 0$ such that $a_n(g) = a_n(\tilde{g})$ for $\|g - \tilde{g}\| < \varepsilon$. And if $a_n(g) = \infty$, then for every $M > 0$, there exists $\varepsilon > 0$ such that $a_n(\tilde{g}) > M$ if $\|g - \tilde{g}\| < \varepsilon$. □

Proposition 1.18. *$\rho(g)$ is a topological invariant.*

Proof. Suppose h is a homeomorphism of the circle, and $\tilde{g} = h \circ g \circ h^{-1}$. It is easy to check that g and $\tilde{g}$ are combinatorially equivalent, therefore $a_n(g) = a_n(\tilde{g})$. □

The previous lemma demonstrates the principle that combinatorial equivalence is weaker than topological equivalence. The converse is not true, without further assumptions. That is, $K^+(g) = K^+(\tilde{g})$ does not imply the existence of a homeomorphism that conjugates g to $\tilde{g}$. The best we can hope for without further assumptions is

Theorem 1.19 (Poincaré). *Suppose $f \in S(I)$ has irrational rotation number α. There is a continuous, increasing map h of I onto I such that*

$$h \circ f = R_\alpha \circ h. \tag{1.30}$$

Proof. 1. $\rho(f) = \rho(R_\alpha) = \alpha = [0; a_1, a_2, \ldots]$. Therefore, by Lemma 1.14, $K^+(f) = K^+(R_\alpha)$, and f and R_α are combinatorially equivalent by Proposition 1.8. Let c and $\tilde{c}$ denote the critical points of f and R_α respectively, and c_n and $\tilde{c}_n$ their orbits. Then the map $h : O_f(c) \to O_{R_\alpha}(\tilde{c})$ defined by $h(c_n) = \tilde{c}_n$, $n \geq 0$ is order-preserving and satisfies (1.30).

2. We now extend h to an increasing map on I by defining

$$h(x) = \sup\{h(c_n) \,|c_n \leq x\}.$$

We claim that h is continuous. Suppose not: then there exists x such that $h(x_-) < h(x_+)$. Since the orbit $O_{R_\alpha}(\tilde{c})$ is dense in I, there exists c_n such that $h(x_-) < h(c_n) < h(x_+)$. But this implies both $x < c_n$ and $c_n < x$ (since h is increasing). □

Poincaré's theorem provides a semi-conjugacy, but not a conjugacy. That is, h while continuous, need not have a continuous inverse. If the closed set $F = \overline{O_f(c)}$ is I then h is strictly increasing. Thus, it is an open mapping, hence a homeomorphism. However, the proof does not preclude the possibility that F is a strict subset of I. In this case, $I \backslash F$ is a countable union of open intervals, and h is constant on each of these intervals. We now explore this matter in greater detail.

1.7 Topological conjugacy

Theorem 1.20. *Suppose g is a circle map with irrational rotation number. Then there is a unique minimal set $K \subset S^1$.*

Recall that a set is minimal if it is non-empty, compact, and invariant with no nonempty, compact, invariant proper subset.

Proof. Pick $x \in S^1$. The ω-limit set $\omega(x)$ is closed, invariant and minimal. If $\omega(x) = S^1$ there is nothing to prove (why?). If $\omega(x) \neq S^1$, consider the open set $G = S^1 \backslash \omega(x)$. Since G is open, it may be written as as a union of disjoint, open intervals $G = \cup_{j \in \mathbb{Z}} I_j$. The proof consists in showing that $\alpha(y), \omega(y) \subset \omega(x)$ for every $y \in G$. This inclusion follows from the

Claim: The orbit of y visits each I_j only once.

Proof of the claim: G is also invariant. Thus, if $f^{k_1}(y) \in I_{j_1}$ and $f^{k_2}(y) \in I_{j_2}$, then $f^{k_2-k_1} : I_{j_1} \to I_{j_2}$ is a homeomorphism. If $j_1 = j_2$ then this homeomorphism has a fixed point, contradicting the assumption that $\rho(f)$ is irrational. □

Theorem 1.21. *Suppose g is a circle map with irrational rotation number and minimal set $K \neq S^1$. Then g is not conjugate to a rotation.*

Proof. If g is conjugate to a rotation, then the rotation must be R_α with $\alpha = \rho(g)$. Suppose $h : S^1 \to S^1$ is a homeomorphism. If D is a dense subset of S^1, then so is $h(D)$. Thus, if g is conjugate to R_α, $\overline{O_g(x)} = S^1$ for every $x \in S^1$. Thus, $K = S^1$. □

If the minimal set K has an interior point, then $K = S^1$. Thus, if $K \neq S^1$, it is a Cantor set (ie. perfect and totally disconnected). This is a genuine obstruction to topological conjugacy:

Proposition 1.22. *For every Cantor set $K \subset S^1$ and irrational $\alpha \in (0,1)$ there is a circle map with rotation number α and minimal set K.*

Proof. This is part of HW 3. □

A finer consequence of the proof of Theorem 1.21 is that if $\rho(g)$ is irrational and g is conjugate to a rotation, then there is no open interval J such that the intervals $\{f^k(J)\}_{k\in\mathbb{Z}}$ are pairwise disjoint.

Definition 1.23. An open interval J is *wandering* for the map f if:

1. The intervals J, $f(J), \ldots$ are pairwise disjoint.
2. The ω-limit set of J is not a single periodic orbit.

The second condition is included to rule out the case when f has periodic orbits. A circle map with periodic points may have an interval J that satisfies condition (1), but these are not wandering because of (2).

1.8 Rigidity and non-rigidity of circle maps

We may rephrase Theorem 1.21 as the assertion that a circle map with irrational rotation number cannot be conjugate to a rotation if it has a wandering interval. Proposition 1.22 shows that the assumption of continuity is not enough to rule out wandering intervals. Differentiability does not suffice either!

Theorem 1.24 (Bohl). *For every irrational $\alpha \in (0,1)$ there exists a C^1 diffeomorphism of the circle with a wandering interval.*

Theorem 1.25 (Denjoy). *Suppose f is a C^1 circle diffeomorphism such that $\log Df$ has bounded variation. Then f does not have a wandering interval.*

Remark 1.26. Note that $0 < Df = |Df|$ for an orientation preserving circle diffeomorphism. If f is a C^2 circle diffeomorphism, it satisfies the hypothesis of the theorem since the variation of $\log f'$ on an interval $[a,b]$ is estimated by

$$\log f'(b) - \log f'(a) = \int_a^b \frac{f''}{f'}\,dx \leq \frac{\max|f''|}{\min f'}|b-a|.$$

Remark 1.27. These theorems are loosely called rigidity (and non-rigidity) theorems. Here a circle map is 'rigid' if it is topologically equivalent to a rotation. The surprising aspect of the theorems is the importance of 'metric' smoothness assumptions for topological conclusions. The smoothness gap between these theorems was closed by Herman: for every $\varepsilon > 0$ and irrational $\alpha \in (0,1)$ there is a $C^{2-\varepsilon}$ diffeomorphism with rotation number α that has a wandering interval.

Proof of Theorem 1.24. 1. Fix a bi-infinite sequence of positive numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ such that

$$\sum_{k\in\mathbb{Z}} \lambda_k = 1, \qquad \lim_{|k|\to\infty} \frac{\lambda_{k+1}}{\lambda_k} = 1. \tag{1.31}$$

Let $\{c_k\}_{k\in\mathbb{Z}}$ denote the orbit of the critical point for R_α. For any positive integer n, we may choose closed, disjoint intervals $I_{k,n}$, $|k| \leq n$, with length $|I_{k,n}| = \lambda_k$ that are ordered in the same way as $c_k, |k| \leq n$. Since S^1 is compact, there exists a subsequence $n_j^{(0)}$ such that the intervals $I_{0,n_j^{(0)}}$ converge to an interval I_0 with length λ_0 as $j \to \infty$. We may now extract a subsequence $n_j^{(1)}$ of $n_j^{(0)}$ such that $I_{\pm 1,n_j^{(1)}}$ also converge to intervals denoted $I_{\pm 1}$. Since the intervals $I_{-1,n}$, $I_{0,n}$ and $I_{1,n}$ are ordered in the same way as c_{-1},c_0 and c_1, the limiting intervals I_-,I_+ and I_0 respect the same ordering. We proceed in this manner, obtaining subsequences $n_j^{(N)}$ such that $I_{k,n_j^{(N)}}$ converges to an interval I_k with length λ_k for $|k| \leq N$. Since the subsequences are nested, ie. $n_j^{(m)} \subset n_j^{(m)}$, I_k is independent of N for all $|k| < N$ and I_k is ordered in the same way as c_k.

2. Let $A = \cup_{k=0}^{\infty} I_k^o$. By construction, A is open and has full measure (thus, it is dense). We use bump functions to construct a C^∞ diffeomorphism f_k from $I_k \to I_{k+1}$ such that $f_k' = 1$ at the endpoints of I_k and $\|f_k' - 1\| \leq 2\lambda_{k+1}/\lambda_k$. We the define $f : A \to A$ and $h : A \to S^1$ by

$$f(x) = f_k(x), \quad h(x) = c_k, \quad x \in I_k.$$

It is easy to check that $h \circ f = R_\alpha \circ h$ for $x \in A$. Since A and $\{c_k\}$ are dense in S^1, h and f extend to continuous maps of the circle. Thus, $\rho(f) = \alpha$. Since f_k is a diffeomorphism, so is f restricted to A with $f^{-1}(x) = f_{k-1}^{-1}(x)$, $x \in I_k$. The inverse also satisfies $h \circ f^{-1} = R_{-\alpha} \circ h$.

3. We must show that f is a C^1 diffeomorphism. It will suffice to show that f is differentiable at each $y \in S^1 \backslash A$ with derivative 1, that is

$$\lim_{z \to y} \frac{f(z) - f(y)}{z - y} = 1. \tag{1.32}$$

We will prove that the derivative from above

$$\lim_{z \downarrow y} \frac{|f([y,z])|}{|[y,z]|} = 1 \tag{1.33}$$

A similar argument for the lower derivative establishes (1.32). Finally, this estimate also establishes that f^{-1} has derivative 1 on A.

4. If y is a left endpoint of I_n then (1.33) follows from the construction of f_n. Thus, we consider the case when $[y,z] \cap I_k \neq \phi$ for infinitely many k. Let $N_1 = \{n \,|I_n \cap [y,z] \neq \phi\}$ and $N_2 = \{n \,|I_n \subset [y,z]\}$. Notice that $N_2 \subset N_1$ and $\# N_1 \backslash N_2 \leq 2$. It follows from these definitions that

$$\sum_{n \in N_2} \lambda_n \leq |[y,z]| \leq \sum_{n \in N_1} \lambda_n.$$

Similarly, since $f(I_n) = I_{n+1}$ we find,

$$\sum_{n \in N_2} \lambda_{n+1} \leq |f([y,z])| \leq \sum_{n \in N_1} \lambda_{n+1}.$$

Therefore,

$$\frac{\sum_{n \in N_2} \lambda_{n+1}}{\sum_{n \in N_1} \lambda_n} \leq \frac{|f([y,z])|}{|[y,z]|} \leq \frac{\sum_{n \in N_1} \lambda_{n+1}}{\sum_{n \in N_2} \lambda_n}$$

As $z \to y$, we must have $m = \min\{n | n \in N_i\} \to \infty$. Thus, the second condition in (1.31) implies (1.33). □

1.9 Denjoy's theorem

Definition 1.28. The *distortion* of a C^1 map g on an interval J is

$$\operatorname{Dist}(g, J) = \sup_{x,y \in J} \log \frac{|Dg(y)|}{|Dg(x)|}. \tag{1.34}$$

The introduction of the logarithm is natural for iterated maps.

Lemma 1.29. $\operatorname{Dist}(f^n, J) \leq \sum_{k=0}^{n-1} \operatorname{Dist}(f, f^k(J))$.

Proof. By the chain rule,

$$Df^n(x) = Df(f^{n-1}(x))Df(f^{n-2}(x)) \ldots Df(x).$$

Therefore,

$$\begin{aligned} \log \frac{|Df^n(y)|}{|Df^n(x)|} &= \log \prod_{k=0}^{n-1} \frac{\left|Df(f^k(y))\right|}{|Df(f^k(x))|} \\ &= \sum_{k=0}^{n-1} \log \frac{\left|Df(f^k(y))\right|}{|Df(f^k(x))|} \leq \sum_{k=0}^{n-1} \operatorname{Dist}(f, f^k(J)). \end{aligned}$$

□

As a corollary, if $Df(x) \neq 0$ in J and $\log|Df(x)|$ is Lipschitz with constant C we have

$$\operatorname{Dist}(f^n, J) \leq C \sum_{k=0}^{n-1} \left| f^k(J) \right|. \tag{1.35}$$

Proof of Theorem 1.25. 1. Suppose J is a wandering interval. Then the intervals $f^k(J)$, $k \in \mathbb{Z}$ are disjoint and have finite total length

$$\sum_{k \in Z} \left| f^k(J) \right| \leq 1. \tag{1.36}$$

2. The main geometric observation is that for every $x \in S^1$ the intervals $f^k([x, f^{-q_n}(x)])$ are disjoint for $0 \leq k \leq q_{n-1}$. This follows from Lemma 1.12 when x is c. It would then also hold for every x if f were simply $R_{\rho(f)}$. But the ordering of points by f is equivalent to $R_{\rho(f)}$, thus it also holds for f.

3. We let T be the smallest subinterval containing both J and $f^{-q_n}(J)$ that is contained in $S^1 \backslash f^{q_n}(J)$. Then $f^k(T)$ are disjoint for $0 \leq k \leq q_{n-1}$. This again follows by considering $R_{\rho(f)}$ as in step (2).

4. Let $V = \mathrm{var}(\log|Df|, S^1)$ denote the total variation of $\log|Df$ on S^1. The distortion of f^{q_n} on T is uniformly controlled by V as follows. By Definition 1.28 and Lemma 1.29, we have

$$\mathrm{Dist}(f^{q_n}, T) \leq \mathrm{var}\left(\log|Df|, \cup_{k=0}^{q_n-1} f^k(T)\right) \leq \mathrm{var}\left(\log|Df|, S^1\right) = V. \tag{1.37}$$

5. By the mean value theorem, there exists $x \in J$ and $y \in f^{-q_n}(J)$ such that

$$|Df^{q_n}(x)||J| = |f^{q_n}(J)|, \quad |Df^{q_n}(x)||f^{-q_n}(J)| = |J|$$

Now both $x, y \in T$, therefore

$$\log \frac{|Df^{q_n}(y)|}{|Df^{q_n}(y)|} \leq \mathrm{Dist}(f^{q_n}, T) \leq V.$$

Therefore, in contradiction to step (1) we have

$$\frac{|J||J|}{|f^{q_n}(J)||f^{-q_n}(J)|} \leq e^V. \tag{1.38}$$

□

1.10 Smooth conjugacy and KAM theory

Denjoy's theorem settles the question of topological conjugacy, and opens the door to questions of smooth conjugacy. If a circle map g with $\rho(g)$ irrational is smooth enough to rule out wandering intervals, is it true that the conjugacy h is smooth too? For example, if g is (real) analytic, is it also true that h is analytic? The answers to this question are surprisingly delicate. In order to understand the basic obstruction let us proceed formally at first.

In this section we will work on the covering space $\mathbb{R}$. We consider $g : \mathbb{R} \to \mathbb{R}$ such that $g(x+1) = g(x)+1$. An equivalent definition of the rotation number is $\rho(g) = \lim_{n\to\infty} \sum_{k=0}^{n-1} g^k(x)/n$. We shall consider the smoothness of the conjugacy for small perturbations of rotations. That is, we consider maps of the form

$$g = x + \alpha + u(x) \tag{1.39}$$

where $u(x+1) = u(x)$. Suppose $\rho(g) = \alpha$, and consider the conjugacy

$$g \circ h = h \circ R_\alpha. \tag{1.40}$$

When $u \equiv 0$, h is the identity. Therefore, we substitute $h(x) = x + v(x)$ in (1.40) to obtain

$$v(x+\alpha) - v(x) = u(x + v(x)). \tag{1.41}$$

For small u we expect that the first step in the solution of this functional equation is to solve the linearized problem

$$v(x+\alpha) - v(x) = u(x). \tag{1.42}$$

The linearized problem is called the *homological equation.* A formal solution is easily obtained using Fourier series. We substitute the Fourier expansions

$$u(x) = \sum_{k \in Z} e^{2\pi i k x} \hat{u}_k, \quad v(x) = \sum_{k \in Z} e^{2\pi i k x} \hat{v}_k, \tag{1.43}$$

in (1.42) to obtain

$$\hat{v}_k = \frac{\hat{u}_k}{e^{2\pi i k \alpha} - 1}, \quad k \neq 0. \tag{1.44}$$

In addition, we must assume $\hat{u}_0 = 0$. Elegant as this formal solution may be, it comes with a subtle convergence problem. The denominators $e^{2\pi i k \alpha} - 1$ never vanish because α is irrational. Nevertheless, if we consider the convergents p_n/q_n of α and set $k = q_n$ we find the *small divisors*

$$\left| e^{2\pi i k \alpha} - 1 \right| = \left| e^{2\pi i q_n \alpha} - e^{2\pi i p_n} \right| \leq 2\pi q_n \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{2\pi}{q_n}.$$

Worse yet, there are irrational numbers (the Liouville numbers) which are arbitrarily badly approximated by rational numbers, that is for every positive integer m there are integers p_m, q_m such that $|p_m/q_m - \alpha| \leq 1/q^m$. We can no longer prove results such as Theorems 1.24 and 1.25 that hold for all irrational numbers. Further assumptions on α are required. The fundamental assumption is that α is badly approximated by rational numbers in the following sense:

Definition 1.30. An irrational number α is *Diophantine of class* (K, σ) if

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{K}{|q|^{2+\sigma}} \tag{1.45}$$

for all integers p, q and some fixed $K, \sigma > 0$.

Not all irrational numbers satisfy Diophantine conditions. However the exceptions form a set of measure zero (but of first category).

We shall now construct a solution to the conjugacy equation q1.41, basing our analysis on the homological equation(1.42). A decay assumption on $\hat{u}_k$ is clearly necessary. This is essentially an assumption on the smoothness of u. We assume that u is analytic. More precisely, let S_ρ denote the horizontal strip $\{x + iy \,||y| < \rho\}$. For u defined on S_ρ we define the norm $\|u\|_\rho = \sup_{z \in S_\rho} |u(z)|$. An analytic function is *univalent* if it is one to one. A univalent map $g : \Omega \to \mathbb{C}$ defines a conformal map from Ω to $g(\Omega)$. The fundamental result on conformal conjugacy is

Theorem 1.31 (Arnol'd). *Let α be Diophantine of class (K, σ) and $\rho > 0$. There exists $\varepsilon(K, \sigma, \rho) > 0$ such that if g is a circle map with rotation number α that extends to a univalent function on S_ρ and $u = g - R_\alpha$ satisfies $\|u\|_\rho < \varepsilon$, then g is conformally conjugate to R_α on the strip $S_{\rho/2}$.*

This is the first of a circle of results called the KAM theory (for Kolmogorov, Arnol'd, and Moser). The unifying aspects of KAM theorems are (a) loss of smoothness in the linear problem because of small divisors, (b) Diophantine assumptions, and (c) accelerated convergence as in Newton's method. The proof of Theorem 1.31 demonstrates all these aspects. We first study the linear problem, then the iteration scheme.

At an abstract level, Theorem 1.31 has the character of an implicit function theorem. Roughly, solvability of a linear problem at a point implies solvability of a nonlinear problem in a neighborhood of the point (look up the statement and proof in a book on advanced calculus). The catch here is the loss of smoothness in the linear problem. This affects the 'implicit function theorem' as we obtain a solution to the nonlinear problem in a worse space.

Lemma 1.32. *Assume u is a 1-periodic function analytic in S_ρ and continuous in $\overline{S_\rho}$. Then its Fourier coefficients satisfy*

$$|\hat{u}_k| \leq e^{-2\pi\rho|k|}\|u\|_\rho, \quad k \in \mathbb{Z}. \tag{1.46}$$

Proof. The Fourier coefficients are defined by

$$\hat{u}_k = \int_0^1 e^{-2\pi ikx} u(x)\, dx.$$

Assume $k > 0$. We shift the contour of integration down by $-i\rho$ and use Cauchy's theorem to obtain

$$\hat{u}_k = e^{-2\pi k\rho} \int_0^1 e^{-2\pi ikx} u(x - i\rho)\, dx.$$

□

The Diophantine condition is used in the following lower estimate.

Lemma 1.33. *Assume α is of type (K,σ). Then for every integer $k \neq 0$ we have*

$$\left| e^{2\pi i k\alpha} - 1 \right| \geq \frac{4K}{|k|^{1+\sigma}}. \tag{1.47}$$

Proof. Suppose $\theta \in (0,\pi)$. Then (draw a picture!)

$$\theta \geq |e^{i\theta} - 1| \geq \frac{2\theta}{\pi}.$$

Now apply the Diophantine condition (1.45). □

We now apply this estimate to the solution of the linear equation (1.42). In all that follows, we assume that u is analytic in S_ρ. We do not assume that u has mean zero, and must consider the modified homological equation

$$v(x+\alpha) - v(x) = u(x) - \hat{u}_0. \tag{1.48}$$

Lemma 1.34. *Assume (1.48) holds and α satisfies the Diophantine condition (1.45). There exists a constant $C(K,\sigma)$ such that*

$$\|v\|_{\rho-\delta} \leq C\|u\|_\rho \, \delta^{-(2+\sigma)}, \quad 0 < \delta < \rho. \tag{1.49}$$

Proof. Since $\hat{u}_0 = 0$ we may solve the homological equation (1.44). We fix $z \in S_{\rho-\delta}$ and estimate $|v(z)|$ using Lemmas 1.32 and 1.33 to obtain

$$|v(z)| \leq \sum_{k\in\mathbb{Z}} |\hat{v}_k(z)||e^{2\pi i k z}| \leq \frac{\|u\|_\rho}{4K} \sum_{k\in\mathbb{Z}} |k|^{1+\sigma} e^{-2\pi|k|\delta}.$$

Let $S(\delta)$ denote the sum $\sum_{k=1}^{\infty} k^{1+\sigma} e^{-2\pi k\delta}$. $S(\delta)$ is a continuous and decreasing function of δ with the following asymptotics.

$$\lim_{\delta\to 0} \delta^{2+\sigma} S(\delta) = \int_0^\infty x^{1+\sigma} e^{-2\pi x}\, dx.$$

As $\delta \to \infty$, $S(\delta)$ decays exponentially with any rate less than 2π. Thus, there is $c(\sigma)$ such that $S(\delta) \leq c\delta^{-(2+\sigma)}$, $\delta > 0$ and $C = c/2K$. □

The solution to the nonlinear equation (1.48) is built out of a sequence of linear approximations. Having solved for v, we construct a circle homeomorphism $h = x + v(x)$ and a new circle map $g^1 = x + u^1$ via the conjugacy $g^1 = h^{-1} \circ g \circ h$. We then solve the homological equation (1.48). This process is then iterated. The difficulty we face is the unavoidable loss of smoothness in the solution of the linear problem: (1.49) diverges as $\delta \to 0$. It is crucial to obtain accelerated convergence so that the deterioting linear estimates do not sabotage the scheme. The margin of victory is the *quadratic* factor $\|u\|_\rho^2$ below. This is akin to Newton's method for solving an equation. Suppose $f : \mathbb{R} \to \mathbb{R}$ and we are trying to find a zero x_* of f via Newton's iteration scheme $x_{n+1} = x_n - f(x_n)/f'(x_n)$. For x_n near x_* the scheme converges rapidly, because $|x_{n+1} - x_*| \le C|x_n - x_*|^2$.

Lemma 1.35. *There are positive constants c,C and δ_* such that if*

$$\|u\|_\rho \le c\delta^{3+\sigma}, \quad 0 < \delta < \delta_*, \tag{1.50}$$

then

$$\|u^1\|_{\rho-\delta} \le C\|u\|_\rho^2\, \delta^{-(3+\sigma)}. \tag{1.51}$$

Proof. 1. We first verify that $h = x + v$ defines an analytic diffeomorphism on a strip $S_{\rho-\delta}$ for suitable $0 < \delta < \rho$. Let C be as in Lemma 1.34 and suppose η and $\|u\|_\rho$ are chosen so small that

$$\|u\|_\rho \le \frac{\eta^{3+\sigma}}{2C} \le \frac{\eta}{2}. \tag{1.52}$$

Then we use Lemma 1.34 and Cauchy's formula to obtain

$$\|v\|_{\rho-\eta} \le \frac{\eta}{2}, \quad \|v'\|_{\rho-2\eta} \le \frac{1}{2}. \tag{1.53}$$

Since $h(z) = z + v(z)$, for $z_1, z_2 \in S_{\rho-2\eta}$ we then have

$$\frac{1}{2}|z_1 - z_2| \le |h(z_1) - h(z_2)| \le \frac{3}{2}|z_1 - z_2|.$$

Thus, h is one-one on $S_{\rho-2\eta}$ with a derivative uniformly bounded away from 0. The implicit function theorem guarantees the existence of an analytic inverse on the image $h(S_{\rho-2\eta})$. In particular, h^{-1} is defined on $S_{\rho-3\eta}$.

Let $\delta = 5\eta < \rho/2$. Then $h(S_{\rho-\delta}) \subset S_{\rho-4\eta}$ and $g\circ h(S_{\rho-\delta}) \subset S_{\rho-4\eta+\|u\|_\rho} \subset S_{\rho-3\eta}$. Thus, $g^1 = h^{-1} \circ g \circ h$ is analytic on $S_{\rho-\delta}$.

2. We now estimate u^1. Since $h \circ g^1 = g \circ h$, we have

$$u^1(z) = v(z) - v(z + \alpha + u^1(z)) + u(z + v(z)). \tag{1.54}$$

Since v solves the homological equation (1.48), we have

$$u^1(z) = (u(z + v(z)) - u(z)) - \big(v(z + \alpha + u^1(z)) - v(z + \alpha)\big) + \hat{u}_0. \tag{1.55}$$

We apply Lemma 1.34 and Cauchy's theorem to the first term to obtain

$$\|u(z+v(z))-u(z)\|_{\rho-\delta} \le \|u'\|_{\rho-\delta}\|v\|_{\rho-\delta} \le \frac{\|u\|_\rho}{\delta}\|v\|_{\rho-\delta} \le C\|u\|_\rho^2\,\delta^{-(3+\sigma)}.$$

The second term admits the estimate

$$\|v(z+\alpha+u^1(z))-v(z+\alpha)\|_{\rho-\delta} \le \|v'\|_{\rho-\delta}\|u^1\|_{\rho-\delta} \le \frac{\|v\|_{\rho-\delta/2}}{\delta/2}\|u^1\|_{\rho-\delta}.$$

We combine these estimates and Lemma 1.34 to find

$$\left(1-C\|u\|_\rho\delta^{-(3+\sigma)}\right)\|u^1\|_{\rho-\delta} \le C\|u\|_\rho^2\,\delta^{-(3+\sigma)}+|\hat{u}_0|.$$

3. In order to estimate $\hat{u}_0$ we use the fact that g^1 and g have the same rotation number. It follows that $u^1(x)=0$ for some x. Set $x=z$ in (1.55) to obtain

$$|\hat{u}_0| = |u(z+v(z))-u(z)| \le \|u(z+v(z))-u(z)\|_{\rho-\delta} \le C\|u\|_\rho^2\delta^{-(3+\sigma)}.$$

If we choose $c>0$ sufficiently small that (1.50) holds, we obtain (1.51) □

Lemmas 1.34 and Lemma 1.35 constitute the main estimates for the iterative scheme. We construct mappings u^n defined on the strip S_{ρ_n} where $\rho_n = \rho_{n-1}-\delta_{n-1}$ and $\rho_0=\rho$, $\delta_0=\delta$. We fix $\beta\in(1,2)$ and choose

$$\delta_n = \delta_{n-1}^\beta, \quad \delta_0 < \min(\delta_*, 1) \tag{1.56}$$

In this case, by further reducing δ_0 if necessary, we have

$$\delta_n = \delta^{\beta^n}, \quad \text{and} \quad \sum_{n=0}^\infty \delta_n < \frac{\rho}{2}. \tag{1.57}$$

Therefore, $\rho_n > \rho/2$, $n\ge 0$ and the limiting map will be defined on $S_{\rho/2}$. Lemma 1.35 yields the estimate

$$\|u^n\|_{\rho_n} \le C\|u^{n-1}\|_{\rho_{n-1}}^2\delta_{n-1}^{-(3+\sigma)}. \tag{1.58}$$

We solve this recurrence to obtain

$$\|u^n\|_{\rho_n} \le C^{1+2+\ldots+2^n}\left(\|u\|_\rho\right)^{2^n}\delta^{-(3+\sigma)r_n}, \tag{1.59}$$

where the exponent r_n is

$$r_n = 2^n\left(1+\frac{\beta}{2}+\ldots+\frac{\beta^n}{2^n}\right) \le \frac{2^n}{1-\beta/2}.$$

Therefore, we have

$$\|u^n\|_{\rho_n} \leq \left(C^2\|u\|_\rho \delta^{-(3+\sigma)/(1-\beta/2)}\right)^{2^n}.$$

If we choose $\|u\|_\rho \leq \delta^\kappa$ for κ large enough, we obtain a constant denoted $\gamma > 0$ and estimates of the form

$$\|u^n\|_{\rho_n} \leq \delta^{-\gamma 2^n},$$

along with the estimate

$$\|v^n\|_{\rho_n} \leq \delta_n = \delta^{\beta^n}.$$

Finally, we consider the analytic diffeomorphim $H^n = h^0 \circ h^1 \ldots \circ h^{n-1}$ on the strip $S_{\rho/2}$.By the chain rule, the derivative

$$(H^n)'(z) = \prod_{k=0}^{n-1} (h^k)'(z_k)$$

for some points $z_k \in S_{\rho/2}$. Since $h^k(z) = z + v^k(z)$ this yields the upper and lower bounds

$$\prod_{k=0}^{n-1}(1-\delta_k) \leq \|H^n\|_{\rho/2} \leq \prod_{k=0}^{n-1}(1+\delta_k).$$

The choice of δ_k ensures both infinite products converge. This bound ensures that H_n converges. Indeed,

$$\left|H^n(z) - H^{n+1}(z)\right| = |H^n(z) - H_n(h^n(z))| \leq \|(H^n)'\|_{\rho/2}\|v^n\|_{\rho/2}.$$

This completes the proof of Arnol'd's theorem.