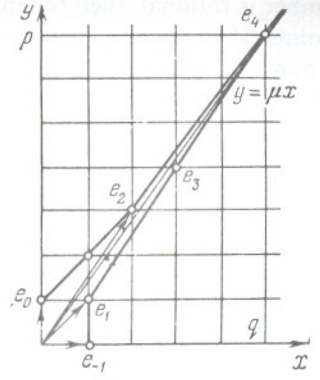

Figure 81.

L. Approximation of Irrational Numbers by Rational Ones

Theorem. *For any irrational number* μ, *there exist arbitrarily accurate rational approximations whose error is less than the reciprocal value of the square of the denominator:*

$$\left|\mu - \frac{p}{q}\right| < \frac{1}{q^2}.$$

For example, the number π can be approximated with an error of the order of one millionth by a rational fraction with a three-digit numerator and denominator: $\pi \approx 335/113$.

Before proving the theorem, we indicate a geometric method of finding an infinite sequence of such approximations (the method is called the algorithm of *continuous fractions*, or the algorithm of stretching the noses, or simply the *Euclidean algorithm*).

Consider the plane with the coordinate system (x, y) (Fig. 81).

We draw the line $y = \mu x$. For the sake of definiteness, we assume $\mu > 0$. In the first quadrant we mark all points with integral coordinates. Except for the point O, they do not lie on our line, since μ is irrational. We consider the convex hulls of lattice points of the quadrant lying on one side of our line ("below" it) and on the other side ("above" it). [In order to construct these convex hulls, we may visualize a thread fastened at infinity and lying on our line. Let us imagine that a nail is hammered in at every lattice point of our quadrant other than O. Pull the free end O of the thread downward (upward). Then the thread will touch some nails and stretch, forming the boundary of the lower (upper) convex hull.] The vertices of the convex polygonal lines thus constructed give the required approximations of the irrational number μ. If the integers (q, p) are coordinates of a vertex, then the fraction p/q corresponding to the vertex is called a *convergent fraction for* μ. It turns out that for any convergent fraction we have

$$\mu - \left(\frac{p}{q}\right) < \frac{1}{q^2}.$$

To prove this inequality, we describe the construction of our convex polygonal lines by another method. Denote by e_{-1} the basis vector (1, 0) and by e_0 the vector (0, 1). These vectors lie on different sides of the line $y = \mu x$. We construct a sequence of vectors $e_1, e_2, \cdots$ in the following way. Let e_{k-1} and e_k be already constructed and lie on different sides of our line. We add e_k to e_{k-1} as many times as we can in such a way that the sum lies on the same side of the line $y = \mu x$ as e_{k-1}.

In this way, we obtain a sequence of natural numbers a_k and a sequence of lattice vectors

$$e_1 = e_{-1} + a_0 e_0, \ \ldots, \qquad e_{k+1} = e_{k-1} + a_k e_k, \ \ldots.$$

The vectors e_k are the vertices of our two convex hulls (the upper one for even k and the lower one for odd k).

Lemma. *The oriented area of the parallelogram spanned by the vectors* (e_{k+1}, e_k) *is equal to* $(-1)^k$ *(taking into account the orientation).*

◀ For the initial parallelogram (e_0, e_{-1}), this is evident. Every following parallelogram has a common side with the preceding one and equal altitude, and gives an opposite orientation of the plane. ▶

Corollary. *Denote by* q_k *and* p_k *the coordinates of the point* e_k. *The difference of two subsequent convergent fractions is equal to*

$$\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} = \frac{(-1)^k}{q_k q_{k+1}}.$$

◀ In bringing the fractions to a common denominator, it turns out that the numerator is the determinant of the components of e_{k+1} and e_k, which is equal to the oriented area of the parallelogram. ▶

◀ *Proof of Theorem.* The vectors e_k lie alternately on one or the other side of the line $y = \mu x$.

Therefore, the convergent fractions are alternately larger or smaller than μ. Consequently, the difference between μ and a convergent fraction is smaller than the modulus of the difference between the convergent fraction and the subsequent convergent fraction. By the corollary, the absolute value of this difference is equal to $1/q_k q_{k+1}$, which is not larger than $1/q_k^2$, since $q_{k+1} \geqslant q_k$ for $k \geqslant 0$. ▶