PDE, HW 4 solutions

1. We only need show $W_0^{1,p}(\mathbb{R}^n) \supset W^{1,p}(\mathbb{R}^n)$. Let $u \in W^{1,p}(\mathbb{R}^n)$. For any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B(0,R)} (|u|^p + |Du|^p)\, dx < \varepsilon.$$

Let χ be a smooth cut-off function with $\chi = 1$ in $B(0,R)$ and $\chi = 0$ outside $B(0,R+2)$ and $|D\chi| \leq 1$. One may then compute that for every $1 \leq p < \infty$, there is a constant C_p such that

$$\|u - u\chi\|_{W^{1,p}(\mathbb{R}^n)} \leq C_p \varepsilon.$$

Let ψ be a standard mollifier, and for any $\eta > 0$ consider $u^\eta = \psi_\eta \star (u\chi)$. Then $u^\eta \in C_c^\infty(\mathbb{R}^n)$, and $u^\eta \to u\chi$ in $W^{1,p}(\mathbb{R}^n)$ as $\eta \to 0$. Thus, we have for sufficiently small $\eta > 0$

$$\|u^\eta - u\| \leq (C_p + 1)\varepsilon.$$

$\square$

2. Suppose there is $A > 0$ such that $|u(y) - u(z)| \leq A|y - z|^\alpha$ for $y, z \in \Omega$. We then have for $y \in B(x,r)$

$$\begin{aligned} |u(y) - \bar{u}_B| &= \left| u(y) - ⨍_{B(x,r)} u(z)\, dz \right| \\ &\leq ⨍_{B(x,r)} |u(y) - u(z)|\, dz \leq A ⨍_{B(x,r)} |y - z|^\alpha\, dz \leq A(2r)^\alpha, \end{aligned}$$

using the uniform estimate $|y - z| \leq 2r$ for $y, z \in B(x,r)$. We integrate this estimate over $y \in B(x,r)$ to obtain Campanato's inequality. $\square$

3. If $u = |x|^\alpha$ we need $\int_B (|u|^p + |Du|^p)\, dx < \infty$. Here

$$\int_B |u|^p\, dx = \omega_n \int_0^1 r^{\alpha p} r^{n-1}\, dr < \infty, \iff \quad \alpha > -\frac{n}{p}.$$

Similarly, $Du = D|x|^\alpha = \alpha |x|^{\alpha-2} x$, and $|Du| = |\alpha||x|^{\alpha-1}$ which is in $L^p(B)$ if and only if

$$\alpha > 1 - \frac{n}{p}.$$

$\square$

4. This is a good illustration of scale-invariance. We have to show that

$$\sup_{x\in\mathbb{R}^n, r>0} \fint_{B(x,r)} \left| \log|y| - \fint_{B(x,r)} \log|z|\,dz \right| dy < \infty.$$

It is instructive to compute the integrand explicitly when $x = 0$ to see the cancellation of $\log r$ that shows the supremum over r is irrelevant. This is a particular manifestation of invariance under the rescaling. If we set

$$x' = \frac{x}{r}, \quad y' = \frac{y}{r}, \quad z' = \frac{z}{r},$$

the integral above is

$$I(x') := \fint_{B(x',1)} \left| \log|y'| - \fint_{B(x',1)} \log|z'|\,dz' \right| dy',$$

and we now have to show that the supremum over $x' \in \mathbb{R}^n$ is finite. Since $\log|x| \in L^1_{loc}(\mathbb{R}^n)$, $I(x')$ is a continuous function of x', and it will suffice to show it stays bounded as $|x'| \to \infty$. We clearly have

$$I(x') \le \fint_{B(x',1)} \fint_{B(x',1)} \left| \log\left(\frac{|y'|}{|z'|}\right) \right| dz'dy'.$$

For $y', z' \in B(x',1)$ we have $|x'| - 1 \le |y'|, |z'| \le |x'| + 1$, thus for large enough $|x'|$ we have

$$\log\left(\frac{|x'|-1}{|x'|+1}\right) \le \log\left(\frac{|y'|}{|z'|}\right) \le \log\left(\frac{|x'|+1}{|x'|-1}\right).$$

Both sides converge uniformly to zero, which shows that $I(x') \to 0$ uniformly as $x' \to \infty$. $\square$

5. A 'soft' function-analytic proof (ie. deferring to harder theorems proved in analysis) is the following. The set of nowhere differentiable functions in $C^{0,\alpha}$ is of full category; however functions in $W^{1,p}$ are differentiable a.e (this part of Rademacher's theorems works for any p). Thus, 'most' functions in $C^{0,\alpha}$ are not in $W^{1,p}$. $\square$

6. The hard part of showing $BV(\Omega)$ is a Banach space is completeness. This requires a proof that BV functions have measures as derivatives. Precisely, suppose $u \in BV(\Omega)$. Define the linear functional $L : C^1_c(\Omega;\mathbb{R}^n) \to \mathbb{R}$

$$L(\mathbf{v}) := -\int_\Omega u \operatorname{div} \mathbf{v}\,dx.$$

Observe that by the definition of $\int_\Omega |Du|\,dx$, we have

$$|L(\mathbf{v})| \leq \left(\int_\Omega |Du|\,dx\right) \|v\|_\infty.$$

Therefore, L extends to a continuous linear functional on $C_0(\Omega;\mathbb{R}^n)$ and by the Riesz Representation theorem (a vector-valued version), there exists a positive finite Radon measure, $\mu(dx)$ (the magnitude), and a unit vector $\sigma(x)$ (the direction) representing the extension of L. In particular,

$$L(\mathbf{v}) = -\int_\Omega u\,\mathrm{div}\mathbf{v}\,dx = \int_\Omega \mathbf{v}\cdot\sigma(x)\mu(dx), \quad \mathbf{v}\in C_c^1(\Omega;\mathbb{R}^n).$$

Completeness is obtained by using completeness of the space of vector-valued Radon measures. You must check that u_k is a Cauchy sequence in BV if and only if u_k is a Cauchy sequence in $L^1(\Omega)$ and the measures $\sigma_k\mu_k(dx)$ are a Cauchy sequence in the space of vector-valued Radon measures on Ω.

If $u \in W^{1,1}(\Omega)$, we have

$$\int_\Omega u\,\mathrm{div}\,\mathbf{v}\,dx = -\int_\Omega Du\cdot\mathbf{v}\,dx \leq \|Du\|_{L^1(\Omega)}\|\mathbf{v}\|_\infty.$$

Chasing definitions, we see that for $u \in W^{1,1}(\Omega)$

$$\|Du\|_{L^1(\Omega)} = \int_\Omega |Du|dx.$$

That is, total variation *is* the L^1 norm of Du in this case. Hence, $W^{1,1}(\Omega) \subset BV(\Omega)$ and a Cauchy sequence (in the BV norm) of a sequence in $W^{1,1}$ is also a Cauchy sequence in $W^{1,1}$. Thus, $W^{1,1}$ is a closed subspace. Note that it is a proper closed subspace. For example, the function $\mathbf{1}_{|x|\leq 1}$ is in BV but not in $W^{1,1}$. □

7. As always, we have to mollify the u_k. However, things are a little tricky because of the boundary, and you have to consider a partition of unity. A detailed proof may be found in 'Measure theory and fine properties of functions' by L. C. Evans and R. F. Gariepy, p. 172. □