PDE, HW 4 solutions

Problem 2, John p. 213. Let α be a multi-index with $|\alpha| \le s$. Formally,

$$\begin{aligned}\partial_x^\alpha u(x,t) &= \int_{\mathbb{R}^n} \partial_x^\alpha k(x-y,t) f(y)\,dy \\ &= \int_{\mathbb{R}^n} (-1)^{|\alpha|} \partial_y^\alpha k(x-y,t) f(y)\,dy = \int_{\mathbb{R}^n} k(x-y,t)\partial_y^\alpha f(y)\,dy. \quad (0.1)\end{aligned}$$

The formal calculation is justified as follows. As explained on p.218, u is analytic in x for every $t > 0$, thus $\partial_x^\alpha u(x,t)$ exists and equals the first term in (0.1) for every $t > 0$. Integration by parts $|\alpha|$ times is legitimate because of the assumption that $\sup_x |\partial^\alpha f| < \infty$ for $|\alpha| \le s$. The issue here is the continuity at $t = 0$, and here we may apply the theorem on p. 209 with f replaced by $\partial^\alpha f$. □

Problem 3, John p. 213. The first step is to show that $u(x,t)$ is well-defined for $x = \xi + i\eta$, $t = \sigma + i\tau$ in the given range, that is $\int_{\mathbb{R}^n} |k(x,y,t)f(y)|dy < \infty$. Combine the assumption $|f(x)| \le Me^{a|x|^2}$ with the estimate above (1.16) on p.211 to obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} |k(x,y,t)f(y)|dy \\ &\quad \le \frac{M}{(4\pi)^{n/2}} \frac{e^{|\eta|^2/4\sigma}}{(\sigma^2+\tau^2)^{n/4}} \int_{\mathbb{R}^n} \exp(a|y|^2 - \frac{|\sigma(\xi-y)+\tau\eta|^2}{4(\sigma^2+\tau^2)\sigma}\,dy.\end{aligned}$$

The integral is finite if and only if the coefficient of $|y|^2$ in the exponent is negative, that is

$$a - \frac{\sigma}{\sigma^2+\tau^2} < 0. \quad (0.2)$$

To prove analyticity, we must check that u is (complex) differentiable in x and t. The derivatives of the kernel are the analytic functions

$$D_x k(x,y,t) = \frac{x-y}{t} k(x,y,t), \quad \partial_t k(x,y,t) = \left(-\frac{n}{2t} + \frac{(x-y)^t(x-y)}{2t^2}\right) k(x,y,t).$$

Formally, the derivatives of u are obtained by differentiating under the integral sign

$$D_x u(x,t) = \int_{\mathbb{R}^n} D_x k(x,y,t) f(y)\,dy, \quad \partial_t u(x,t) = \int_{\mathbb{R}^n} \partial_t k(x,y,t) f(y)\,dy.$$

Now observe that we may combine the estimate on f and the estimates on k as above to show that these integrals are well-defined if (0.2) holds. Finally, to justify this calculation, one may use finite differences and pass to the limit via the mean value theorem and the dominated convergence theorem. $\square$

Problem 7, John p.213. (a) Since f and $k(x,y,t)$ are positive, for $\xi \in \mathbb{R}$

$$|u(\xi,t)| \le \int_{\mathbb{R}} k(\xi,y,t) f(y)\, dy = u(\xi,t) \le |u(\xi,t)|.$$

Now use estimate (1.16) on p.218 with $\sigma = t$, $\tau = 0$ to obtain directly,

$$|u(\xi + i\eta, t)| \le e^{\eta^2/4t} \int_{\mathbb{R}^n} k(\xi,y,t) f(y)\, dy = e^{\eta^2/4t} u(\xi,t).$$

(b) The estimate in part (a) tells us that u is an entire function in x. Therefore, for any $x \in \mathbb{C}$ and $r > 0$ we have by Cauchy's integral formula

$$D_x u(x,t) = \frac{1}{2\pi i} \int_{|x-z|=r} \frac{u(z)}{(x-z)^2}\, dz.$$

This always yields the estimate

$$|D_x u(x,t)| \le \frac{1}{r} \sup_{|x-z|=r} |u(z)| = \frac{1}{r} \sup_{|z|=r} |u(x+z)|.$$

In particular, if $x \in \mathbb{R}$, and $z = \xi' + i\eta'$, by part (a)

$$\sup_{|z|=r} |u(x+z)| \le \sup_{\xi'^2 + \eta'^2 = r^2} e^{\eta'^2/4t} u(x+\xi') \le e^{r^2/4t} \sup_{|\xi'| \le r} u(x+\xi').$$

If we choose $r = \sqrt{ct}$ we obtain an estimate independent of time. The constant $c = 2$ is simply convenient.

$$|D_x u(x,t)| \le \sqrt{\frac{e}{2t}} \sup_{|\xi'| \le 2t} u(x+\xi', t).$$

$\square$

Problem 12. John p. 214. (a) This is called the Cole-Hopf transformation. Once discovered, it is routine to verify: plug and chug.

$$\theta_x = -2\mu \left(\frac{u_{xx}}{u} - \frac{u_x^2}{u^2} \right)$$

$$\theta_{xx} = -2\mu \left(\frac{u_{xxx}}{u} - \frac{3u_x u_{xx}}{u^2} + \frac{2u_x^3}{u^3} \right) \tag{0.3}$$

$$\theta_t = -2\mu \left(\frac{u_{xt}}{u} - \frac{u_t u_x}{u^2} \right) = -2\mu^2 \left(\frac{u_{xxx}}{u} - \frac{u_x u_{xx}}{u^2} \right), \tag{0.4}$$

using $u_t = \mu u_{xx}$. Now combine terms to find that θ solves Burgers equation

$$\theta_t + \theta\theta_x = \mu\theta_{xx}.$$

(b) The change of variables is inverted as follows. Since $\theta = -2\mu u_x/u = -2\mu(\log u)_x$ we have

$$u(x,t) = \exp\left(-\frac{1}{2\mu}\int_{-\infty}^{x} \theta(y,t)\,dy\right).$$

(The antiderivative can be defined on a finite interval too, if the integral is divergent). The inner integral $\int_{-\infty}^{x} \theta(y,t)\,dy$ is called the potential. Suppose $\theta(x,0) = \varphi(x)$ with $\varphi \in C_0^2(\mathbb{R})$. Let $\Phi(x) = \int_{-\infty}^{x} \varphi(y)\,dy$, and $u(x,0) = \exp(-(2\mu)^{-1}\Phi(x))$. The solution to the heat equation with this initial data is

$$u(x,t) = \frac{1}{\sqrt{4\pi\mu t}}\int_{\mathbb{R}} \exp\left(-\frac{1}{2\mu}\left(\frac{(x-y)^2}{2t} + \Phi(y)\right)\right)\,dy.$$

Since $\theta = -2\mu u_x/u$ we have

$$\theta(x,t) = \frac{1}{t}\frac{\int_{\mathbb{R}} y\exp(-(2\mu)^{-1}A(x,y,t))\,dy}{\int_{\mathbb{R}} \exp(-2\mu)^{-1}A(x,y,t)\,dy.},$$

where we have defined

$$A(x,y,t) = \frac{y^2}{2t} + \Phi(x-y).$$

Since $\varphi \in C_0^2(\mathbb{R})$, the integral $\sup_x |\Phi(x)| < \infty$, and there is a constant $C > 0$ such that

$$-C\frac{\int_{\mathbb{R}} |y|e^{-y^2/4\mu t}\,dy}{\int_{\mathbb{R}} e^{-y^2/4\mu t}\,dy} \le t\theta(x,t) \le C\frac{\int_{\mathbb{R}} |y|e^{-y^2/4\mu t}\,dy}{\int_{\mathbb{R}} e^{-y^2/4\mu t}\,dy}.$$

Rescale by setting $y' = y/\sqrt{4\mu t}$ in the integrals to obtain the decay estimate,

$$-2C\sqrt{\frac{\mu}{t}} \le \theta(x,t) \le 2C\sqrt{\frac{\mu}{t}}.$$

The assumption that $\varphi \in C_0^2(\mathbb{R})$ is not necessary. As long as Φ is bounded we have the estimate above. In fact, one can trade in a growth assumption on Φ as $x \to \infty$ for a weaker decay estimate in time than the one above. □

Problem, John p.220. Suppose u solves the heat equation in an open subset Ω of the x,t plane. Suppose $(x,t) \in \Omega$. Since $u(x,t)$ is analytic in x, there is a neighborhood $B(x,r) \subset \mathbb{C}$ and $M > 0$ such that

$$\left|\partial_x^l u\right| \le \frac{M l!}{r^l}, \qquad l \in \mathbb{Z}_+.$$

Iterate the heat equation to obtain $\partial_t^k u = \partial_x^{2k} u$, for every $k \in \mathbb{Z}_+$; thus

$$\left|\partial_t^k u\right| \le \frac{M(2k)!}{r^{2k}}.$$

□

Problem 6. There are several examples of solutions that are defined for all t. For example, (i) $u \equiv$ constant, (ii) any Tychonoff solution defined to be zero for $t < 0$. A more interesting class of solutions is provided by exponentials (see John (1.4), p. 207). Examples of *positive* solutions defined for all time are the traveling waves $u(x,t) = e^{bx+b^2t}$ for any $b \in \mathbb{R}$. Things are more interesting when we consider superpositions of the traveling waves

$$u(x,t) = \int_{\mathbb{R}} e^{bx+b^2t} \mu(db),$$

for some positive measure μ on $\mathbb{R}$. As long as μ decays fast enough, that is $\int_{\mathbb{R}} e^{b^2t}\mu(db) < \infty$ for every $t \in \mathbb{R}$, the integral is finite and $u(x,t)$ is well-defined. Quite remarkably, it was shown by Widder that these are the only positive solutions of the heat equation defined for all $t \in \mathbb{R}$. If you assume this result, you obtain a Liouville theorem: every bounded positive solution of the heat equation on the entire plane $(x,t) \in \mathbb{R}^2$ is constant. □

Problem 7. Assertion: Assume f is integrable, and $\int_{\mathbb{R}^n} f(x)dx = 1$, and $u(x,t) = \int_{\mathbb{R}^n} k(x-y,t) f(y)dy$. Then

$$\lim_{t\to\infty} (4\pi t)^{n/2} u(2x\sqrt{t}, t) = e^{-|x|^2},$$

uniformly for $x \in \mathbb{R}^n$.

Here the standard meaning of integrable is adopted: f is measurable and $\|f\|_{L^1} = \int_{\mathbb{R}^n} |f(y)|dy < \infty$. The main step is to recognize that every integrable function gives rise to an approximate identity through rescaling. It is also necessary to recall the following: for every $\varepsilon > 0$ there exists $R > 0$ such that $\int_{|x|>R} |f(x)|dx < \varepsilon$. This is an immediate consequence of the monotone convergence theorem and $\int_{\mathbb{R}^n} |f(x)|dx < \infty$. By the definition of u we have

$$(4\pi t)^{n/2} u(x\sqrt{4t}, t) = \int_{\mathbb{R}^n} \exp\left(-\frac{|2x\sqrt{t} - y'|^2}{4t}\right) f(y')\,dy' = \int_{\mathbb{R}^n} e^{-|x-y|^2} f_t(y) dy,$$

where we have rescaled $y' = 2y\sqrt{t}$ and defined

$$f_t(y) = (4t)^{n/2} f(2y\sqrt{t}).$$

The properties of an approximate identity required of f_t are

$$\int_{\mathbb{R}^n} f_t(y) dy = 1, \quad \int_{|y|>R/2\sqrt{t}} |f_t(y)| dy < \varepsilon.$$

Now consider the difference

$$\left|(4\pi t)^{n/2} u(x\sqrt{2t}, t) - e^{-|x|^2/2}\right| = \left|\int_{\mathbb{R}^n} \left(e^{-|x|^2} - e^{-|x-y|^2}\right) f_t(y)\,dy\right|$$
$$\leq \int_{\mathbb{R}^n} \left|e^{-|x|^2} - e^{-|x-y|^2}\right| |f_t(y)|\ dy.$$

As always, consider the integral separately on $|y| \leq R/2\sqrt{t}$ and $|y| > R/2\sqrt{t}$. In the first region, the integral is bounded by

$$\sup_{|y|\leq R/2\sqrt{t}} |e^{-|x|^2} - e^{-|x-y|^2}| \|f_t\|_{L^1} \leq \sup_x |D_x e^{-x^2}| \|f_t\|_{L^1} \frac{R}{2\sqrt{t}} = R\|f\|_{L^1}\sqrt{\frac{e}{2t}}.$$

On the region $|y| > R/2\sqrt{t}$ we have

$$\sup_{y\in\mathbb{R}^n} \left|e^{-|x|^2} - e^{-|x-y|^2}\right| \int_{|y|>R/\sqrt{4t}} |f_t(y)| dy < 2\varepsilon.$$

$\square$