# Linear Algebra

MA 242 (Spring 2013)

Instructor: M. Chirilus-Bruckner

## EIGENVALUES, EIGENVECTORS, DIAGONALIZATION

– $3 \times 3$ example —

Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$ compute eigenvalues, eigenspaces and its diagonalization.

**"characteristic equation":**

$$\det(A - \lambda I) = \lambda(\lambda - 1)(\lambda - 2) = 0$$

**"eigenvalues":** characteristic equation is solved by $\lambda = 0, 1, 2$, eigenvalues are

$$\lambda_1 = 0, \qquad \lambda_2 = 1, \qquad \lambda_3 = 2$$

**"eigenspace for $\lambda_1 = 0$" = $\mathrm{Nul}(A - \lambda_1 I)$:**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 4 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \mathrm{Nul}(A - 0 \cdot I) = \mathrm{Span}\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\}$$

**"eigenspace for $\lambda_2 = 1$" = $\mathrm{Nul}(A - \lambda_2 I)$:**

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \mathrm{Nul}(A - 1 \cdot I) = \mathrm{Span}\{\begin{bmatrix} \frac{1}{2} \\ -2 \\ -1 \end{bmatrix}\}$$

**"eigenspace for $\lambda_3 = 2$" = $\mathrm{Nul}(A - 2\lambda_1 I)$:**

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 4 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \mathrm{Nul}(A - 2 \cdot I) = \mathrm{Span}\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\}$$

**"diagonalization":** Compile the matrix of eigenvectors $V = [v_1\ v_2\ v_3] = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & -2 & 1 \\ -1 & -1 & 1 \end{bmatrix}$, which is invertible since $v_1, v_2$ and $v_3$ are linearly independent (check that!).

The matrix $A$ can be transformed into a diagonal matrix by

$$V^{-1}AV = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 2 & 0 & 0 \\ 3 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & -2 & 1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

When executing these multiplications note the following structure:

$$V^{-1}AV = V^{-1}[Av_1\ Av_2\ Av_3] = V^{-1}[\lambda_1 v_1\ \lambda_2 v_2\ \lambda_3 v_3] = V^{-1}V\mathrm{diag}[\lambda_1, \lambda_2, \lambda_3] = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]$$