Propagation Failure

Intro

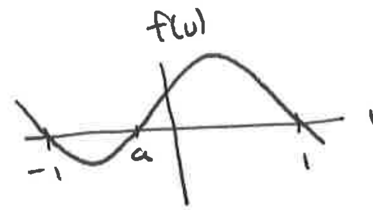
Consider the system

$$\dot{u}_j = d(u_{j+1} + u_{j-1} - 2u_j) + f(u_j) \qquad j \in \mathbb{Z}$$

where $\cdot$ indicates differentiation wrt time t, and f is a bistable nonlinearity, e.g.,

$$f(u) = (1-u^2)(u-a),$$

$$-1 < a < 1$$



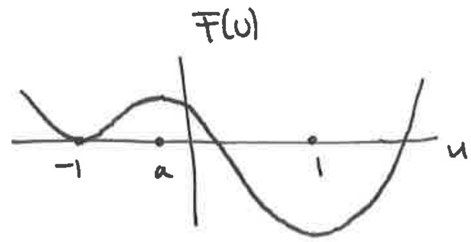
d can either be considered as a coupling constant – for example, if derived from nerve signal propagation as Bjorn described previously – or if we write $d = \frac{1}{h^2}$ as discretization of PDE

$$u_t = u_{xx} + f(u)$$

Start by recalling how we analyzed the continuous case:

Considered potential

$$F(u) := -\int_{-1}^{u} f(w)\,dw$$

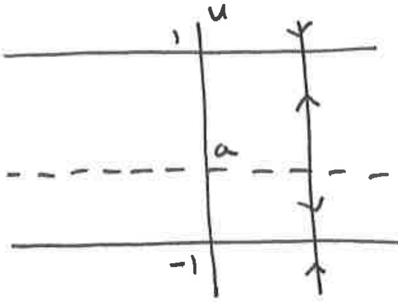


Can write $u_t = u_{xx} - F'(u)$

If we first look for spatially homogeneous solns

$$u(x,t) = u(t) \quad \forall x$$

$\hookrightarrow$ $u_t = -F'(u)$ $\rightarrow$ gradient system with energy $F(u)$



Meanwhile, if look at $u_t = u_{xx}$, acts to smooth out gradients in IC

Now look for traveling front solutions

$$\varphi(\xi) = \varphi(x-ct) = u(x,t) \qquad \left[\overset{c<0}{\longleftarrow} \quad \overset{c>0}{\longrightarrow} \right]$$

We want

$$\varphi(\xi) \to \begin{cases} -1 & \xi \to -\infty \\ 1 & \xi \to +\infty \end{cases}$$

Substituting, we get

$$-c\varphi'(\xi) = \underbrace{\varphi''(\xi) + f(\varphi(\xi))}_{\text{conservative}}$$

[$c=0$ conservative
$c<0$ forcing
$c>0$ damping]

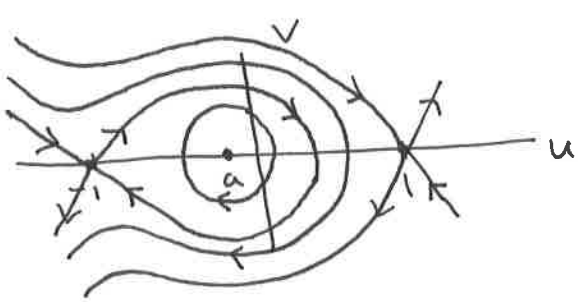
which we can rewrite as a 1st order system

$$\begin{pmatrix} u \\ v \end{pmatrix}_\xi = \begin{pmatrix} v \\ -cv - f(u) \end{pmatrix}$$

equilibria are $(-1,0)$, $(1,0)$ [saddles] and $(a,0)$ [center]

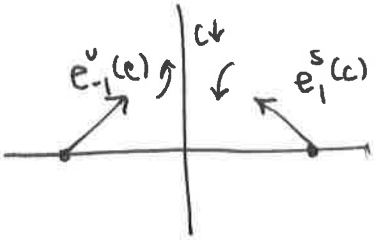
Define energy $H(u,v) = \frac{v^2}{2} + \int_{-1}^{u} f(w)\,dw$

Level sets of H:



For $c<0$, $H(u(t), v(t))$ will increase strictly along trajectories

So if we consider unit vectors tangent to unstable & stable manifolds at $(-1,0)$ & $(1,0)$ respectively:



Idea: fix b s.t. $a < b < 1$

show $h^u(c) :=$ first intersection of $W^u(-1,0)$ w/ $u=b$

and $h^s(c) :=$ " " $W^s(1,0)$ "

are well-defined, continuous in c, $h^u(0) < h^s(0)$,

and $h^u(c) > h^s(c)$ for $c \ll -1$.

$\Rightarrow \exists c_* $ s.t. $h^u(c_*) = h^s(c_*)$

Can also show this is unique, and c_* varies continuously as a function of a.

Note that since we have a continuous flow, if the invariant manifolds $W^u(-1,0)$ & $W^s(1,0)$ intersect, they must be tangent where they intersect.

=

Now let's return to our LDE

$$\dot{u}_j = d(u_{j+1} + u_{j-1} - 2u_j) + f(u_j) \qquad j \in \mathbb{Z}$$

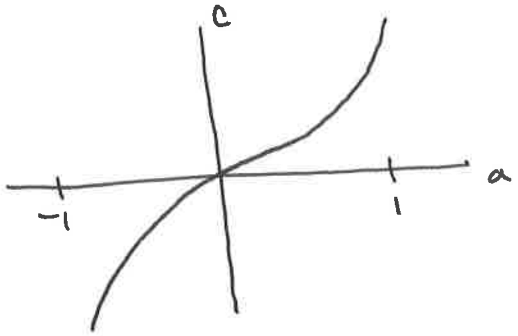
We again want to find traveling front solutions

$$u_j(t) = \varphi(j - ct)$$

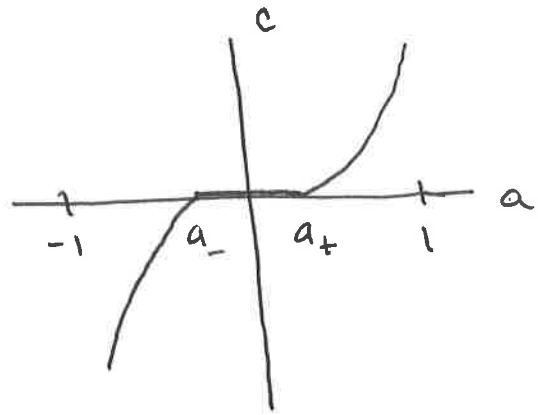
where φ is the wave profile $[\varphi : \mathbb{R} \to \mathbb{R}]$, c the wave speed, satisfying

$$\lim_{\xi \to \pm\infty} \varphi(\xi) = \pm 1$$

We find that $c = c(a)$ varies continuously and is nondecreasing wrt a, but now we may find that $c(a) = 0$ on a nontrivial interval. Pictorially we have



continuous case



LDE case

Define $[a_-, a_+] = \{ a \in (-1,1) : c(a) = 0 \}$.

If $a_- < a_+$, we say propagation failure has occured

Some history:

Bell (1981)

Bell & Cosner (1984)

[Britton (1984)]

Keener (1987) → where I'll start

Keener (1987)

$$\frac{du_n}{dt} = d(u_{n+1} - 2u_n + u_{n-1}) + f(u_n) \qquad n \in \mathbb{Z} \qquad (1)$$

~~where $f(0)=f(a)=f(1)=0$; $0<a<1$; $f(u)<0$ for $0<u<a$, $f(u)>0$ for $a<u<1$~~

Thm 2.1 Suppose the function $f(x)$ is continuously differentiable for $x \in [0,1]$ and that

(i) $f(0)=f(a)=0 \quad 0<a<1$; $f(x) \neq 0 \quad x \neq 0, a, 1$

(ii) $f(x)<0 \quad 0<x<a$

$\quad\; >0 \quad a<x<1$

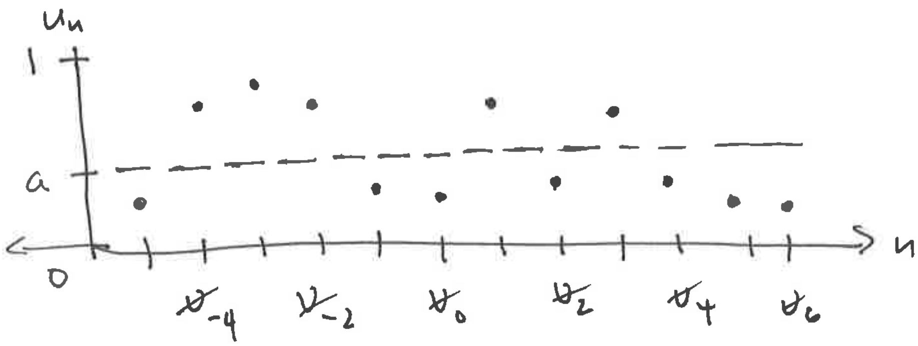
(iii) $f'(x_0)=f'(x_1)=0$ for some x_0, x_1 with $0<x_0<a<x_1<1$

$\quad f'(x) \neq 0$ for $x \neq x_0, x_1$

Then, for every $d>0$ sufficiently small, to every doubly infinite sequence $\{s_n\}$ with $s_n \in \{0,1\}$, there corresponds at least one steady solution of (1) with $u_n \in [0,a)$ when $s_n = 0$, and $u_n \in [a,1]$ when $s_n = 1$.

e.g. $\{\ldots, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, \ldots\}$

$s_{-4} \quad s_{-2} \quad s_0 \quad s_2 \quad s_4 \quad s_6$



Will show all these solns stable & block propagation.

But first pf of 2.1

Rewrite as steady-state, 1st order system

$$\left.\begin{aligned} U_{n+1} &= 2U_n - V_n - f(U_n)/d \\ V_{n+1} &= U_n \end{aligned}\right\} \quad (2)$$

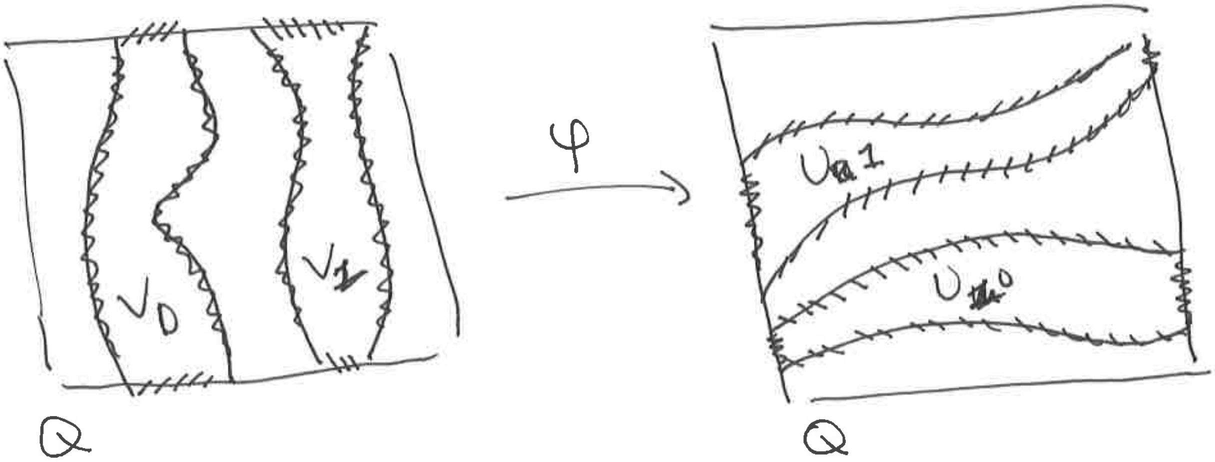
Write $\varphi : (U_n, V_n) \mapsto (U_{n+1}, V_{n+1})$

Wts φ has as a subsystem the shift on the sequence of symbols $\{0,1\}$.

Def $Q := [0,1] \times [0,1]$

A curve $v = v(u)$ is a horizontal curve in Q if $0 \leq v(u) \leq 1$ for $0 \leq u \leq 1$. The set lying between two nonintersecting horizontal curves is called a horizontal strip. Similarly, $u = u(v)$ is called a vertical curve in Q if $0 \leq u(v) \leq 1$ for $0 \leq v \leq 1$, and the set between two nonintersecting vertical curves is a vertical strip.

Thm [Moser] Let φ be a homeomorphism on Q. Suppose U_i, V_i, $i = 0,1$, are disjoint horizontal & vertical strips (resp.) in Q, and that $\varphi(V_i) = U_i$, $i = 0,1$. Further suppose the vertical boundaries of V_i are mapped ~~into~~ to vertical boundaries of U_i, and the horizontal boundaries of V_i are mapped to horizontal boundaries of U_i. Then φ posesses the shift on sequences $\{0,1\}$ as a subsystem.



We're using a somewhat weaker version of the theorem, not requiring contraction, since only require existence, not uniqueness.

For φ as given by (2), note we have $\varphi^{-1}(u_n, v_n) \to (u_{n-1}, v_{n-1})$ given by

$$\left.\begin{aligned} u_{n-1} &= v_n \\ v_{n-1} &= 2v_n - u_n - \frac{f(v_n)}{d} \end{aligned}\right\} (3)$$

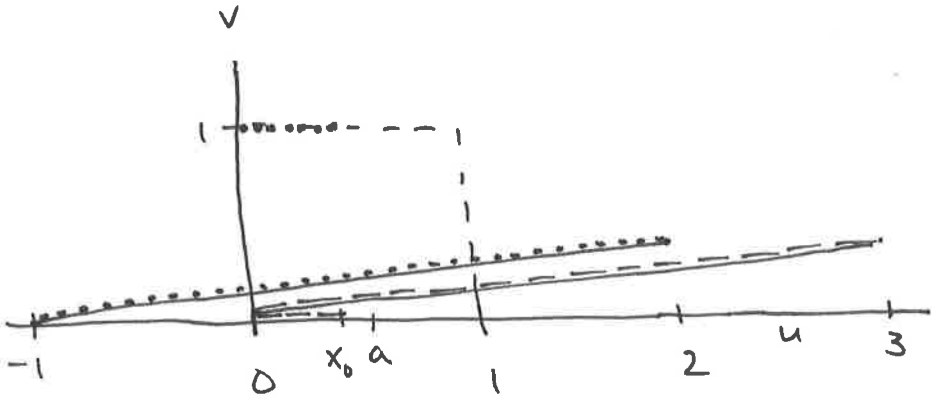
Recall we defined x_0, x_1 as the pts s.t.

$$f'(x_0) = f'(x_1) = 0, \quad 0 < x_0 < a < x_1 < 1$$

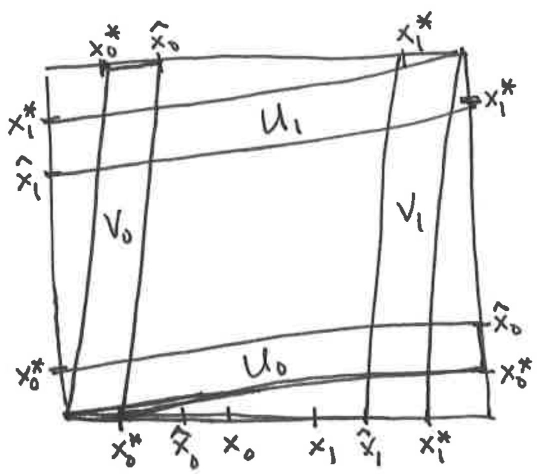
So $v = 0$, $0 \leq u \leq x_0$ is a piece of a horizontal curve that is mapped by φ to a monotone increasing horizontal curve $v = v_0(u)$ defined for $0 \leq u \leq F_0 := 2x_0 - \frac{f(x_0)}{d}$ with $v_0(0) = 0$, $v_0(F_0) = x_0$

Similarly $v = 1$, $0 \leq u \leq x_0$ is mapped by φ to $v = v_0(u+1)$ defined for $-1 \leq u \leq F_0 - 1$

$\Rightarrow$ If $F_0 > 2$, the curves $v = v_0(u)$ and $v = v_0(u+1)$, $0 \leq u \leq 1$ are the boundaries of a horizontal strip U_0



Now let x_0^* s.t. $2(x_0^*) - f(x_0^*)/d = 1$
$\hat{x}_0$ s.t. $2(\hat{x}_0) - f(\hat{x}_0)/d = 2$
$\Big\}\ 0 \le x_0^*, \hat{x}_0 < x_0$



Follow similar procedure for $v = 0,\ x_1 \le u_1 \le 1$
$v = 1,$ "

Define $F_1 := 2x_1 - \frac{f(x_1)}{d}$

Now require $F_1 < -1$

let x_1^* s.t. $2(x_1^*) - \frac{f(x_1^*)}{d} = 1$
$\hat{x}_1$ s.t. $2(\hat{x}_1) - \frac{f(\hat{x}_1)}{d} = 0$
$\Big\}\ x_1 < \hat{x}_1, x_1^* < 1$

$\Rightarrow \varphi$ has the shift on 2 symbols as a subsystem.

Also note in particular we have for any seq. $(s_i)_{i\in\mathbb{Z}}$, $s_i \in \{0,1\}$ $\exists (u_i)_{i\in\mathbb{Z}}$ which is a stationary solution of (1) with $u_i \in [0, \hat{x}_0]$ if $s_i = 0$ and $u_i \in [\hat{x}_1, 1]$ if $s_i = 1$. $\square$

As we saw above, needed $F_0 > 2$ and $F_1 < -1$, which can occur only for d suff. small : this leads us to following

<u>Cor 2.2</u> Suppose $f(x)$ is continuously differentiable for $x \in [0,1]$ and suppose $f(0) = f(1) = 0$. If $\exists \hat{x}_0, \hat{x}_1$ s.t.

(i) $2(\hat{x}_0 - 1) - \frac{f(\hat{x}_0)}{d} = 0$ (clearly $\Leftrightarrow 2\hat{x}_0 - \frac{f(\hat{x}_0)}{d} = 2$)

(ii) $2\hat{x}_1 - \frac{f(\hat{x}_1)}{d} = 0$

(iii) $f'(x) < 2d$ on $0 \le x \le \hat{x}_0$ and $\hat{x}_1 \le x \le 1$

then the conclusions of Thm 2.1 hold.

[Note also weakens (requirements on) f slightly]

(i) & (ii) ensure existence of horizontal and vertical strips and (iii) ensures boundary curves are monotone.

To understand stability, will use a comparison result:

Thm 2.3 Suppose $\{u_n(t)\}$ and $\{v_n(t)\}$ satisfy the eqn's

$$\frac{du_n}{dt} \le d(u_{n+1} - 2u_n + u_{n-1}) + f(u_n)$$

$$\frac{dv_n}{dt} \ge d(v_{n+1} - 2v_n - v_{n-1}) + f(v_n)$$

If $u_n(t_0) \le v_n(t_0)$ $\forall n$ then $u_n(t) \le v_n(t)$ $\forall t \ge t_0$.

[Skip proof.]

Now using hypotheses of Cor 2.2, establish global stability result.

Thm 2.4 Suppose $\exists\, \hat{x}_0, \bar{x}_0, \bar{x}_1, \hat{x}_1$ s.t.

(i) $2(\hat{x}_0 - 1) - \frac{f(\hat{x}_0)}{d} = 0$, $\quad 2(\bar{x}_0 - 1) - \frac{f(\bar{x}_0)}{d} = 0$

(ii) $2\hat{x}_1 - \frac{f(\hat{x}_1)}{d} = 0$, $\quad 2\bar{x}_1 - \frac{f(\bar{x}_1)}{d} = 0$

and $\frac{f(x)}{d} - 2x > 0$ for $\bar{x}_1 < x < \hat{x}_1 < 1$

Suppose $0 \le u_n(0) \le 1$ $\forall n$. If $u_k \in [0, \bar{x}_0)$ at $t=0$ then $u_k(t) \in [0, \bar{x}_0)$ $\forall t \ge 0$, whereas if $u_k \in (\bar{x}_1, 1]$ at $t=0$ then $u_k(t) \in (\bar{x}_1, 1]$ $\forall t \ge 0$.

Pf [Skip.]

Keener goes on to show can weaken hypo's somewhat by looking specifically at monotone soln's, and finds in particular that for $f(u) = u(1-u)(u-a)$, if $0 < a < \frac{1}{2}$, the cond's of Cor 2.2 hold for $d < a^2/8$ and weaker conds for $d < a^2/4$.

So we've seen one very nice explanation & exploration of propagation failure, but from the point of view of dynamical systems, what's "really" going on?

- compare w/ continuous case
- connection to chaos/ horseshoe map?

Invariant manifolds

Consider the equation

$$-c\varphi'(\xi) = d(\varphi(\xi+1) + \varphi(\xi-1) - 2\varphi(\xi)) + f(\varphi(\xi);a)$$

Which we get if we substitute $u_j(t) = \varphi(j-ct) = \varphi(\xi)$ in our original LDE. If $c=0$ this reduces to

$$0 = d(\varphi(j+1) + \varphi(j-1) - 2\varphi(j)) + f(\varphi(j);a) \quad j \in \mathbb{Z}$$

which is just a difference equation

We can rewrite it as

$$p_{j+1} = r_j$$
$$r_{j+1} = -p_j + 2r_j - \frac{f(r_j;a)}{d}$$

i.e., a discrete-time dynamical map. The front sol'ns were looking for are heteroclinic connections between $(-1,-1)$ and $(1,1)$, and in general a heteroclinic connection may lie in the transverse intersection of $W^u(-1,-1)$ and $W^s(1,1)$. If the intersection is transverse, stationary solutions will persist for a in some nbd of 0.

Of course, that's not really the end of the story.

- Subtle dependence of behavior on form of f
- Interesting & more complicated phenomena when consider LDE posed on higher dimensional lattices.
- Predicting region $[a_+, a_+]$ on which failure (pinning) occurs?

...

Continuing on with papers | history

- Erneux & Nicolis (1993)

- Erneux & Laplante (1992)
  - ↳ Propagation failure in arrays of coupled bistable chemical reactors
  - ↳ clever: hard to vary d if working with nerve cells or cardiac tissue
    but with chemical reactors can vary exchange rate
  - ↳ results show impact of noise?

- Anderson & Sleeman (1995)

Now sticking with lattice on $\mathbb{Z}$ | skipping ahead chronologically
(come back to crystallographic pinning)

Hupkes, Pelinovsky & Sandstede (2011)

Start concretely; consider

$$\dot{u}_j = \frac{1}{h^2}(u_{j+1}+u_{j-1}-2u_j) + f(u_j;a) \quad j\in\mathbb{Z} \qquad (1)$$

with

$$f(u;a) = 2(1-u^2)(u-a) \quad -1<a<1 \qquad (2)$$

Look for stationary soln's; rewrite as

$$\left.\begin{aligned} p_{j+1} &= r_j \\ r_{j+1} &= -p_j + 2r_j - h^2 f(r_j;a) \end{aligned}\right\} (3)$$

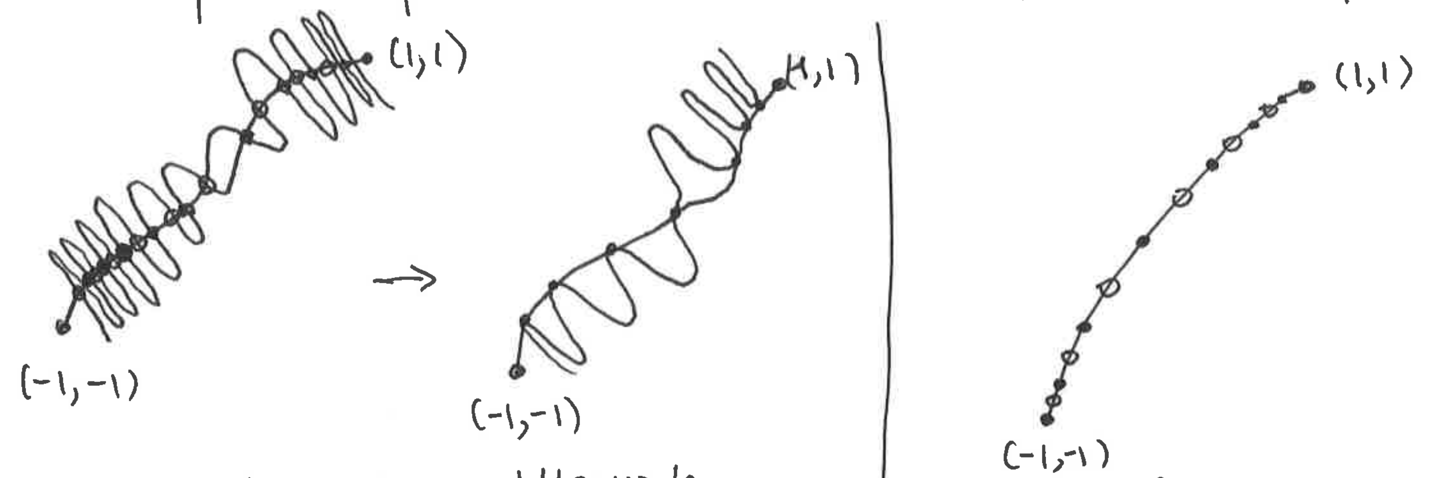
When $a=0$, equilibria $(-1,-1)$ and $(+1,+1)$ are both saddles, and (3) admits solns $p^{(s)}$ and $p^{(b)}$ satisfying

$$\lim_{j\to\pm\infty} p_j^{(s)} = \lim_{j\to\pm\infty} p_j^{(b)} = \pm 1, \quad p_{-j}^{(s)} = -p_j^{(s)}, \quad p_{-j+1}^{(b)} = -p_j^{(b)} \qquad (4)$$

These are refered to as site-centered and bond-centered solutions.

For each $j\in\mathbb{Z}$, $(p_j^{(s)}, p_{j+1}^{(s)})$ and $(p_j^{(b)}, p_{j+1}^{(b)})$ lie in the intersection of the unstable manifold $W^u(-1,-1)$ and the stable manifold $W^s(1,1)$.

There are now two possibilities: either the intersections of $W^u(-1,-1)$ are transverse, in which case $p^{(s)}$ and $p^{(b)}$ persist for $a\approx 0$ (though only identify (s) & (b) by continuation) or else $W^s(1,1)$ and $W^u(-1,-1)$ coincide entirely at $a=0$, and $p^{(a)}$ and $p^{(b)}$ are part of a smooth family of stationary solns.



Case 1 $\rightarrow$ saddle-node bifurcation

(stationary solns persist for $a \approx 0$)

Case 2

(persitence for $a \neq 0$??)

Do not expect the type of degeneracy in case 2 to exist for f as in (2), but give example where it does:
[think of as alternate discretization of $u_t = u_{xx} + f(u;a)$]

$$\dot{u}_j = \frac{1}{h^2}(u_{j-1}+u_{j+1}-2u_j) + (1-u_j^2)(u_{j+1}+u_{j-1}-2a), \quad j \in \mathbb{Z} \qquad (5)$$

Can directly verify that for any $a, \ell \in \mathbb{R}$, (5) is satisfied by

$$u_j(t) = \tanh(\operatorname{arcsinh}(h)(j-ct+\ell)), \quad c = \frac{2a}{\operatorname{arcsinh}(h)} \qquad (6)$$

so that we have a branch of stationary solutions at $a=0$ parameterized by $\ell \in \mathbb{R}$, i.e., in case 2.

It follows from [M-P 1999, Thm 2.1] that $a_- = a_+ = 0$ holds for (5), so we ~~also~~ do not see propagation failure.

Note this means the manifolds $W^u(-1,-1)$ & $W^s(1,1)$ separate completely for $a \neq 0$.

[Suggest using (5) as discretization for $u_t = u_{xx} + f(u)$; Q: Daubechies]

Main idea now is to show that if we are in case 2, for a broad class of nonlinearities f, we'll have complete separation of the stable & unstable manifolds so that propagation failure does not occur.

In particular, consider

$$\dot{u}_j = g(u_{j-1}, u_j, u_{j+1}; a) \quad j \in \mathbb{Z}, \ u_j \in \mathbb{R} \tag{7}$$

and define the conditions:

(Hg1) The nonlinearity g is C^3 smooth, with $\partial_{u_1} g(u_1, u_2, u_3; a) > 0$ and $\partial_{u_3} g(u_1, u_2, u_3; a) > 0$ $\forall (u_1, u_2, u_3) \in \mathbb{R}^3$ and $a \in (-1, 1)$.
In addition

$$\partial_a g(u_1, u_2, u_3; a) < 0$$

$\forall a \in (-1,1)$ and $(u_1, u_2, u_3) \in \mathbb{R}^3$ with $-1 < u_1 < u_2 < u_3 < 1$.

(Hg2) Setting $\bar{g}(u; a) := g(u, u, u; a)$ we have

$\bar{g}(\pm 1; a) = 0$, $\quad \bar{g}(a; a) = 0$
$\bar{g}(u; a) < 0$ for $u \in (-1, a) \cup (1, \infty)$
$\bar{g}(u; a) > 0$ for $u \in (-\infty, -1) \cup (a, 1)$
for every $a \in (-1, 1)$. Also
$\partial_u \bar{g}(\pm 1; a) < 0 \qquad \partial_u \bar{g}(a; a) > 0$
$\partial_{ua} \bar{g}(-1; a) < 0 \qquad \partial_{ua} \bar{g}(1; a) > 0$.

[So $\bar{g}(u; a)$ "looks like" original cubic.]

Also need to impose degeneracy condition on stationary solutions:

(Hp) There exists a $\bar{p} \in BC^3(\mathbb{R},\mathbb{R})$ s.t. for any $l \in \mathbb{R}$, the constant function $u(t) = p^{(l)}$ given by

$$p_j^{(l)} = \bar{p}(j+l)$$

satisfies (7) with $a = a_*$ for some $a_* \in (-1,1)$.
In addition, $\bar{p}$ has $\bar{p}'(\xi) > 0$ $\forall \xi \in \mathbb{R}$ and satisfies

$$\lim_{\xi \to \pm\infty} \bar{p}(\xi) = \pm 1$$

Thm 1.1 Consider the system (7) and suppose (Hg1), (Hg2) & (Hp) holds.
Then for every $a \in (-1,1)$, (7) admits a solution of the form

$$u_j(t) = \bar{u}(j - ct)$$

for some $c \in \mathbb{R}$ and $\bar{u} \in C^1(\mathbb{R},\mathbb{R})$ with $\bar{u}'(\xi) > 0$ $\forall \xi \in \mathbb{R}$ and

$$\lim_{\xi \to \pm\infty} \bar{u}(\xi) = \pm 1.$$

The wave speed $c = c(a)$ depends C^1-smoothly on a with $c(a_*) = 0$ and $\partial_a c(a) > 0$ $\forall a \in (-1,1)$.

[so we don't get prop. failure w/ given conditions.]

Cor 1.2 Consider (7) and suppose (Hg1), (Hg2), (Hg3) hold.
There exists const. $\delta > 0$ s.t. the following holds:
Suppose (7) admits a stationary soln

$$u_j(t) = u_j$$

for some $a \in (-1,1)$. Suppose $u_{j_1} \le u_{j_2}$ holds $\forall j_1 \le j_2$ together with $\lim_{j \to \pm\infty} u_j = \pm 1$, or alternatively that $|a - a_*| < \delta$ and $|u - p^{(l)}| < \delta$ for some $l \in \mathbb{R}$. Then we must have $a = a_*$.

[So for $a \neq a_*$ can't have j-monotonic stationary solns; also for a close to a_* if u is stationary & close to $p^{(l)}$, must have $a = a_*$].

Cor 1.3 (follows from M-P 1999) Consider (7) and suppose (Hg1), (Hg2) and (Hg3) are satisfied. Pick any $a \in (-1,1)$ and suppose (7) admits a soln of the form

$$u_j(t) = \bar{u}(j-ct)$$

for $c \neq 0$ and $\bar{u} \in C^1(\mathbb{R},\mathbb{R})$ that satisfies

$$\lim_{\xi \to \pm\infty} \bar{u}(\xi) = \pm 1$$

Then u must be a temporal translate of the soln in Thm 1.1.

[So waves in thm 1.1 unique among solns connecting ±1].

Idea of pf of Thm 1.1

Focus on dynamics of (7) for a near a_*

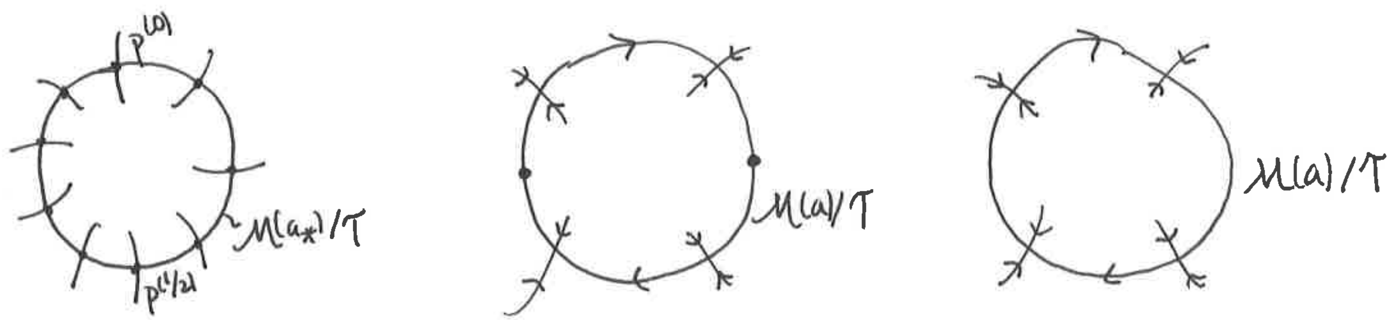
Define $\mathcal{M}(a_*) = \{p^{(l)}\}_{l \in \mathbb{R}}$, manifold of equilibria

Show persists as an invariant manifold $\mathcal{M}(a)$ for a near a_*

To simplify, write T for the right-shift operator with $(Tu)_j = u_{j-1}$ and note

$$p^{(l)} = Tp^{(l+1)}$$

$\forall l \in \mathbb{R}$. After factoring out the symmetry T, the manifold $\mathcal{M}(a_*)$ is a circle of equilibria (really $\mathcal{M}(a_*)/T$, see (i))



(i) every point represents an equivalence class $p^{(l)}$, $l \in S^1$; all equilibria

(ii) one or more equilibria could survive for $a \neq a_*$

(iii) if (7) is a normal family equilibria don't survive

In theory some equilibria could survive for $a \neq a_*$.
But by computing flow on $\mathcal{M}(a)$ to leading order, show situation in (iii) holds, so that the traveling waves described in 1.1 can be read off from the shift-periodic soln to (7) induced by the flow on $\mathcal{M}(a)$.

In final section give 3 examples:

- normal family (no prop. fail)
- nonnormal family w/o prop. fail
- nonnormal family w/ prop. fail

Crystallographic Pinning

Now consider higher dimensional lattices, for example

$$\dot{u}_{i,j} = (\Delta u)_{i,j} - f(u_{i,j}, a) \qquad (i,j) \in \mathbb{Z}^2$$

where Δ is the discrete laplacian in $\mathbb{Z}^2$

$$(\Delta u)_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}$$

Turns out will again see propagation failure (pinning), but now <u>direction matters</u>.

(very) Loosely, M-P ~~1999~~ 2001 showed that for rational propagation directions θ (where rational means if $x = \cos\theta$, $y = \sin\theta$, then $\frac{x}{y}$ is rational or infinite) we have a nontrivial pinning region, whereas for θ irrational the pinning region is trivial, as long as the nonlinearity f is sufficiently close to a sawtooth function [MP proved directly in 1998 paper with Cahn & Van Vleck for sawtooth fcn itself.]

$$\left(\Leftarrow f(u,a) = \begin{cases} u+1 & u < a \\ u-1 & u > a \end{cases} \right)$$

In 2008, Hoffman & M-P showed that crystallographic pinning occurs in the horizontal (or vertical) direction for almost all f with properties similar to our familiar bistable cubic nonlinearity (will of course make precise.)

Hoffman & Mallet-Paret 2008

Focus on traveling wave (front) soln's of LDEs posed on $\mathbb{Z}^2$:

<u>Def</u> The nonlinearity f is of <u>bistable</u> type if

$$\left.\begin{array}{llll} f(\pm 1)=0 & f(a)=0 & f'(\pm 1)>0 & f'(a)<0 \\ f(u)>0 & \text{for } u\in(-1,a)\cup(1,+\infty) & & \\ f(u)<0 & \text{for } u\in(-\infty,-1)\cup(a,1) & & \end{array}\right\} \quad (1)$$

for some $a\in(-1,1)$.

Further consider a family of bistable functions $f=f(u,a)$ parameterized by $a\in(-1,1)$ satisfying the monotonicity condition

$$\frac{\partial f}{\partial a}(u,a)>0 \quad \text{for } u\in(-1,1),\ a\in(-1,1) \qquad (2)$$

<u>Def</u> Let $\mathcal{N}$ be the set of functions

$\mathcal{N}=\{f:[-1,1]\times(-1,1)\to\mathbb{R} : f(\cdot,\cdot)$ is C^2 smooth with $f(\cdot,a)$ satisfying (1) for every $(u,a)\in[-1,1]\times(-1,1)$, and (2) holding$\}$.

f is a <u>normal family</u> if $f\in\mathcal{N}$.

Consider in particular the lattice $\mathbb{Z}^2$ and the system

$$\dot{u}_{i,j} = (\Delta u)_{i,j} - f(u_{i,j}, a) \qquad (3)$$

A traveling wave soln of (3) is of the form

$$u_{i,j}(t) = \phi(ik + j\sigma - ct) \qquad (4)$$

where $(\kappa,\sigma) \in \mathbb{R}^2/\{0,0\}$ is the direction vector, c is the wave speed, ϕ the wave profile.

Substituting (4) in (3) we have

$$-c\phi'(\xi) = \phi(\xi+\sigma) + \phi(\xi-\sigma) + \phi(\xi+\kappa) + \phi(\xi-\kappa) - 4\phi(\xi) - f(\phi(\xi), a) \qquad (5)$$

From M-P Global Struct. of Traveling Waves, we have that for each direction vector $(\kappa,\sigma) \in \mathbb{R}^2/\{0\}$ and each $a \in (-1,1)$ $\exists$ a unique wave speed $c = c(a,(\kappa,\sigma))$ s.t. (5) admits a monotone soln satisfying the boundary conditions

$$\phi(-\infty) = -1, \qquad \phi(+\infty) = 1 \qquad (6)$$

Moreover for $c \neq 0$, this soln $\phi = \phi(\xi, a, (\kappa,\sigma))$ is unique up to translation. The wavespeed $c(a,(\kappa,\sigma))$ depends continuously on a and (κ,σ). For each (κ,σ) it is nondecreasing in a, smooth in a, and satisfies $\frac{\partial c(a,(\kappa,\sigma))}{\partial a} > 0$ when $c(a,(\kappa,\sigma)) \neq 0$. Also we have quantities $a_{\pm} = a_{\pm}(\kappa,\sigma)$ characterized by

$$[a_-(\kappa,\sigma), a_+(\kappa,\sigma)] \cap (-1,1) = \{a \in (-1,1) : c(a,(\kappa,\sigma)) = 0\}$$

Since we can check by rescaling ξ by r that

$$a_{\pm}(r\cos\theta, r\sin\theta) = a_{\pm}(\cos\theta, \sin\theta)$$

$r > 0$, we can write $a_{\pm}(\theta)$ with slight abuse of notation.

Now focus on $a_+(\theta)$ (though could do same for $a_-(\theta)$).

$a_+(\theta)$ need not depend continuously on θ, but it is upper semi-continuous in θ, i.e.,

$$\limsup_{\theta \to \theta_0} a_+(\theta) \leq a_+(\theta_0) \qquad (7)$$

for every θ_0.

<u>Def</u> <u>Crystallographic pinning</u> occurs for (3) in the direction of θ_0 if the inequality in (7) is strict or the analogous inequality for $a_-(\theta)$ is strict.

Goal of paper is to give conditions on f s.t. crystallographic pinning occurs for $\theta_0 = 0$ $[(k,\delta) = (1,0)]$.

Use two conditions:

(HA) $\exists\, p \in \ell^\infty(\mathbb{Z})$ denoted $p = \{p_n\}_{n\in\mathbb{Z}}$ satisfying

$$p_{n+1} + p_{n-1} - 2p_n = f(p_n, a_+(0)) \qquad n \in \mathbb{Z} \qquad (8)$$

and which also satisfies the bdry & monotonicity cond's

$$\lim_{n\to\pm\infty} p_n = \pm 1, \quad p_n \leq p_{n+1} \text{ for } n \in \mathbb{Z} \qquad (9)$$

Moreover, p is unique up to a shift in the index n.

(HB) A holds. Further, if $v \in \ell^\infty(\mathbb{Z}) \setminus \{0\}$ satisfies

$$v_{n+1} + v_{n-1} - 2v_n = f'(p_n, a_+(0))v_n \qquad n \in \mathbb{Z} \qquad (10)$$

where p is as in A and $f'(u,a) := \frac{\partial f(u,a)}{\partial u}$, then

$$B = \frac{1}{2} \sum_{n=-\infty}^{\infty} f''(p_n, a_+(0))v_n^3 \qquad (11)$$

satisfies $B \neq 0$.

Before stating main results, pause to see what these cond's mean / that they make sense.

First note (8) is the wave profile eq'n with $c=0$ and $(\kappa,\sigma)=(1,0)$:

$$-c\phi'(\xi)=\phi(\xi+\sigma)+\phi(\xi-\sigma)+\phi(\xi+\kappa)+\phi(\xi-\kappa)-4\phi(\xi)-f(\phi(\xi),a)$$

$$\hookrightarrow \quad 0=\phi(i)+\phi(i)+\phi(i+1)+\phi(i-1)-4\phi(i)-f(\phi(i),a)$$

$p_i=\phi(i)$

$$\hookrightarrow \quad p_{i+1}+p_{i-1}-2p_i=f(p_i,a)$$

Prop 1.5 Assume $c(a,(1,0))=0$, so $a\in[a_-(0),a_+(0)]$. Then $\exists p\in\ell^\infty(\mathbb{Z})$ satisfying

$$p_{n+1}+p_{n-1}-2p_n=f(p_n,a) \quad n\in\mathbb{Z}$$

along with (9). Moreover, any such monotone p is strictly monotone, i.e., $p_n<p_{n+1}$ $\forall n\in\mathbb{Z}$.

Pf (easy) Existence & monotonicity follow from previous comments. To show strict monotonicity, suppose $p_{m+1}=p_m$ for some $m\in\mathbb{Z}$. Then $f(p_m,a)=p_{m-1}-p_m\le 0$ by monotonicity. But also $f(p_m,a)=f(p_{m+1},a)=p_{m+2}-p_{m+1}\ge 0 \Rightarrow f(p_m,a)=0$. Thus $p_{m-1}=p_m=p_{m+1}=p_{m+2}$, and continuing in this fashion we see p_n is constant in n, which violates the fact that it is a heteroclinic connection between -1 and 1. ☐

Now consider (HB)...

(10) can be expressed with the operator $L \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ given by

$$L = S + S^{-1} - 2I - f'(p, a_+(0)) \qquad (12)$$

where $S \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ is the shift operator

$$(SX)_n = x_{n+1} \quad \text{for } X = \{x_n\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$$

and we denote by $f'(p, a_+(0)) \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ the operator with entries $f'(p_n, a_+(0))$ on the diagonal.

Prop 1.4 Assume (HA) with p as stated there, and $L \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ as in (12) above. Then $\exists\ v \in \ell^\infty(\mathbb{Z}) \setminus \{0\}$ satisfying (10), i.e. $Lv = 0$. v is unique up to a scalar multiple, and thus

$$\ker(L) = \{av : a \in \mathbb{R}\}$$

Further, v can be chosen to satisfy

$$v_n > 0, \quad n \in \mathbb{Z}$$

and has the coordinatewise estimate

$$v_n \le K\mu^{|n|}, \quad n \in \mathbb{Z} \qquad (13)$$

for some $K > 0$, $0 < \mu < 1$. Thus $v \in \ell^1(\mathbb{Z})$ and we may normalize it to satisfy $\langle v, v \rangle = 1$ where $\langle \cdot, \cdot \rangle$ denotes the dot product (duality) between $\ell^1(\mathbb{Z})$ and $\ell^\infty(\mathbb{Z})$. The operator L is Fredholm with index zero, and range

$$\operatorname{ran}(L) = \{w \in \ell^\infty(\mathbb{Z}) : \langle v, w \rangle = 0\}$$

and its spectrum satisfies $\sigma(L) \cap (0, \infty) = \emptyset$.

[Pf later in paper.] Note estimate (13) implies sum in def. of B, as given in (11) is absolutely convergent.

Now state the main results:

Thm 1.1 Assume (HB). Then the inequality (7) is strict at $\theta_0 = 0$ and so crystallographic pinning occurs in the direction $\theta_0 = 0$.

Thm 1.2 (HB) is generic in the following sense: fix any $f_0 \in \mathcal{N}$ and define the set

$$C_+^2 = \{\gamma \in C^2[-1,1] : \gamma(u) > 0 \text{ for every } u \in [-1,1]\}$$

noting that for every $\gamma \in C_+^2$, we have $f \in \mathcal{N}$, where $f(u,a) = \gamma(u) f_0(u,a)$. Let C_+^2 be endowed with the usual C^2 topology, so making it an open subset of the Banach space $C^2[-1,1]$.
Then the set $G(f_0) \subseteq C_+^2$, defined as

$$G(f_0) = \{\gamma \in C_+^2 : \gamma f_0 \text{ satisfies (HB)}\}$$

is a residual subset of C_+^2.

[residual in sense of Baire category thm: complement of a meager set, i.e., countable union of nowhere dense sets]