- Nonlinear Dynamics of Networks: The Groupoid Formalism
  Comments on what we're doing & where we're going.
- Processes evolving on networks

- Coupled ODEs — this is our focus, but could develop similar theory for
- Markov chains / processes — countable state space; discrete or cont. time
- Cellular automata — discrete in space & time

- Network architecture fixed — assume throughout
  - generally have been assuming this

- Nework symmetry ⇒ specific behavior of solns irrespective of precise form of ODEs
  i.e., as long as ODEs respect the network architecture (will formalize via concept of admiss.) eq'ns can have any form
- Symmetry sufficient but not necessary
  - many naturally occurring networks have trivial symmetry yet display synchrony phase relations etc.
  - understanding this is heart of paper

- ODEs on networks not just high-D d.s.
  - nodes have meaning & we can compare properties & behavior: synchrony, phase relns, etc.

Ex 1 FitzHugh-Nagumo

single cell

$x = (v, w) \in \mathbb{R}^2$ — recall before we had fast & slow ionic curr. — won't worry about that here

membrane potential — ionic current

$\dot{v} = v(a-v)(v-1) - w$

$\dot{w} = bv - \gamma w$

a, b, γ param's

Now go to 2 cell coupled system:

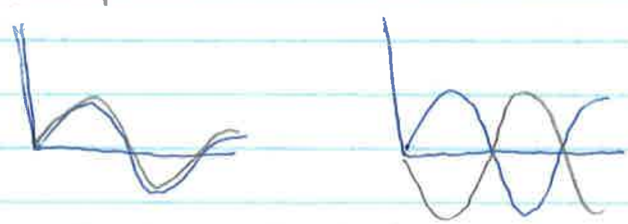
$(x_1, x_2) = (v_1, w_1; v_2, w_2) \in \mathbb{R}^4$

$\dot{v}_1 = v_1(a-v_1)(v_1-1) - w_1 - cv_2$

$\dot{w}_1 = bv_1 - \gamma w_1$

c is coupling strength

A few things to note: already seen existence of synchronous solutions & phase related sol'ns



note: will be drawing all systems as though cells were 1-dim; think of as one component in general internal dynamics $\in \mathbb{R}^k$

Second, we can describe the system by defining

$g: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$

$(x_1, x_2) \to (v_1(a-v_1)(v_1-1) - w_1 - cv_2,\ bv_1 - \gamma w_1)$

$\dot{x}_1 = g(x_1, x_2)$

$\dot{x}_2 = g(x_2, x_1)$

and this is a complete description of the system

So now having made these observations, let's look more generally, and see why 'really' these behaviors are a consequence of network symmetry rather than the precise form of the FitzHugh-Nagumo model

Start by reviewing def. of symmetry

Symmetry ~~and synchrony~~

↳ as it says on your handout, symmetry of an ODE is a transformation that maps sol'ns to sol'ns

↳ equivalent to the equivariance equation

$$f(\gamma x) = \gamma f(x)$$

(actually in Bjorn's class introduced equivarience eq'n first and then noted that equivalent to mapping sol'ns to sol'ns)

Also introduce $\mathrm{Fix}(\gamma)$

$$\mathrm{Fix}(\gamma) := \{x \in \mathbb{R}^n : x = \gamma x\}$$

Easy to see that if γ is a symmetry then $\mathrm{Fix}(\gamma)$ is a flow-invariant subspace:

$$\gamma f(x) = f(\gamma x) = f(x) \Rightarrow f: \mathrm{Fix}(\gamma) \to \mathrm{Fix}(\gamma)$$

All we need re: symmetries on ODEs

On the other hand, symmetry of a network is a purely combinatorial concept

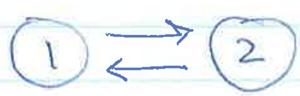
- can define as a pair of permutations one on nodes & one on edges s.t. incidence relations between nodes & edges are preserved

But the point is that because of the constraints a network structure places on the form of the ODEs (and we'll formalize this through the concept of admissible vector fields)

$\Rightarrow$ ODEs inherit network symmetries

So start to make these ideas concrete w/ a very simple example

Ex 2



identical cells with identical coupling; note we'll always be using directed arrows to indicate the input of one cell to another

$$\dot{x}_1 = g(x_1, x_2)$$
$$\dot{x}_2 = g(x_2, x_1)$$

symmetry group $\Gamma = \mathbb{Z}_2$

Any system obeying this graph can be written in this form (again will formalize)

Looks familiar - this is how we said we could rewrite Fitzhugh-Nagumo

Now show synchrony & phase-locking result from network architecture & not precise form of g

① Synchrony The reason we have synchronous sol'ns is the existence of a flow-invariant subspace

$$\Delta = \{(x_1, x_2) : x_1 = x_2\}$$

Obvious: set $x_1 = x_2$ then $\dot{x}_1$ & $\dot{x}_2$ governed by same equation so same I.C. $\Rightarrow$ same sol'n

But we can explain the existence of this subspace in terms of symmetry

$\mathbb{Z}_2 = \{\sigma, \sigma^2 = 1\}$ where $\sigma = (1\ 2)$

i.e., $\sigma(x_1, x_2) = (x_2, x_1)$

and actually $\Delta = \mathrm{Fix}(\sigma)$

(obvious but to make sure on same page ;)

$\mathrm{Fix}(\sigma) = \{x \in \mathbb{R}^2 : x = \sigma(x)\} = \{x \in \mathbb{R}^2 : x_1 = x_2\} = \Delta$

so existence of symmetry $\Rightarrow$ existence of synchrony for <u>any</u> g

(2) Phase-locking

Also a consequence of

Doesn't ne

Suppose $\exists$ soln $x(t)$ T-periodic

then for any symmetry γ either $\gamma x(t)$ is same trajectory or a different one that is also a soln.

Assuming it's the same trajectory uniqueness of soln implies

$\gamma x(t) = x(t+\theta)$ $\forall t$, some θ

Now in particular consider permutation σ

$\sigma x(t) = x(t+\theta)$

$$x(t) = \sigma^2 x(t) = \sigma x(t+\theta) = x(t+2\theta)$$

$$\Rightarrow 2\theta = 0 \pmod{T} \Rightarrow \theta = 0 \text{ or } T/2 \pmod{T}$$

So for $\theta = T/2$ $\quad x_2(t) = x_1(t+\theta)$

(and $\theta = 0 \Rightarrow$ synchrony)

Questions?

Return to more general setting

$\dot{x} = f(x)$, finite symmetry group Γ

Define 2 subgroups of Γ:

$H := \{ \gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\} \}$ spatiotemporal symmetries

$K := \{ \gamma \in \Gamma : \gamma x(t) = x(t) \ \forall t \}$ spatial symmetries

Now we'll make some observations about the structure of H & K

And the really cool thing is that if we can find H & K satisfying these properties then the implication goes the other way, i.e., H & K will be spatiotemporal & spatial symm. resp.

First note that $\forall h \in H \ \exists \theta(h) \in S^1$ associated phase shift
s.t. $hx(t) = x(t + \theta(h))$

Also $\theta : H \to S^1$ is a group homomorphism with $\ker \theta = K$

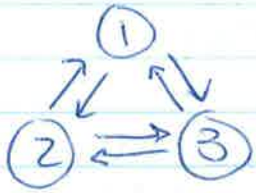
$\Rightarrow H \backslash K$ is isomorphic to a finite subgroup of S^1
$\Rightarrow H \backslash K$ is cyclic

Further (trivial but need to go in other direction),
K is isotropy subgroup of action of Γ on $\mathbb{R}^n$

So we'll go through a few examples to get a feel for these H, K subgroups, but first want to introduce H K theorem
- see handouts pg. 2

Examples to see what this means & why it's useful

Ex 3



$\Gamma = S_3$ - all permutations of nodes (dihedral grp order 6)

All ODEs of the form

$$\begin{aligned} \dot{x}_1 &= g(x_1, \overline{x_2, x_3}) \\ \dot{x}_2 &= g(x_2, \overline{x_3, x_1}) \\ \dot{x}_3 &= g(x_3, \overline{x_1, x_2}) \end{aligned}$$

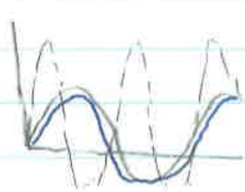
$\overline{\quad}$ means invariant under permutation (property of g)

$x_1, x_2, x_3 \in \mathbb{R}^k$, $g : (\mathbb{R}^k)^3 \to \mathbb{R}^k$

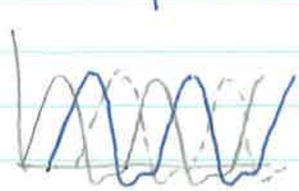
Generators of $\Gamma = S_3$ are $\sigma = (1\ 2)$ and $\tau = (1\ 2\ 3)$
obviously $\sigma^2 = 1$ $\tau^3 = 1$
get all permutations by combining

3 types of asynchronous periodic solns w/ some symmetry - write down then explain
$(H, K) = (\mathbb{Z}_3(\tau), 1) \to$ rotating waves
$(H, K) = (\mathbb{Z}_2(\sigma), \mathbb{Z}_2(\sigma)) \to$ 2 cells in phase
$(H, K) = (\mathbb{Z}_2(\sigma), 1) \to$ two cells out of phase by half period

(2)



(1)



(3) Hold off on drawing until analyzed

$\boxed{\text{All}}$

(1) $Tx(t) = x(t+\theta)$

$T^3 x(t) = x(t) = x(t+3\theta) \Rightarrow 3\theta = 0 \pmod{T}$

$\theta = 0, T/3, 2T/3$

know $\theta \neq 0$ as $K = 1$

$Tx(t) = (x_3, x_1, x_2)$

$x_3(t) = x_1(t + T/3)$

$x_1(t) = x_2(t + T/3) = x_1(t + T) = x_1(t)$

$x_2(t) = x_3(t + T/3) = x_1(t + 2T/3)$

$\boxed{\text{All}}$

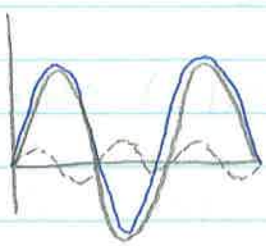
(3) $\bar{\sigma} x(t) = x(t + \frac{T}{2})$ (same analysis know $\theta \neq 0$ as $K = 1$)

$x_2(t) = x_1(t + \frac{T}{2})$

but also $x_3(t) = x_3(t + \frac{T}{2})$

$$\begin{pmatrix} x_2(t) \\ x_1(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t+\frac{T}{2}) \\ x_2(t+\frac{T}{2}) \\ x_3(t+\frac{T}{2}) \end{pmatrix}$$

$\Rightarrow$ x_3 oscillates at twice the frequency of x_1 & x_2

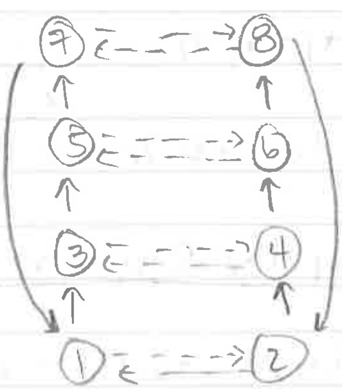


relative amplitude doesn't matter – point is just don't need to be same

3. Animal Locomotion

Will now consider quadruped gaits – walk, trot, etc. (and the now famous prank
↳ it's thought there's a central pattern generator (CPG)
= collection of cells in the spinal cord that controls locomotion

Golubitsky et al. previously showed using symmetry arguments that 4 cells do not suffice to explain typical combinations of gaits & their stability
(basic argument: symmetry groups of trot and pace cannot be conjugate since they have different stability as shown in certain experiments w/ binding dogs' legs. However, in any 4 cell network that also produces a walk, trot and pace must be conjugate)
Not going into details, but will look in some detail at the 8-cell network which they showed previously was necessary



NOTE: This does not necessarily indicate actual connections – just shows minimal connections to define symm. group

So if ∃ edge 1 to 4, also ∃ edge btw all other nodes one apart on opp. sides (3-6, 5-8, 7-2, 2-3, 4-5, 6-7, 8-1)

So we have contralateral symmetry κ (interchange left & right) and ipsalateral symmetry ω which simultaneously cyclically permutes all cells on left & right
$\Rightarrow$ symmetry group $\Gamma = \mathbb{Z}_2(\kappa) \times \mathbb{Z}_4(\omega)$

If $H = \Gamma$, there are 6 possible choices for K that make H\K cyclic, and these correspond to unique gaits

(Copy table)

Let's look at the walk and trot to see how this all works

First note that for all of these the spatiotemporal symmetry of H implies $\kappa x_i(t) = x_i(t+\theta)$

$x_i(t) = \kappa^2 x_i(t) = x_i(t+2\theta)$ so $2\theta = 0 \pmod{T}$

$\Rightarrow$ always shift of 0 or $\frac{1}{2}$ across sides

and $\omega x_i(t) = x_i(t+\theta')$

$x_i(t) = \omega^4 x_i(t) = x_i(t+4\theta')$ so $4\theta = 0 \pmod{T}$

$\Rightarrow$ always shift $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ on same side

Now for the walk in particular

$K = \mathbb{Z}_2(\kappa\omega^2) \Rightarrow \kappa\omega^2 x_i(t) = x_i(t)$ so $x_6 = x_1$, $x_5 = x_2$, $x_7 = x_4$, $x_8 = x_3$

and since $\kappa\omega^2$ and 1 are only spatial symmetries (vs. spatiotemporal) this completely determines form

For the trot

$K = \mathbb{Z}_4(\kappa\omega) \Rightarrow \kappa\omega x_i(t) = x_i(t)$ $x_1 = x_4 = x_5 = x_8$

$x_2 = x_3 = x_6 = x_7$

again this completely determines solution

4. Is Symmetry Necessary for Synchrony?

So far we've been building intuition about what symmetry groups of networks mean for the types of solns we're able to find

But now we note that many 'real world' networks do not exhibit global symmetry, even in an approximate sense and so the question is whether the type of symmetries we've been discussing are actually necessary to observe these kinds of phase relations

The point of the paper & what we're about to see is that the global group theoretic symmetries aren't actually necessary

We'll look at 4 examples to illustrate this point

Ex. 4.1

$$\begin{tikzcd} & 1 \arrow[dl] & \\ 2 \arrow[rr] & & 3 \arrow[ul] \end{tikzcd}$$

directed ring (recall previous ex. was bidirec.)
$\mathbb{Z}_3$ equivarient (previous was S_3)

We lose the double frequency periodic state given by
$(H,K) = (\mathbb{Z}_2(0), 1)$
but we still have rotating wave soln's given by
$(H,K) = (\mathbb{Z}_3, 1)$
So that $x_2(t) = x_1(t + T/3)$
$x_3(t) = x_2(t + 2T/3)$

($x_1(t)$ any T-periodic solution)

Ex 4.2

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7$$

$$\dot{x}_1 = h(x_1)$$
$$\dot{x}_j = g(x_j, x_{j-1}) \quad j = 2, .. 7$$

trivial symmetry $\rightarrow$ no traveling wave solutions

Ex 4.3

just add an edge

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend right] \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \end{tikzcd}$$

Now if we consider states s.t.
$x_1 = x_4 = x_7$
$x_2 = x_5$
$x_3 = x_6$

Then these ODEs reduce to exactly those of ex. 4.1
so we have a "polydiagonal"
$\Delta = \{ (x_1, x_2, x_3, x_1, x_2, x_3, x_1) : x_1, x_2, x_3 \in \mathbb{R} \}$
that is flow invariant for any choice of g

And we can take our solution (call it "lifting" the soln)

so $x_7(t) = x_4(t) = x_1(t)$

$x_6(t) = x_3(t) = x_1(t + 2T/3)$

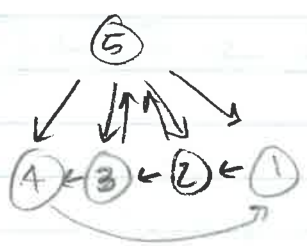
$x_5(t) = x_2(t) = x_1(t + T/3)$

So the point is we get nontrivial phase locking even though no global network symmetry exists.

The unidirectional ring will be called the 'quotient' network of the chain - a notion Kelly will formalize in the second part

Ex 4.4. last one in this series is slightly more complicated

$$\dot{x}_1 = g(x_1, \overline{x_4, x_5})$$
$$\dot{x}_2 = g(x_2, \overline{x_1, x_5})$$
$$\dot{x}_3 = g(x_3, \overline{x_2, x_5})$$
$$\dot{x}_4 = g(x_4, \overline{x_3, x_5})$$
$$\dot{x}_5 = g(x_5, \overline{x_2, x_3})$$



would have symm. $\mathbb{Z}_4$ but doesn't b/c of feedback from 2 & 3

<u>Def</u> A <u>polysynchronous subspace</u> is a flow-invariant polydiag.

Turns out there are 5 in this network

each corresponds to a <u>partition</u> = decomp. into disjoint subsets where components are identical w/i each subset

Call a partition a <u>pattern of synchrony</u>

(Copy table 3) these are the 5 partitions for our network

Note not every partition leads to a flow invariant subspace. For example, if we tried $\{1,2,3\}, \{4,5\}$ then we'd have the polydiagonal $\{y,y,y,z,z\}$ and the eq'ns

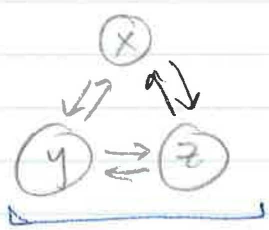
$$\left[\begin{aligned}
\dot y &= g(y, \overline{z,z}) \\
\dot y &= g(y, \overline{y,z}) \\
\dot y &= g(y, \overline{y,z}) \\
\dot z &= g(z, \overline{y,z}) \\
\dot z &= g(z, \overline{y,y})
\end{aligned}\right.$$

> not flow invariant for general g

let's look in particular at $\{1,3\}, \{2,4\}, \{5\}$,

These correspond to our 3 cell bidirectional ring w/ symmetry group S_3 since the eq'ns become

$$\left[\begin{aligned}
\dot x &= g(x, \overline{y,z}) \\
\dot y &= g(y, \overline{x,z}) \\
\dot x &= g(x, \overline{y,z}) \\
\dot y &= g(y, \overline{x,z}) \\
\dot z &= g(z, \overline{y,x})
\end{aligned}\right.$$



Recall we can have a state where one cell rotates at twice the frequency of the other two (this was the soln we couldn't get w/ unidirectional ring)

Show fig. 17

~~Note these soln's will have same stability inside the synchrony subspace, but transverse stability may be different~~

Finally consider 2^{nd} to last partition $\{1,3\}, \{2,4,5\}$ corresponds to $\{x, y, x, y, y\}$

$$\Rightarrow \quad \begin{aligned}
\dot x &= g(x, \overline{y,y}) \\
\dot y &= g(y, \overline{x,y})
\end{aligned}$$

so 'makes sense' to think of multiple arrows & self-loops