AM117 Sample Solutions, HW#1

Hi, I'm Andreas Kloeckner, your TA for this course. Feel free to come to my office hours

> Tuesdays 2:30-4:30 in Room 004 in the Applied Math Castle at 182 George St.

Directions to Room 004: Go in by the main entrance, walk into the hallway behind the mailboxes. Turn left and go down the flight of steps. At the bottom, take a left and then a right. 004 should be straight ahead. (There's a sign on the door.)

Every week, you may download solutions (including code) from

> `http://www.dam.brown.edu/people/kloeckner/am117`

Feel free to email me at `kloeckner@dam.brown.edu`.

The code for the sample solutions is written so that it will run in both Matlab and Octave. (I test it in both.) Octave is a language for scientific computing that is generally very similar to Matlab. Unlike Matlab, Octave is *freely downloadable*. You may grab a copy at `http://www.gnu.org/software/octave`.

1. Problem 6, p.28

We are attempting to find α and λ such that

$$|e_{n+1}| \approx \lambda |e_n|^\alpha.$$

Taking the logarithm of this, we obtain

$$\underbrace{\log|e_{n+1}|}_{\text{"}y\text{"}} \approx \underbrace{\log\lambda}_{\text{"}b\text{"}} + \underbrace{\alpha}_{\text{"}a\text{"}} \underbrace{\log|e_n|}_{\text{"}x\text{"}} \tag{1}$$

If we look closely, we can see that this reduces the problem to finding a line (given by slope a and y-intercept b). We need to find two parameters, and since we have three data points for each method, we may write two equations of the form (1) to find them:

$$\begin{aligned} \log|e_2| &= \log\lambda + \alpha\log|e_1|, \\ \log|e_3| &= \log\lambda + \alpha\log|e_2|. \end{aligned}$$

Subtracting the two, we find

$$\frac{\log|e_2| - \log|e_3|}{\log|e_1| - \log|e_2|} = \alpha.$$

Knowing α, we solve the first one for λ:

$$\begin{aligned} \log\lambda &= \log|e_2| - \alpha\log|e_1|, \\ \lambda &= \exp(\log|e_2| - \alpha\log|e_1|). \end{aligned}$$

This yields:

```
Method 1
alpha =  1.9950
lambda =  0.55965
Method 2
alpha =  3.9083
lambda =  0.031034
```

```
Method 3
alpha =  2.9953
lambda =  0.22063
```

From these calculations it seems the orders of convergence are roughly 2, 4 and 3, respectively.

Here's some code to perform this computation:

```
function p1
    verify_order_of_convergence('Method 1', [4.0e-2, 9.1e-4, 4.8e-7], 2)
    verify_order_of_convergence('Method 2', [3.7e-4, 1.2e-15, 1.5e-60], 3.9)
    verify_order_of_convergence('Method 3', [4.3e-3, 1.8e-8, 1.4e-24], 3)
end

function verify_order_of_convergence(label, seq, order)
    disp(label)
    alpha = (log(seq(2))-log(seq(3))) / (log(seq(1))-log(seq(2)))
    lambda = exp(log(seq(2))-alpha*log(seq(1)))
end
```

2. Problem 11, p.28

Let

$$f(x) := \frac{x^3 + 3x\,a}{3x^2 + a}$$

and observe that $f(\sqrt{a}) = \sqrt{a}$. Further, it is not hard to see that

- for $x \in (0, \sqrt{a})$, we have $f(x) > x$ and $f(x) < \sqrt{a}$,
- for $x \in (\sqrt{a}, \infty)$, we have $f(x) < x$ and $f(x) > \sqrt{a}$.

Convergence of the sequence

$$x_{n+1} := f(x_n)$$

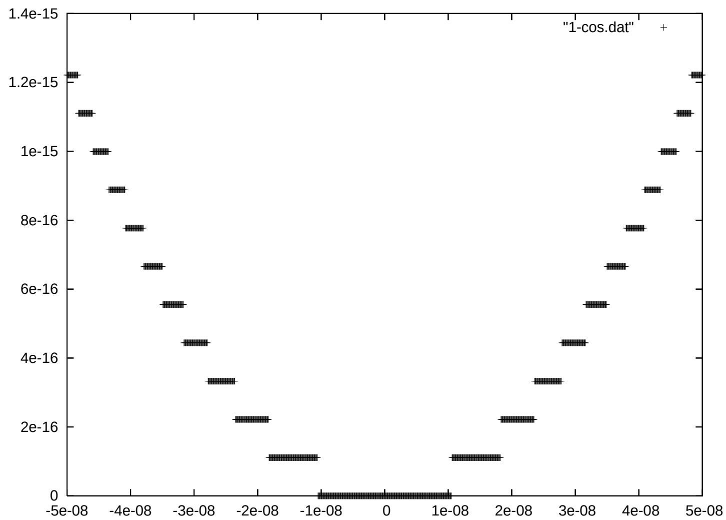
to $\sqrt{a}$ is thus established. Now consider

$$\begin{aligned}
\frac{|e_{n+1}|}{|e_n|^3} &= \frac{|\sqrt{a} - x_{n+1}|}{|\sqrt{a} - x_n|^3} = \frac{\left|\sqrt{a} - \frac{x_n^3 + 3x_n a}{3x_n^2 + a}\right|}{\left|\sqrt{a} - x_n\right|^3} = \left|\frac{\sqrt{a} - \frac{x^3 + 3x_n a}{3x_n^2 + a}}{(\sqrt{a} - x_n)^3}\right| \\
&= \left|\frac{3\sqrt{a}x_n^2 + a^{3/2} - x^3 - 3x_n a}{(\sqrt{a} - x_n)^3(3x_n^2 + a)}\right| = \left|\frac{(\sqrt{a} - x)^3}{(\sqrt{a} - x_n)^3(3x_n^2 + a)}\right| \\
&= \frac{1}{3x_n^2 + a} \to \frac{1}{3a + a} = \frac{1}{4a}
\end{aligned}$$

as $n \to \infty$. The asymptotic error constant is $1/4a$.

3. Problem 9, p.52

a)



Code:

```
function p3
    p3_do_plot(-5e-8, 5e-8, 1000)
    pause
end

function do_plot(a, b, n)
    x = linspace(a, b, n);
    y = 1-cos(x);
    plot(x,1-cos(x));
end
```

b) Plot $\sin^2 x$ and note that it looks a bit like the above. Remember

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

or, for our case,

$$\cos(2x) = \cos^2 x - \sin^2 x. \quad (2)$$

Also remember

$$1 = \cos^2 x + \sin^2 x. \quad (3)$$

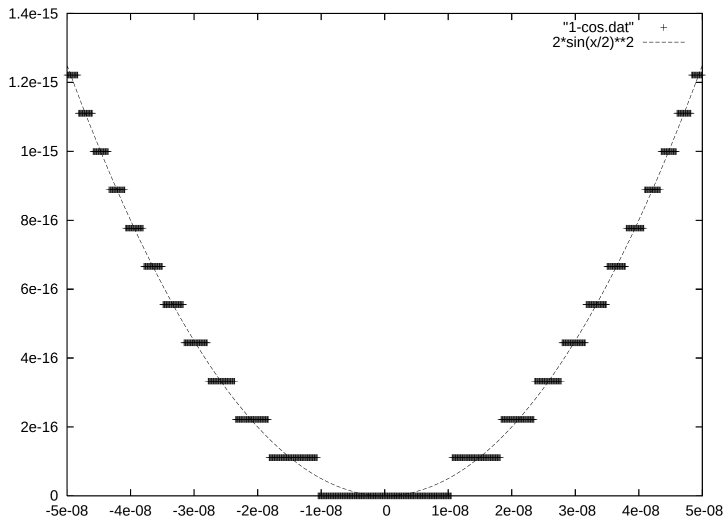
Now compute $(3) - (2)$:

$$1 - \cos(2x) = 2\sin^2(x),$$

which, after substituting $\xi = 2x$, yields

$$1 - \cos(\xi) = 2\sin^2(\xi/2)$$

and a much cleaner plot:



4. Problem 13, p.52

a)

$$g(x) - f(x) = \frac{\cos^2 x}{1+\sin x} - (1-\sin x) = \frac{\cos^2 x - (1-\sin x)(1+\sin x)}{1+\sin x} = \frac{\cos^2 x - (1-\sin^2 x)}{1+\sin x} = 0.$$

In preparation for b) and c), we create this table:

x	$\sin(x)$	$\cos^2(x)$
$\pi/2$	1	0
$3\pi/2$	-1	0

b) Near $\pi/2$, $\sin(x) \approx 1$, so $f(x) = 1 - \sin(x)$ subtracts two values of order 1 to obtain a result very close to zero, and so it will encounter significant cancellation error. The logical choice is $g(x) = \cos^2(x)/(1+\sin(x))$, for which we observe no such phenomena.

c) Near $3\pi/2$, $\sin(x) \approx -1$, so the denominator of g (which is $1+\sin x$) will encounter cancellation error.

5. Problem 6, p.69

The length of the initial interval is $|b-a|$. After each iteration, the length of the interval is halved, so we have $|b-a|\cdot 2^{-1}$ after the first, $|b-a|\cdot 2^{-2}$ after the second, ..., $|b-a|\cdot 2^{-n}$ after the nth.

Once we know that the length of our candidate interval is shorter than ε, our tolerance, i.e.

$$|b-a|\cdot 2^{-n} < \varepsilon,$$

we are done. We can now compute the number of iterations needed to achieve this precision:

$$\begin{aligned} |b-a|\cdot 2^{-n} &< \varepsilon \\ \Leftrightarrow \frac{|b-a|}{\varepsilon} &< 2^n \\ \Leftrightarrow \log\frac{|b-a|}{\varepsilon} &< n\cdot\log 2 \\ \Leftrightarrow \frac{\log\frac{|b-a|}{\varepsilon}}{\log 2} &< n. \end{aligned}$$

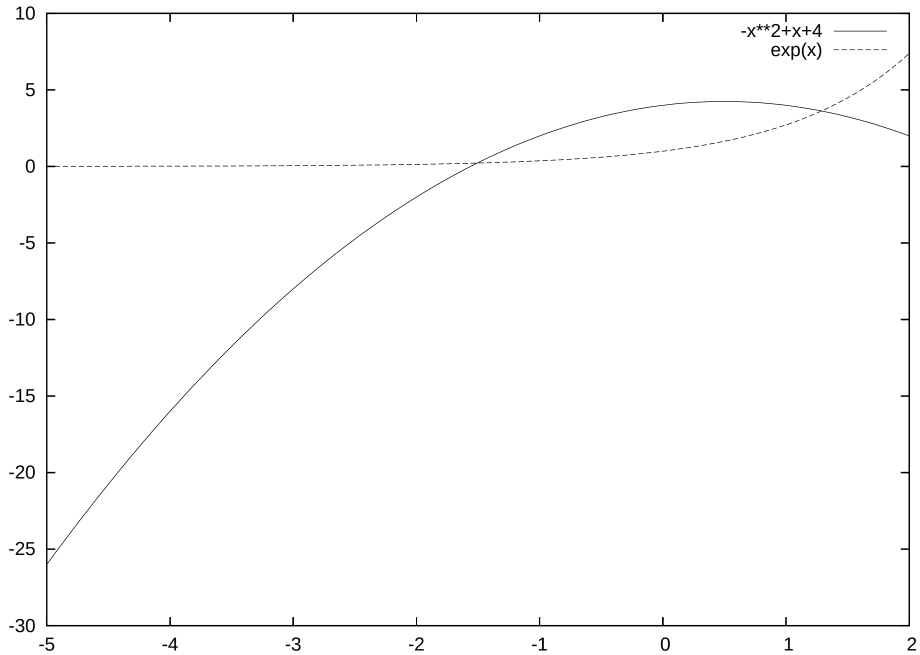
6. Problem 16, p.70

a) $f(x) = e^x + x^2 - x - 4$. Consider this function in two parts.

$$\begin{aligned} f_1(x) &= e^x \\ f_2(x) &= -x^2 + x + 4 = (x-1/2)^2 - 17/4 \end{aligned}$$

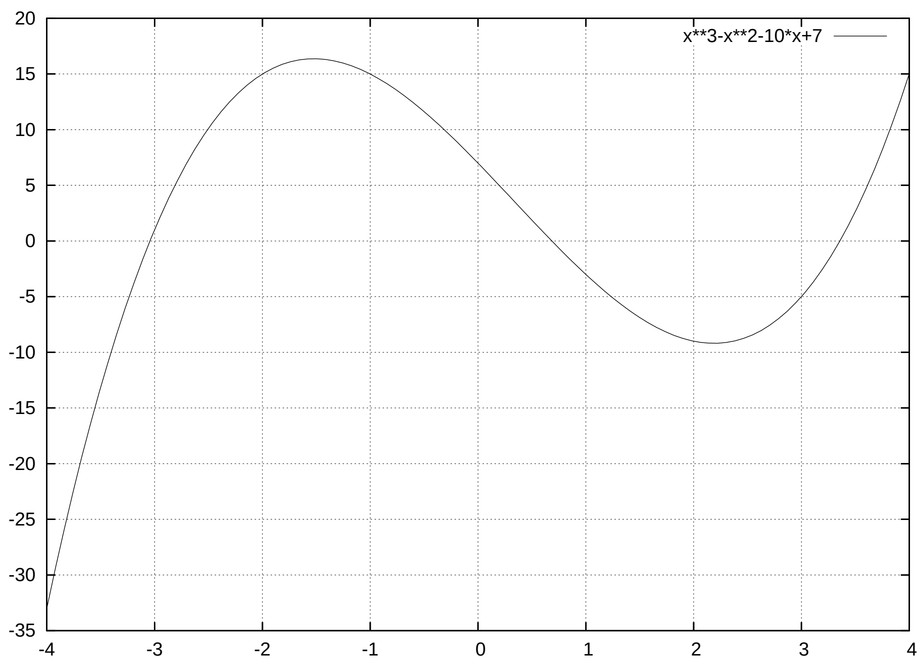
$$f(x) = f_1(x) - f_2(x),$$

so that the zeros we seek are the intersections of the graphs of f_1 and f_2, as shown below:



Using our knowledge of the behavior of the exponential function and of parabolas, it is not hard to see that there will be two zeros, one each in the intervals $[-2, -1]$ and $[1, 2]$. Bisection search results are listed at the end of the problem in one big batch.

b) $f(x) = x^3 - x^2 - 10x + 7$. Theory says that this polynomial cannot have more than three zeros. Plot it:

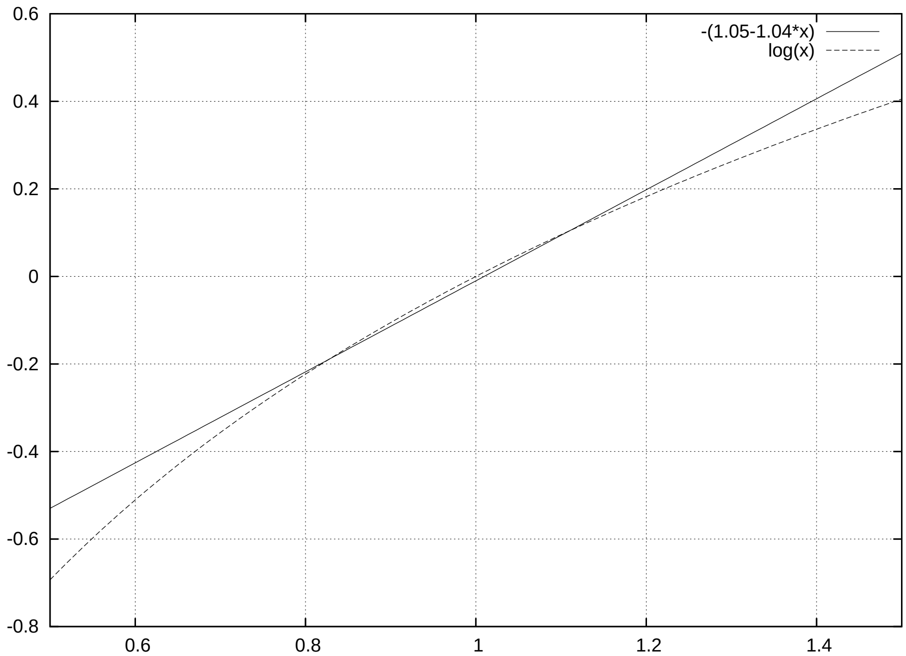


It's not hard to see that we have three candidate intervals, $[-4, -3]$, $[0, 1]$ and $[3, 4]$.

c) $f(x) = 1.05 - 1.04x + \log x$. Once again consider this function as composed of two parts:

$$\begin{aligned} f_1(x) &= -(1.05 - 1.04x), \\ f_2(x) &= \log x, \\ f(x) &= f_2(x) - f_1(x). \end{aligned}$$

Again, we can view the sought zeros as intersections of the graphs of two well-understood functions:



We are led to believe that we should look in $[0.8, 1]$ and $[1, 1.2]$.

Code:

```
function p6
    disp('----------------------')
    disp('Part a)')
    disp('----------------------')
    bisect(-2, -1, @f_part_a)
    bisect(1, 2, @f_part_a)

    disp('----------------------')
    disp('Part b)')
    disp('----------------------')

    bisect(-4, -3, @f_part_b)
    bisect(0, 1, @f_part_b)
    bisect(3, 4, @f_part_b)

    disp('----------------------')
    disp('Part c)')
    disp('----------------------')

    bisect(0.8, 1, @f_part_c)
    bisect(1, 1.2, @f_part_c)
```

```
end

function bisect(a, b, f)
    tol = 1e-6

    fa = f(a);
    fb = f(b);

    disp('Interval at start:')
    a
    b

    % intermediate value theorem must guarantee existence of a
    % zero in (a,b).
    if (fa*fb > 0)
      error('Potentially no zero in search interval.')
    end

    while (abs(b-a) > tol)
        c = (a+b)/2.;
        fc = f(c);

        if (fc*fa <= 0)
            b = c;
            fb = fc;
        else
            a = c;
            fa = fc;
        end
    end

    disp('Final Interval:')
    a
    b
end

function y = f_part_a(x)
    y = exp(x)+x^2-x-4;
end
function y = f_part_b(x)
    y = x^3-x^2-10*x+7;
end
function y = f_part_c(x)
    y = 1.05 - 1.04*x+log(x);
end
```

Results:

```
----------------------
Part a)
----------------------
tol =  1.0000e-06
Interval at start:
a = -2
```

```
b = -1
Final Interval:
a = -1.5071
b = -1.5071
tol =  1.0000e-06
Interval at start:
a =  1
b =  2
Final Interval:
a =  1.2887
b =  1.2887
----------------------
Part b)
----------------------
tol =  1.0000e-06
Interval at start:
a = -4
b = -3
Final Interval:
a = -3.0427
b = -3.0427
tol =  1.0000e-06
Interval at start:
a = 0
b =  1
Final Interval:
a =  0.68522
b =  0.68522
tol =  1.0000e-06
Interval at start:
a =  3
b =  4
Final Interval:
a =  3.3575
b =  3.3575
----------------------
Part c)
----------------------
tol =  1.0000e-06
Interval at start:
a =  0.80000
b =  1
Final Interval:
a =  0.82718
b =  0.82718
tol =  1.0000e-06
Interval at start:
a =  1
b =  1.2000
Final Interval:
a =  1.1097
b =  1.1097
```