SIAM J. NUMER. ANAL.
Vol. 50, No. 5, pp. 2778–2801

COUPLING OF RAVIART–THOMAS AND HYBRIDIZABLE DISCONTINUOUS GALERKIN METHODS WITH BEM*

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Abstract. We present new a priori error analyses of the coupling of the Raviart–Thomas (RT) method and BEM as well as of the coupling of the hybridizable discontinuous Galerkin (HDG) method with BEM. The novel features of the analysis of the RT-BEM coupling are the superconvergence estimates for the scalar approximation of the interior problem and new bounds in weak and strong norms for the boundary variables. The analysis of the HDG-BEM coupling is the first analysis of this coupling and shows that the coupling provides approximations with the same convergence properties as those of the RT-BEM coupling.

Key words. boundary element methods, discontinuous Galerkin methods, mixed finite element, coupling

AMS subject classifications. 65N30, 65N38, 65N12, 65N15

DOI. 10.1137/100818339

1. Introduction.

In this paper, we continue the work started in [9], where a new, systematic approach was proposed for coupling a wide variety of FEM (old and new) with BEM, and present the first a priori error analysis of two of the couplings proposed therein. The first coupling, proposed in [17, 5], uses the Raviart–Thomas (RT) mixed method [19] to solve the interior problem, whereas the second is new and uses a hybridizable discontinuous Galerkin (HDG) method, namely, the so-called LDG-H method [7].

Let us relate our results with those of the available literature. The analysis of RT-BEM (carried out in two dimensions only, but easily extendable to three dimensions) was done for the first time by Meddahi et al. [17]. It was rediscovered by Carstensen and Funken [5], who explicitly dealt with more general mixed methods. Both papers treat the coupling method as a Galerkin method for a boundary-field formulation that uses the four integral operators associated to the problem in the exterior domain. In this sense, the methods fit in the family of symmetric coupling of BEM and FEM devised by Costabel [11] and Han [15].

Our analysis of the RT-BEM method takes a different approach and is able to provide new convergence results. Since it is based on energy and duality arguments, it allows us to obtain convergence properties for the flux variable $\mathbf{q} := -\nabla u$ in a *weaker* norm than in [17] or [5]. Indeed, we work on $L^2(\Omega)^d$, whereas the traditional variational approach gives an analysis in the smaller space $H(\mathrm{div}, \Omega)$. It has to be pointed out, however, that the error control for the divergence represents a minor advantage since the divergence of $\mathbf{q}$ is actually data of the problem.

*Received by the editors December 15, 2010; accepted for publication (in revised form) August 31, 2012; published electronically October 25, 2012.
http://www.siam.org/journals/sinum/50-5/81833.html

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Moreover, we provide *new* bounds in weak and strong norms for the boundary variables, including *superconvergence* estimates for the postprocessed exterior solution. We also are able to extend to this setting the well-known superconvergence estimate of the RT approximation of the scalar variable u; see [2]. This allows us to construct an *element-by-element* postprocessed new approximation to u converging with an additional order.

Let us now compare the HDG-BEM coupling with other DG-BEM methods. In this case, and just as in the case of the RT-BEM coupling, there are *two* different approximations to the *same* variable at the coupling boundary, namely, the numerical flux of the HDG method and the BEM approximation. In contrast, the LDG-BEM method proposed in [13], which contains the first analysis of the coupling of a discontinuous Galerkin and a BEM, employs a *mortar* variable to enforce the coupling. The only other DG-BEM coupling is the LDG-BEM method considered in [12]. Therein, the approximate solution of the DG method is forced to become continuous right on the coupling interface. Yet another difference between the HDG-BEM coupling under consideration and the above-mentioned LDG-BEM coupling is that in our case, a hybridized formulation of the full system, in the spirit of [9], is available which extends to the realm of BEM-FEM the ideas of [7].

To analyze the HDG-BEM coupling, we take advantage of the strong relation between mixed methods and HDG methods uncovered in [7] to proceed in an analogous, if more involved, manner. We obtain for this coupling the same convergence properties obtained for the above-mentioned RT-BEM coupling. In particular, this implies that *both* the approximate flux and the postprocessing of the scalar solution converge to the exact solution with orders of convergence that are one more than the orders provided by the approximations of the LDG-BEM methods proposed in [13] and [12].

Our analysis is carried out for the exterior Yukawa equation ($-\Delta v + v = 0$) to avoid several minor technicalities related to the Laplace equation: energy-free solutions (constant functions are nontrivial solutions of the interior Neumann problem) and the important difference of behavior at infinity of the fundamental solution of the Laplacian in two and three dimensions. Just for the case of HDG-BEM we offer two options to deal with the peculiarities of the Laplace operator in two and three dimensions.

The paper is organized as follows. In section 2, we describe the model problem we are going to work on. In section 3, we carry out the analysis of the RT-BEM coupling and in section 4 that of the HDG-BEM coupling. Section 5 is devoted to the study of several technical results needed in the analysis of these two methods. In section 6, we sketch the modifications that need to be taken into consideration when instead of the exterior Yukawa problem we consider the exterior Laplace equation. Finally, section 7 contains some numerical experiments confirming the theoretical results.

Foreword. For properties of the classical Sobolev spaces $H^k(\mathcal{O})$ (where $\mathcal{O}$ is an open set in $\mathbb{R}^d$), we refer the reader to [1]. The norm and seminorm of $H^k(\mathcal{O})$ will be respectively denoted $\|\cdot\|_{k,\mathcal{O}}$ and $|\cdot|_{k,\mathcal{O}}$. The norm $L^2(\mathcal{O}) = H^0(\mathcal{O})$ will omit the 0 subindex. On the polygon/polyhedron Γ, Sobolev spaces $H^s(\Gamma)$ can be defined for any $-1 \le s \le 1$. Their classical norms will be denoted $\|\cdot\|_{s,\Gamma}$. For other values of the index s, we will clarify some necessary details in section 5. We will use some of the basic properties of layer potentials and boundary integral operators that can be found in [16] and were first gathered in [10]. For quantities depending on the discrete parameter h, we will write $a \lesssim b$ when there exists $C > 0$, independent of h, such that $a \le C\, b$. When $a \lesssim b \lesssim a$, we simply write $a \approx b$.

2. Model problem. As a first model problem we consider the following one. The space $\mathbb{R}^d$ is divided into a Lipschitz bounded domain Ω and its exterior Ω_+. Its common boundary Γ is taken to be a polygon/polyhedron. For simplicity, we assume that the exterior domain Ω_+ is connected. The continuous equations are (for $f \in L^2(\Omega)$)

$$\mathbf{q} + \nabla u = 0 \quad \text{in } \Omega, \tag{2.1a}$$

$$\nabla \cdot \mathbf{q} = f \quad \text{in } \Omega, \tag{2.1b}$$

$$-\Delta v + v = 0 \quad \text{in } \Omega_+, \tag{2.1c}$$

$$u = v \quad \text{on } \Gamma, \tag{2.1d}$$

$$-\mathbf{q} \cdot \mathbf{n} = \partial_\nu v \quad \text{on } \Gamma. \tag{2.1e}$$

Let

$$E(\mathbf{x}, \mathbf{y}) := \begin{cases} \dfrac{1}{2\pi} K_0(|\mathbf{x} - \mathbf{y}|) & \text{if } d = 2, \\ \dfrac{e^{-|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x} - \mathbf{y}|} & \text{if } d = 3 \end{cases}$$

be the fundamental solution of $u \mapsto -\Delta u + u$. (K_0 is the modified Bessel function of order zero, also known as the Macdonald function of order zero.) Because we have considered the Yukawa operator in the exterior domain, we do not need to impose a radiation condition at infinity. Instead, we ask for u to be in $H^1(\Omega_+)$. The modifications needed to deal with the Laplace equation in the exterior will be dealt with in section 6. We then consider the single and double layer potentials

$$\mathcal{S}\lambda := \int_\Gamma E(\,\cdot\,, \mathbf{y})\lambda(\mathbf{y}) \mathrm{d}\Gamma(\mathbf{y}) \ : \mathbb{R}^d \setminus \Gamma \to \mathbb{R},$$

$$\mathcal{D}\varphi := \int_\Gamma \partial_{\nu(\mathbf{y})} E(\,\cdot\,, \mathbf{y})\varphi(\mathbf{y}) \mathrm{d}\Gamma(\mathbf{y}) \ : \ \mathbb{R}^d \setminus \Gamma \to \mathbb{R}$$

that define solutions of $-\Delta u + u = 0$ in $\mathbb{R}^d \setminus \Gamma$ for arbitrary $\lambda \in H^{-1/2}(\Gamma)$ and $\varphi \in H^{1/2}(\Gamma)$. The four associated integral operators that define functions (distributions) on Γ are denoted as follows:

$$\mathcal{V}\lambda := \int_\Gamma E(\,\cdot\,, \mathbf{y})\lambda(\mathbf{y}) \mathrm{d}\Gamma(\mathbf{y}) \ : \Gamma \to \mathbb{R},$$

$$\mathcal{K}^t\lambda := \int_\Gamma \partial_{\nu(\,\cdot\,)} E(\,\cdot\,, \mathbf{y})\lambda(\mathbf{y}) \mathrm{d}\Gamma(\mathbf{y}) \ : \Gamma \to \mathbb{R},$$

$$\mathcal{K}\varphi := \int_\Gamma \partial_{\nu(\mathbf{y})} E(\,\cdot\,, \mathbf{y})\varphi(\mathbf{y}) \mathrm{d}\Gamma(\mathbf{y}) \ : \Gamma \to \mathbb{R},$$

$$\mathcal{W}\varphi := \partial_\nu \int_\Gamma \partial_{\nu(\mathbf{y})} E(\,\cdot\,, \mathbf{y})\varphi(\mathbf{y}) \mathrm{d}\Gamma(\mathbf{y}) \ : \Gamma \to \mathbb{R}.$$

Note that although the integral expressions of $\mathcal{V}$ and $\mathcal{K}$ coincide with those of $\mathcal{S}$ and $\mathcal{D}$, respectively, their output is defined on different domains: free space minus the boundary for the potentials, the boundary Γ for the integral operators. The mathematically precise definition of these entities [16] is also different. The exterior branch of the solution will be represented using Green's third identity,

$$v = \mathcal{D}\varphi - \mathcal{S}\lambda, \qquad \text{where } \varphi := v \text{ and } \lambda := \partial_\nu v.$$

We also have two identities derived from Calderón's theory:

$$\mathcal{V}\lambda + (\tfrac{1}{2}\mathcal{I} - \mathcal{K})\varphi = 0, \qquad (\tfrac{1}{2}\mathcal{I} + \mathcal{K}^t)\lambda + \mathcal{W}\varphi = 0.$$

Here and in what follows $\mathcal{I}$ will denote a general identity operator. We rearrange the previous two identities in the following form:

$$\mathcal{V}\lambda - \widetilde{\mathcal{K}}\varphi = -\varphi,$$
$$\widetilde{\mathcal{K}}^t\lambda + \mathcal{W}\varphi = 0,$$

where $\widetilde{\mathcal{K}} := \frac{1}{2}\mathcal{I} + \mathcal{K}$.

3. Analysis of RT-BEM. We consider a triangulation $\mathcal{T}_h$ of the polygonal (polyhedral) domain Ω as well as the inherited partition Γ_h of its boundary. The set of edges of $\mathcal{T}_h$ is denoted $\mathcal{E}_h$. The space of polynomials of degree not greater than k restricted to the element $K \in \mathcal{T}_h$ is denoted $\mathbb{P}_k(K)$. The space $\mathbb{P}_k(e)$ for $e \in \mathcal{E}_h$ is similarly defined. We now define five discrete spaces for a given $k \geq 0$:

$$\mathbf{V}_h := \prod_{K\in\mathcal{T}_h} \mathrm{RT}_k(K), \qquad W_h := \prod_{K\in\mathcal{T}_h} \mathbb{P}_k(K), \qquad M_h := \prod_{e\in\mathcal{E}_h} \mathbb{P}_k(e),$$
$$X_h := \prod_{e\in\Gamma_h} \mathbb{P}_k(e), \qquad Y_h := \Big(\prod_{e\in\Gamma_h} \mathbb{P}_{k+1}(e)\Big) \cap \mathcal{C}(\Gamma).$$

The local RT space is defined as $\mathrm{RT}_k(K) = \mathbb{P}_k(K)^d + \mathbf{x}\,\mathbb{P}_k(K)$ (see [4]). For some of the forthcoming expressions, it will be useful to have the function $\mathfrak{h} \in \prod_{e\in\mathcal{E}_h} \mathbb{P}_0(e)$ defined by $\mathfrak{h}|_e = h_e$ (h_e being the diameter of e), as well as the function $\mathfrak{h}_\Gamma := \mathfrak{h}|_\Gamma$.

The discrete equations are as follows: find $\mathbf{q}_h \in \mathbf{V}_h$, $u_h \in W_h$, $\widehat{u}_h \in M_h$, $\lambda_h \in X_h$, $\varphi_h \in Y_h$ such that

$$(\mathbf{q}_h, \mathbf{r})_{\mathcal{T}_h} - (u_h, \nabla\cdot\mathbf{r})_{\mathcal{T}_h} + \langle \widehat{u}_h, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0 \qquad \forall \mathbf{r} \in \mathbf{V}_h, \tag{3.1a}$$
$$(\nabla\cdot\mathbf{q}_h, w)_{\mathcal{T}_h} = (f, w)_{\mathcal{T}_h} \qquad \forall w \in W_h, \tag{3.1b}$$
$$\langle \mathbf{q}_h\cdot\mathbf{n}, \mu\rangle_{\partial\mathcal{T}_h} + \langle \lambda_h, \mu\rangle_\Gamma = 0 \qquad \forall \mu \in M_h, \tag{3.1c}$$
$$\langle \widehat{u}_h, \eta\rangle_\Gamma + \langle \mathcal{V}\lambda_h, \eta\rangle_\Gamma - \langle \widetilde{\mathcal{K}}\varphi_h, \eta\rangle_\Gamma = 0 \qquad \forall \eta \in X_h, \tag{3.1d}$$
$$\langle \widetilde{\mathcal{K}}^t\lambda_h, \psi\rangle_\Gamma + \langle \mathcal{W}\varphi_h, \psi\rangle_\Gamma = 0 \qquad \forall \psi \in Y_h. \tag{3.1e}$$

The first three equations above constitute a hybridizable version of the mixed method (with RT elements for $\mathbf{q}_h$ paired with discontinuous polynomials for u_h) with weakly enforced Neumann boundary conditions on Γ. A solution of (3.1) satisfies the energy identity

$$\|\mathbf{q}_h\|_\Omega^2 + \langle \mathcal{V}\lambda_h, \lambda_h\rangle_\Gamma + \langle \mathcal{W}\varphi_h, \varphi_h\rangle_\Gamma = (f, w_h). \tag{3.2}$$

This can be easily proved testing these equations with $\mathbf{q}_h$, u_h, $-\widehat{u}_h$, λ_h, and φ_h, respectively, and finally adding the result. For some of the arguments below, we will need the space $\mathbf{V}_h^{\mathrm{div}} := \mathbf{V}_h \cap \mathbf{H}(\mathrm{div}, \Omega)$. For easy reference, let us recall that the operator

$$\begin{array}{rcl} \mathrm{RT}_k(K) & \longrightarrow & M_K := \prod_{e\subset\partial K} \mathbb{P}_k(e), \\ \mathbf{r} & \longmapsto & \mathbf{r}\cdot\mathbf{n}, \end{array} \tag{3.3}$$

is onto. Therefore, so is the operator $\mathbf{V}_h^{\mathrm{div}} \to X_h$ that takes the normal component of a discrete vector field on Γ.

PROPOSITION 3.1. *Equations* (3.1) *are uniquely solvable.*

Proof. Because (3.1) is equivalent to a square linear system, we only have to prove uniqueness. Then take $f = 0$ in (3.1). By (3.2), it follows that $\mathbf{q}_h = \mathbf{0}$, $\lambda_h = 0$, and $\varphi_h = 0$ because $\mathcal{V}$ and $\mathcal{W}$ are elliptic (see (3.10) below). Going back to (3.1d), tested with $\widehat{u}_h|_\Gamma \in X_h$, we obtain that $\widehat{u}_h = 0$ on Γ.

We now pick $\mathbf{r} \in \mathbf{V}_h^{\text{div}}$ such that $\nabla \cdot \mathbf{r} = -u_h$ and use it to test (3.1a). This can be done because the divergence operator is onto from $\mathbf{V}_h^{\text{div}}$ to W_h. Since $\widehat{u}_h = 0$ on Γ, $\widehat{u}_h$ is single-valued on interior faces and $\mathbf{r} \cdot \mathbf{n}$ is continuous across element interfaces, it follows that

$$0 = (u_h, u_h)_{\mathcal{T}_h} + \langle \widehat{u}_h, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = \|u_h\|_\Omega^2$$

and therefore $u_h = 0$. Equation (3.1a) is thus reduced to

$$\langle \widehat{u}_h, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = 0 \qquad \forall \mathbf{r} \in \mathbf{V}_h,$$

which implies that $\widehat{u}_h = 0$. (To prove this, we only have to consider this equation element by element and choose $\mathbf{r}$ such that $\mathbf{r} \cdot \mathbf{n} = \widehat{u}_h$ on ∂K.) ∎

In the following result we prove that (3.1) are a hybridizable formulation for the RT-BEM coupling of [17] (see also [5]). A partially hybrid formulation can be reached by eliminating $(\mathbf{q}_h, u_h)$ from the system (3.1) using local solvers. In that case, the resulting system involves a variable on the skeleton $\widehat{u}_h$ and two variables on the boundary (λ_h, φ_h). A fully hybridized formulation, exclusively written in terms of $\widehat{u}_h$, can be found by eliminating the boundary variables from the system (see [9]).

PROPOSITION 3.2. *Equations* (3.1) *are equivalent to the problem of finding* $\mathbf{q}_h \in \mathbf{V}_h^{\text{div}}$, $u_h \in W_h$, *and* $\varphi_h \in Y_h$ *such that*

(3.4a)
$$(\mathbf{q}_h, \mathbf{r})_\Omega + \langle \mathcal{V}(\mathbf{q}_h \cdot \mathbf{n}), \mathbf{r} \cdot \mathbf{n} \rangle_\Gamma - (u_h, \nabla \cdot \mathbf{r})_\Omega + \langle \widetilde{\mathcal{K}} \varphi_h, \mathbf{r} \cdot \mathbf{n} \rangle_\Gamma = 0 \qquad \forall \mathbf{r} \in \mathbf{V}_h^{\text{div}},$$

(3.4b)
$$(\nabla \cdot \mathbf{q}_h, w)_\Omega = (f, w)_\Omega \quad \forall w \in W_h,$$

(3.4c)
$$-\langle \widetilde{\mathcal{K}}^t (\mathbf{q}_h \cdot \mathbf{n}), \psi \rangle_\Gamma + \langle \mathcal{W} \varphi_h, \psi \rangle_\Gamma = 0 \qquad \forall \psi \in Y_h.$$

More precisely, a solution of (3.1) *solves* (3.4) *and a solution of* (3.4) *defines a solution of* (3.1) *by taking* $\lambda_h := -\mathbf{q}_h \cdot \mathbf{n}$ *on* Γ *and by solving the problem*

$$\langle \widehat{u}_h, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = -(\mathbf{q}_h, \mathbf{r})_{\mathcal{T}_h} + (u_h, \nabla \cdot \mathbf{r})_{\mathcal{T}_h} \qquad \forall \mathbf{r} \in \mathbf{V}_h. \tag{3.5}$$

Proof. Given a solution of (3.1), (3.1c) implies that $\mathbf{q}_h \in \mathbf{V}_h^{\text{div}}$ and that $\mathbf{q}_h \cdot \mathbf{n} + \lambda_h = 0$ on Γ. Therefore (3.4c) follows from (3.1e). Note (3.1b) and (3.4b) are the same equation. Testing (3.1a) with $\mathbf{r} \in \mathbf{V}_h^{\text{div}}$ and using (3.1d) simplifies

$$\begin{aligned} \langle \widehat{u}_h, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = \langle \widehat{u}_h, \mathbf{r} \cdot \mathbf{n} \rangle_\Gamma &= -\langle \mathcal{V} \lambda_h, \mathbf{r} \cdot \mathbf{n} \rangle_\Gamma + \langle \widetilde{\mathcal{K}} \varphi_h, \mathbf{r} \cdot \mathbf{n} \rangle_\Gamma \\ &= \langle \mathcal{V}(\mathbf{q}_h \cdot \mathbf{n}), \mathbf{r} \cdot \mathbf{n} \rangle_\Gamma + \langle \widetilde{\mathcal{K}} \varphi_h, \mathbf{r} \cdot \mathbf{n} \rangle_\Gamma, \end{aligned}$$

which gives (3.4a) easily.

We now start with a solution of (3.4) and construct λ_h and $\widehat{u}_h$ as in the statement of the proposition. Then, immediately (3.1b), (3.1c), and (3.1e) are satisfied. Note that (3.5) is equivalent to the local problems of finding $\widehat{u}_K := \widehat{u}_h|_{\partial K} \in M_K$ satisfying

$$\langle \widehat{u}_K, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial K} = -(\mathbf{q}_h, \mathbf{r})_K + (u_h, \nabla \cdot \mathbf{r})_K \qquad \forall \mathbf{r} \in \mathrm{RT}_k(K).$$

Solvability of these local problems and coincidence of values on interior faces follow from the fact that $(\mathbf{q}_h, u_h)$ is part of the solution of (3.4). This shows that (3.1a) is satisfied. Given $\eta \in X_h$ we can take $\mathbf{r} \in \mathbf{V}_h^{\rm div}$ such that $\mathbf{r}\cdot\mathbf{n} = \eta$. Using this test function $\mathbf{r}$ in (3.4a) and substituting (3.1a) in the resulting expression, we verify (3.1d). ∎

Equations (3.4) can be analyzed by a stability argument for a Galerkin discretization of a mixed problem that is well posed in $\mathbf{H}(\mathrm{div},\Omega)\times L^2(\Omega)\times H^{1/2}(\Gamma)$. We will follow a different approach which will provide stronger estimates for u_h as well as estimates for the hybrid variable $\widehat{u}_h$, weak norm estimates for λ_h and φ_h, and some postprocessing strategies. None of these results appear in [17] or in its generalization [5].

Energy estimate. We consider five discrete projections. First, $\Pi^{\rm RT}$ is the RT projection (see [4], for example) onto $\mathbf{V}_h$, that can be defined in $\prod_{K\in\mathcal{T}_h}\mathbf{H}^1(K)$ or on somewhat weaker spaces. The $L^2(\Omega)$ projection onto W_h is denoted P. The $\prod_{e\in\mathcal{E}_h}L^2(e)$ projection onto M_h is denoted P^∂, while the $L^2(\Gamma)$ projection onto X_h is denoted P^Γ. Finally, we consider the Lagrange interpolation operator $\mathrm{I}^\Gamma : \mathcal{C}(\Gamma)\to Y_h$. We will study the approximation properties of

$$\boldsymbol{\varepsilon}_h^q := \Pi^{\rm RT}\mathbf{q}-\mathbf{q}_h,\qquad \varepsilon_h^u := \mathrm{P}u-u_h,\qquad \widehat{\varepsilon}_h^u := \mathrm{P}^\partial u-\widehat{u}_h,$$
$$\varepsilon_h^\lambda := \mathrm{P}^\Gamma\lambda-\lambda_h,\qquad \varepsilon_h^\varphi := \mathrm{I}^\Gamma\varphi-\varphi_h.$$

In what follows, we will use the following well-known properties that relate some of these projections [4]:

$$\nabla\cdot\Pi^{\rm RT}\mathbf{d} = \mathrm{P}(\nabla\cdot\mathbf{d}),\qquad (\Pi^{\rm RT}\mathbf{d})\cdot\mathbf{n} = \mathrm{P}^\partial(\mathbf{d}\cdot\mathbf{n}). \tag{3.6}$$

To shorten some forthcoming notation we consider the bilinear form $\mathcal{A}$, defined in the product space $H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$, that is obtained by adding the integral terms in (3.1d) and (3.1e):

$$\mathcal{A}\big((\lambda,\varphi),(\eta,\psi)\big) := \langle\mathcal{V}\lambda,\eta\rangle_\Gamma - \langle\widetilde{\mathcal{K}}\varphi,\eta\rangle_\Gamma + \langle\widetilde{\mathcal{K}}^t\lambda,\psi\rangle_\Gamma + \langle\mathcal{W}\varphi,\psi\rangle_\Gamma. \tag{3.7}$$

Note that

$$\mathcal{A}\big((\lambda,-\varphi),(\eta,-\psi)\big) = \mathcal{A}\big((\eta,\psi),(\lambda,\varphi)\big), \tag{3.8}$$
$$|\mathcal{A}\big((\lambda,\varphi),(\eta,\psi)\big)| \le C\big(\|\lambda\|_{-1/2,\Gamma}+\|\varphi\|_{1/2,\Gamma}\big)\big(\|\eta\|_{-1/2,\Gamma}+\|\psi\|_{1/2,\Gamma}\big), \tag{3.9}$$
$$\mathcal{A}\big((\lambda,\varphi),(\lambda,\varphi)\big) = \langle\mathcal{V}\lambda,\lambda\rangle_\Gamma + \langle\mathcal{W}\varphi,\varphi\rangle_\Gamma \ge C\big(\|\lambda\|^2_{-1/2,\Gamma}+\|\varphi\|^2_{1/2,\Gamma}\big). \tag{3.10}$$

These last two inequalities follow from the properties of the boundary integral operators [16].

Proposition 3.3.

$$\|\boldsymbol{\varepsilon}_h^q\|_\Omega + \|\varepsilon_h^\lambda\|_{-1/2,\Gamma} + \|\varepsilon_h^\varphi\|_{1/2,\Gamma} \lesssim \|\Pi^{\rm RT}\mathbf{q}-\mathbf{q}\|_\Omega + \|\mathrm{P}^\Gamma\lambda-\lambda\|_{-1/2,\Gamma} + \|\mathrm{I}^\Gamma\varphi-\varphi\|_{1/2,\Gamma}.$$

Proof. Using (3.6), it is simple to see that the following error equations are satisfied:

(3.11a)
$$(\boldsymbol{\varepsilon}_h^q,\mathbf{r})_{\mathcal{T}_h} - (\varepsilon_h^u,\nabla\cdot\mathbf{r})_{\mathcal{T}_h} + \langle\widehat{\varepsilon}_h^u,\mathbf{r}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} = (\Pi^{\rm RT}\mathbf{q}-\mathbf{q},\mathbf{r})_{\mathcal{T}_h}\qquad \forall\mathbf{r}\in\mathbf{V}_h,$$

(3.11b)
$$(\nabla\cdot\boldsymbol{\varepsilon}_h^q,w)_{\mathcal{T}_h} = 0\qquad \forall w\in W_h,$$

(3.11c)
$$\langle\boldsymbol{\varepsilon}_h^q\cdot\mathbf{n},\mu\rangle_{\partial\mathcal{T}_h} + \langle\varepsilon_h^\lambda,\mu\rangle_\Gamma = 0\qquad \forall\mu\in M_h,$$

(3.11d)
$$\mathcal{A}\big((\varepsilon_h^\lambda,\varepsilon_h^\varphi),(\eta,\psi)\big) + \langle\widehat{\varepsilon}_h^u,\eta\rangle_\Gamma = \mathcal{A}\big((\mathrm{P}^\Gamma\lambda-\lambda,\mathrm{I}^\Gamma\varphi-\varphi),(\eta,\psi)\big)\quad \forall(\eta,\psi)\in X_h\times Y_h.$$

Testing these equations with $\boldsymbol{\varepsilon}_h^q, \varepsilon_h^u, -\widehat{\varepsilon}_h^u$, and $(\varepsilon_h^\lambda, \varepsilon_h^\varphi)$, respectively, and adding them we obtain that

$$\|\boldsymbol{\varepsilon}_h^q\|_\Omega^2 + \langle \mathcal{V}\varepsilon_h^\lambda, \varepsilon_h^\lambda\rangle_\Gamma + \langle \mathcal{W}\varepsilon_h^\varphi, \varepsilon_h^\varphi\rangle_\Gamma = (\Pi^{\mathrm{RT}}\mathbf{q} - \mathbf{q}, \boldsymbol{\varepsilon}_h^q) + \mathcal{A}\big((\mathrm{P}^\Gamma\lambda - \lambda, \mathrm{I}^\Gamma\varphi - \varphi), (\varepsilon_h^\lambda, \varepsilon_h^\varphi)\big).$$

Using now the boundedness of $\mathcal{A}$ (3.9) and the ellipticity of $\mathcal{V}$ and $\mathcal{W}$ (3.10), the result follows readily. ◻

Before we continue with error estimates for the remaining variables as well as with error estimates in different norms for the boundary unknowns, let us comment here on the error estimates that follow from Proposition 3.3 in the case of highest regularity. For the definitions of the broken Sobolev spaces $X^k(\Gamma)$, see section 5. If $\mathbf{q} \in \mathbf{H}^{k+1}(\Omega)$, $\lambda \in X^{k+1}(\Gamma)$ and $\varphi \in X^{k+2}(\Gamma)$, then

$$\|\mathbf{q} - \mathbf{q}_h\|_\Omega + \|\lambda - \lambda_h\|_{-1/2,\Gamma} + \|\varphi - \varphi_h\|_{1/2,\Gamma} \lesssim h^{k+1}|\mathbf{q}|_{k+1,\Omega} + h_\Gamma^{k+3/2}\Big(\|\lambda\|_{X^{k+1}(\Gamma)} + \|\varphi\|_{X^{k+2}(\Gamma)}\Big),$$

where $h_\Gamma := \max\{h_e : e \in \Gamma_h\} \le h$. In principle, this shows how a term of order $h^{1/2}$ in the approximation property of the boundary unknowns is not taken advantage of. Note, however, that all terms have the same order if we reduce the regularity requirement of λ and φ in 1/2, which corresponds to an exact solution satisfying $u \in H^{k+2}(\Omega)$:

$$\|\mathbf{q} - \mathbf{q}_h\|_\Omega + \|\lambda - \lambda_h\|_{-1/2,\Gamma} + \|\varphi - \varphi_h\|_{1/2,\Gamma} \lesssim h^{k+1}\Big(|\mathbf{q}|_{k+1,\Omega} + \|\lambda\|_{X^{k+1/2}(\Gamma)} + \|\varphi\|_{X^{k+3/2}(\Gamma)}\Big).$$

Estimate by duality. To estimate ε_h^u we consider the solution to the problem

$$\mathbf{d} + \nabla\Theta = 0 \quad \text{in } \Omega, \tag{3.12a}$$

$$\nabla\cdot\mathbf{d} = \varepsilon_h^u \quad \text{in } \Omega, \tag{3.12b}$$

$$-\Delta\omega + \omega = 0 \quad \text{in } \Omega_+, \tag{3.12c}$$

$$\Theta = \omega \quad \text{on } \Gamma, \tag{3.12d}$$

$$\mathbf{d}\cdot\mathbf{n} = -\partial_\nu\omega \quad \text{on } \Gamma \tag{3.12e}$$

and denote $H^{-1/2}(\Gamma) \ni \xi = -\mathbf{d}\cdot\mathbf{n} = \partial_\nu\omega$ and $H^{1/2}(\Gamma) \ni \alpha = \Theta = \omega$ (on Γ).

PROPOSITION 3.4. *There exists a constant C_{reg} such that the solution of* (3.12) *satisfies*

$$\|\mathbf{d}\|_{1,\Omega} + \|\Theta\|_{2,\Omega} + \|\omega\|_{2,\Omega_+} \le C_{\mathrm{reg}}\|\varepsilon_h^u\|_\Omega.$$

Proof. Consider the function $v \in H^1(\mathbb{R}^d)$ given by

$$v = \begin{cases} \Theta & \text{in } \Omega, \\ \omega & \text{in } \Omega_+. \end{cases}$$

Notice that $\partial_\nu^- v = \partial_\nu^+ v$ on Γ, which means that the Laplacian of v in $\mathbb{R}^d$ is the same as the function defined by the Laplacian of v on each side of Γ. Therefore

$$-\Delta v + v = g \in L^2(\mathbb{R}^d) \qquad \text{with} \quad g := \begin{cases} -\varepsilon_h^u + \Theta & \text{in } \Omega, \\ \omega & \text{in } \Omega_+. \end{cases}$$

Hence $v \in H^2(\mathbb{R}^d)$ and

$$\|v\|_{2,\mathbb{R}^d} \le C\|g\|_{0,\mathbb{R}^d} \le C'\Big(\|\varepsilon_h^u\|_\Omega + \|\Theta\|_\Omega + \|\omega\|_{\Omega_+}\Big) \le C''\|\varepsilon_h^u\|_\Omega,$$

where in the last inequality we have applied a simple ellipticity estimate for the transmission problem (3.12). ∎

Proposition 3.5.

$$\begin{aligned}\|\varepsilon_h^u\|_\Omega^2 = {} & (\mathbf{q}-\mathbf{q}_h, \Pi^{\mathrm{RT}}\mathbf{d}-\mathbf{d})_{\mathcal{T}_h} + (f - \mathrm{P}f, \Theta - \mathrm{P}\Theta)_{\mathcal{T}_h} \\ & + \mathcal{A}\big((\lambda-\lambda_h, \varphi-\varphi_h), (\mathrm{P}^\Gamma\xi - \xi, \alpha - \alpha_h)\big) \qquad \forall \alpha_h \in Y_h.\end{aligned}$$

Proof. Let us test the error equations (3.11) with $\Pi^{\mathrm{RT}}\mathbf{d}$, $\mathrm{P}\Theta$, $\mathrm{P}^\partial\Theta$, and $(\mathrm{P}^\Gamma\xi, -\alpha_h)$, respectively. After reordering some terms we obtain these four identities:

$$(\mathbf{q}-\mathbf{q}_h, \Pi^{\mathrm{RT}}\mathbf{d})_{\mathcal{T}_h} - (\varepsilon_h^u, \nabla\cdot\Pi^{\mathrm{RT}}\mathbf{d})_{\mathcal{T}_h} + \langle\widehat{\varepsilon}_h^u, (\Pi^{\mathrm{RT}}\mathbf{d})\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.13a}$$

$$(\nabla\cdot\boldsymbol{\varepsilon}_h^q, \mathrm{P}\Theta)_{\mathcal{T}_h} = 0, \tag{3.13b}$$

$$\langle\boldsymbol{\varepsilon}_h^q\cdot\mathbf{n}, \mathrm{P}^\partial\Theta\rangle_{\partial\mathcal{T}_h} + \langle\varepsilon_h^\lambda, \mathrm{P}^\partial\Theta\rangle_\Gamma = 0, \tag{3.13c}$$

$$\langle\widehat{\varepsilon}_h^u, \mathrm{P}^\Gamma\xi\rangle_\Gamma + \mathcal{A}\big((\lambda-\lambda_h, \varphi-\varphi_h), (\mathrm{P}^\Gamma\xi, -\alpha_h)\big) = 0. \tag{3.13d}$$

Note first that by (3.6)

$$(\varepsilon_h^u, \nabla\cdot\Pi^{\mathrm{RT}}\mathbf{d})_{\mathcal{T}_h} = (\varepsilon_h^u, \mathrm{P}(\nabla\cdot\mathbf{d}))_{\mathcal{T}_h} = (\varepsilon_h^u, \nabla\cdot\mathbf{d})_{\mathcal{T}_h} = \|\varepsilon_h^u\|_\Omega^2.$$

The normal component of $\mathbf{d}$ is single-valued on interior faces and hence so is the normal component of $\Pi^{\mathrm{RT}}\mathbf{d}$. Moreover, $(\Pi^{\mathrm{RT}}\mathbf{d})\cdot\mathbf{n} = \mathrm{P}^\partial(\mathbf{d}\cdot\mathbf{n})$ and on Γ this function equals $\mathrm{P}^\Gamma(\mathbf{d}\cdot\mathbf{n})$. Therefore

$$\langle\widehat{\varepsilon}_h^u, (\Pi^{\mathrm{RT}}\mathbf{d})\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} = \langle\widehat{\varepsilon}_h^u, (\Pi^{\mathrm{RT}}\mathbf{d})\cdot\mathbf{n}\rangle_\Gamma = \langle\widehat{\varepsilon}_h^u, \mathrm{P}^\Gamma(\mathbf{d}\cdot\mathbf{n})\rangle_\Gamma = \langle\widehat{\varepsilon}_h^u, \mathbf{d}\cdot\mathbf{n}\rangle_\Gamma.$$

On the other hand the normal component of $\boldsymbol{\varepsilon}_h^q$ is single-valued on interior faces and so is Θ. Therefore

$$\langle\boldsymbol{\varepsilon}_h^q\cdot\mathbf{n}, \mathrm{P}^\partial\Theta\rangle_{\partial\mathcal{T}_h} = \langle\boldsymbol{\varepsilon}_h^q\cdot\mathbf{n}, \Theta\rangle_{\partial\mathcal{T}_h} = \langle\boldsymbol{\varepsilon}_h^q\cdot\mathbf{n}, \Theta\rangle_\Gamma.$$

We also know that $\mathrm{P}^\partial\Theta = \mathrm{P}^\Gamma\Theta$ on Γ. This implies that (3.13) can be rewritten in the following simplified form:

$$(\mathbf{q}-\mathbf{q}_h, \Pi^{\mathrm{RT}}\mathbf{d})_{\mathcal{T}_h} - \|\varepsilon_h^u\|_\Omega^2 + \langle\widehat{\varepsilon}_h^u, \mathbf{d}\cdot\mathbf{n}\rangle_\Gamma = 0, \tag{3.14a}$$

$$-(\nabla\cdot\boldsymbol{\varepsilon}_h^q, \Theta)_{\mathcal{T}_h} = 0, \tag{3.14b}$$

$$\langle\boldsymbol{\varepsilon}_h^q\cdot\mathbf{n}, \Theta\rangle_\Gamma + \langle\varepsilon_h^\lambda, \Theta\rangle_\Gamma = 0, \tag{3.14c}$$

$$\langle\widehat{\varepsilon}_h^u, \xi\rangle_\Gamma + \mathcal{A}\big((\lambda-\lambda_h, \varphi-\varphi_h), (\mathrm{P}^\Gamma\xi, -\alpha_h)\big) = 0. \tag{3.14d}$$

Adding these four equations, using (3.12a) and the definitions of α and ξ, we obtain

$$\begin{aligned}\|\varepsilon_h^u\|_\Omega^2 &= (\mathbf{q}-\mathbf{q}_h, \Pi^{\mathrm{RT}}\mathbf{d})_{\mathcal{T}_h} - (\nabla\cdot\boldsymbol{\varepsilon}_h^q, \Theta)_{\mathcal{T}_h} + \langle\boldsymbol{\varepsilon}_h^q\cdot\mathbf{n}, \Theta\rangle_\Gamma \\ &\quad + \langle\widehat{\varepsilon}_h^u, \mathbf{d}\cdot\mathbf{n}\rangle_\Gamma + \langle\widehat{\varepsilon}_h^u, \xi\rangle_\Gamma \\ &\quad + \langle\varepsilon_h^\lambda, \Theta\rangle_\Gamma + \mathcal{A}\big((\lambda-\lambda_h, \varphi-\varphi_h), (\mathrm{P}^\Gamma\xi, -\alpha_h)\big) \\ &= (\mathbf{q}-\mathbf{q}_h, \Pi^{\mathrm{RT}}\mathbf{d}-\mathbf{d})_{\mathcal{T}_h} + T_1 + T_2, \end{aligned} \tag{3.15}$$

where

$$T_1 := -(\mathbf{q} - \mathbf{q}_h, \nabla\Theta)_\Omega - (\nabla \cdot \boldsymbol{\varepsilon}_h^q, \Theta)_\Omega + \langle \boldsymbol{\varepsilon}_h^q \cdot \mathbf{n}, \Theta\rangle_\Gamma,$$
$$T_2 := \langle \varepsilon_h^\lambda, \alpha\rangle_\Gamma + \mathcal{A}\big((\lambda - \lambda_h, \varphi - \varphi_h), (\mathrm{P}^\Gamma\xi, -\alpha_h)\big).$$

We now apply Green's theorem twice as well as known properties of the RT projection to prove that

$$\begin{aligned} T_1 &= (-\mathbf{q} + \mathbf{q}_h + \Pi^{\mathrm{RT}}\mathbf{q} - \mathbf{q}_h, \nabla\Theta)_\Omega \\ &= -(\nabla\cdot(\Pi^{\mathrm{RT}}\mathbf{q} - \mathbf{q}), \Theta)_\Omega + \langle(\Pi^{\mathrm{RT}}\mathbf{q} - \mathbf{q})\cdot\mathbf{n}, \Theta\rangle_\Gamma \\ &= (\nabla\cdot\mathbf{q} - \mathrm{P}(\nabla\cdot\mathbf{q}), \Theta)_\Omega + \langle \mathrm{P}^\Gamma(\mathbf{q}\cdot\mathbf{n}) - \mathbf{q}\cdot\mathbf{n}, \Theta\rangle_\Gamma \\ &= (f - \mathrm{P}f, \Theta - \mathrm{P}\Theta)_\Omega + \langle \lambda - \mathrm{P}^\Gamma\lambda, \alpha\rangle_\Gamma. \end{aligned} \tag{3.16}$$

Since $\alpha = \omega$ and $\xi = \partial_\nu\omega$ on Γ, where $-\Delta\omega + \omega = 0$ in Ω_+, it follows that

$$\alpha + \mathcal{V}\xi - \widetilde{\mathcal{K}}\alpha = 0, \qquad \widetilde{\mathcal{K}}^t\xi + \mathcal{W}\alpha = 0.$$

This is equivalent to

$$\mathcal{A}\big((\xi,\alpha),(\eta,-\psi)\big) + \langle\alpha,\eta\rangle_\Gamma = 0 \qquad \forall(\eta,\psi)\in H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$$

and also, using (3.8), to

$$\mathcal{A}\big((\eta,\psi),(\xi,-\alpha)\big) + \langle\alpha,\eta\rangle_\Gamma = 0 \qquad \forall(\eta,\psi)\in H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma). \tag{3.17}$$

Applying this expression with $\eta = \lambda - \lambda_h$ and $\psi = \varphi - \varphi_h$ we obtain

$$\begin{aligned} T_2 + \langle \lambda - \mathrm{P}^\Gamma\lambda, \alpha\rangle_\Gamma &= \langle\lambda - \lambda_h, \alpha\rangle_\Gamma + \mathcal{A}\big((\lambda - \lambda_h, \varphi - \varphi_h), (\mathrm{P}^\Gamma\xi, -\alpha_h)\big) \\ &= \mathcal{A}\big((\lambda - \lambda_h, \varphi - \varphi_h), (\mathrm{P}^\Gamma\xi - \xi, \alpha - \alpha_h)\big). \end{aligned} \tag{3.18}$$

Substituting (3.16) and (3.18) in (3.15), the result follows. □

Henceforth we will write $\underline{k} := \min\{k, 1\}$. Proposition 3.5 has prepared the field for an error estimate for u_h, showing superconvergence to the projection of u.

PROPOSITION 3.6.

$$\|\varepsilon_h^u\|_\Omega \lesssim h\Big(\|\mathbf{q} - \mathbf{q}_h\|_\Omega + h^{\underline{k}}\|f - \mathrm{P}f\|_\Omega + \|\lambda - \lambda_h\|_{-1/2,\Gamma} + \|\varphi - \varphi_h\|_{1/2,\Gamma}\Big).$$

Proof. We just have to bound the three terms in the right-hand side of Proposition 3.5 after giving an adequate choice for α_h. First, using properties of the RT projection (see [4])

$$|(\mathbf{q} - \mathbf{q}_h, \Pi^{\mathrm{RT}}\mathbf{d} - \mathbf{d})_\Omega| \lesssim h\|\mathbf{q} - \mathbf{q}_h\|_\Omega\,|\mathbf{d}|_{1,\Omega} \lesssim h\|\mathbf{q} - \mathbf{q}_h\|_\Omega\|\varepsilon_h^u\|_\Omega.$$

Also,

$$\begin{aligned} |(f - \mathrm{P}f, \Theta - \mathrm{P}\Theta)_\Omega| &\lesssim h^{\min\{k+1,2\}}\|f - \mathrm{P}f\|_\Omega\,|\Theta|_{\min\{k+1,2\},\Omega} \\ &\le h^{1+\underline{k}}\|f - \mathrm{P}f\|_\Omega\|\varepsilon_h^u\|_\Omega. \end{aligned}$$

By the boundedness of $\mathcal{A}$ (3.9), we only need to deal with

$$\|\xi - \mathrm{P}^\Gamma\xi\|_{-1/2,\Gamma} + \|\alpha - \alpha_h\|_{1/2,\Gamma}.$$

We will choose $\alpha_h := \mathrm{C}^\Gamma\Theta$ (see section 5). Then, by Proposition 5.1 (formula (5.1) with $t = -1/2$ and $s = 1/2$), Proposition 5.1, and the boundedness of $\partial_\nu : H^2(\Omega) \to X^{1/2}(\Gamma)$ (see section 5), it follows that

(3.19)
$$\|\xi - \mathrm{P}^\Gamma\xi\|_{-1/2,\Gamma} + \|\Theta - \mathrm{C}^\Gamma\Theta\|_{1/2,\Gamma} \lesssim h(\|\partial_\nu\Theta\|_{X^{1/2}(\Gamma)} + \|\Theta\|_{2,\Omega}) \lesssim h\|\Theta\|_{2,\Omega} \lesssim h\|\varepsilon_h^u\|_\Omega.$$

This completes the proof. Note that the choice of α_h produces in a simple way the desired h in the estimate using the assumed regularity. ∎

Next we prove a superconvergence result for $\widehat{\varepsilon}_h^u$.

PROPOSITION 3.7.

$$\|\widehat{\varepsilon}_h^u\|_h := \Big(\sum_{K\in\mathcal{T}_h} h_K\|\widehat{\varepsilon}_h^u\|_{0,\partial K}^2\Big)^{1/2} \lesssim h\|\mathbf{q}_h - \mathbf{q}\|_\Omega + \|\varepsilon_h^u\|_\Omega.$$

Proof. This follows from standard arguments. Using the degrees of freedom of the RT space, we can choose $\mathbf{v} \in \mathrm{RT}_k(K)$ such that $\mathbf{v}\cdot\mathbf{n} = \widehat{\varepsilon}_h^u$ on ∂K and

$$\|\mathbf{v}\|_{0,K} + h_K\|\nabla\cdot\mathbf{v}\|_{0,K} \lesssim h_K^{1/2}\|\mathbf{v}\cdot\mathbf{n}\|_{0,\partial K} = h_K^{1/2}\|\widehat{\varepsilon}_h^u\|_{0,\partial K}.$$

Then by the first of the error equations (3.11), we prove for every $K \in \mathcal{T}_h$ that

$$\begin{aligned}\|\widehat{\varepsilon}_h^u\|_{0,\partial K}^2 &= \langle\widehat{\varepsilon}_h^u, \mathbf{v}\cdot\mathbf{n}\rangle_{\partial K} = (\mathbf{q} - \mathbf{q}_h, \mathbf{v})_K - (\varepsilon_h^u, \nabla\cdot\mathbf{v})_K \\ &\lesssim h_K^{1/2}\|\widehat{\varepsilon}_h^u\|_{0,\partial K}(\|\mathbf{q} - \mathbf{q}_h\|_{0,K} + h_K^{-1}\|\varepsilon_h^u\|_{0,K}).\end{aligned}$$

The result is a straightforward consequence of these inequalities. ∎

We now prove an error estimate for the boundary quantities in weaker norms. Apart from the interest of having an $L^2(\Gamma)$ error estimate for $\varphi - \varphi_h$, the following result is useful for estimating the error in pointwise computation of $v_h := \mathcal{D}\varphi_h - \mathcal{S}\lambda_h \approx v$. In principle, if we want $H^1(\Omega_+)$ estimates for $v - v_h$, we can only take advantage of the bound in natural norms given in Proposition 3.3. However, as is well known,

$$|v(\mathbf{x}) - v_h(\mathbf{x})| \le C(\mathbf{x})(\|\varphi - \varphi_h\|_{0,\Gamma} + \|\lambda - \lambda_h\|_{-1,\Gamma}) \qquad \forall \mathbf{x} \in \Omega_+,$$

where the constant $C(\mathbf{x})$ remains bounded as long as $\mathbf{x}$ does not approach Γ.

PROPOSITION 3.8. *Denoting* $h_{\min,\Gamma} := \min_{e\in\Gamma_h} h_e$ *and* $h_\Gamma := \max_{e\in\Gamma} h_e$*, we can bound*

$$\|\lambda - \lambda_h\|_{-1,\Gamma} + \|\varphi - \varphi_h\|_{0,\Gamma} \lesssim h_\Gamma^{1/2}\Big(\|\lambda - \lambda_h\|_{-1/2,\Gamma} + \|\varphi - \varphi_h\|_{1/2,\Gamma}\Big) + h_{\min,\Gamma}^{-1/2}\|\widehat{\varepsilon}_h^u\|_h.$$

Proof. We start by recalling that the matrix of operators

$$\begin{bmatrix} \mathcal{V} & \widetilde{\mathcal{K}} \\ -\widetilde{\mathcal{K}}^t & \mathcal{W} \end{bmatrix}$$

(which is the transpose of the one that defines the bilinear form $\mathcal{A}$) defines an isomorphism between $H^0(\Gamma) \times H^1(\Gamma)$ and $H^1(\Gamma) \times H^0(\Gamma) = (H^{-1}(\Gamma) \times H^0(\Gamma))'$. Therefore, there exists a constant such that

$$\|e_h^\lambda\|_{-1,\Gamma} + \|e_h^\varphi\|_{0,\Gamma} \le C \sup_{\mathbf{0}\neq(\eta,\psi)\in H^0(\Gamma)\times H^1(\Gamma)} \frac{\mathcal{A}((e_h^\lambda, e_h^\varphi), (\eta,\psi))}{\|\eta\|_{0,\Gamma} + \|\psi\|_{1,\Gamma}}, \tag{3.20}$$

where we have denoted $e_h^\lambda := \lambda - \lambda_h$ and $e_h^\varphi := \varphi - \varphi_h$. Note now that

$$\begin{aligned}
&|\mathcal{A}((e_h^\lambda, e_h^\varphi), (\eta - \mathrm{P}^\Gamma \eta, \psi - \mathrm{Q}^\Gamma \psi))| \\
&\qquad \lesssim \Big(\|e_h^\lambda\|_{-1/2,\Gamma} + \|e_h^\varphi\|_{1/2,\Gamma}\Big)\Big(\|\eta - \mathrm{P}^\Gamma\eta\|_{-1/2,\Gamma} + \|\psi - \mathrm{Q}^\Gamma\psi\|_{1/2,\Gamma}\Big) \\
&\qquad \lesssim \Big(\|e_h^\lambda\|_{-1/2,\Gamma} + \|e_h^\varphi\|_{1/2,\Gamma}\Big) h_\Gamma^{1/2}\Big(\|\eta\|_{0,\Gamma} + \|\psi\|_{1,\Gamma}\Big),
\end{aligned}$$

where $\mathrm{Q}^\Gamma : H^{1/2}(\Gamma) \to Y_h$ is the $H^{1/2}(\Gamma)$ orthogonal projection onto Y_h and we have applied Proposition 5.1. In addition, using the fourth of the error equations (3.11), it follows that

$$\mathcal{A}((e_h^\lambda, e_h^\varphi), (\mathrm{P}^\Gamma\eta, \mathrm{Q}^\Gamma\psi)) = -\langle \widehat{\varepsilon}_h^u, \mathrm{P}^\Gamma\eta\rangle_\Gamma \le h_{\min,\Gamma}^{-1/2} \|\widehat{\varepsilon}_h^u\|_h \|\mathrm{P}^\Gamma\eta\|_{0,\Gamma}.$$

Inserting these last two inequalities in the right-hand side of (3.20), the result follows readily. □

The condition for Proposition 3.8 (coupled with Propositions 3.7 and 3.6) to give an $h^{1/2}$-order superconvergence estimate is $h \lesssim h_{\min,\Gamma}$. This is satisfied if the larger elements of the triangulation are in the vicinity of Γ and the boundary mesh Γ_h is quasi-uniform.

We finally derive two error estimates for $\lambda - \lambda_h$ in a stronger (functional and not dual) norm.

PROPOSITION 3.9. $\|\mathfrak{h}_\Gamma^{1/2} \varepsilon_h^\lambda\|_{0,\Gamma} \lesssim \|\boldsymbol{\varepsilon}_h^q\|_\Omega$.

Proof. By the third error equation in (3.11), it follows that $\varepsilon_h^\lambda = -\boldsymbol{\varepsilon}_h^q \cdot \mathbf{n}$. If $e \in \Gamma_h \cap \mathcal{E}(K)$, then by a scaling argument and finite dimensionality it follows that $h_e^{1/2}\|\boldsymbol{\varepsilon}_h^q\|_e \lesssim \|\boldsymbol{\varepsilon}_h^q\|_K$, which proves the result. □

Note that if the grid Γ_h is quasi-uniform we can obtain by a simple inverse estimate that

$$\|\varepsilon_h^\lambda\|_{0,\Gamma} \lesssim h_\Gamma^{-1/2} \|\varepsilon_h^\lambda\|_{-1/2,\Gamma}. \tag{3.21}$$

4. Analysis of HDG-BEM.

We redefine the space $\mathbf{V}_h$ to be

$$\mathbf{V}_h := \prod_{K\in\mathcal{T}_h} (\mathbb{P}_k(K))^d$$

and leave W_h, M_h, X_h, and Y_h unchanged. We have thus only changed the space $\mathbf{V}_h$ by eliminating the stabilizing degrees of freedom of the RT element. We consider $\tau : \partial\mathcal{T}_h \to \mathbb{R}$ and $\tau_\mathrm{B} : \Gamma \to \mathbb{R}$ satisfying that

$$\tau \in \prod_{K\in\mathcal{T}_h}\left(\prod_{e\subset\partial K} \mathbb{P}_0(e)\right), \quad \tau > 0, \qquad \tau_\mathrm{B} \in \prod_{e\in\Gamma_h} \mathbb{P}_0(e), \quad \tau_\mathrm{B} \ge 0.$$

In order to keep track of how these stabilization parameters affect the convergence properties, we will consider the quantities

$$T_{\max} := \|\tau_\mathrm{B}\|_{L^\infty(\Gamma)} \quad \text{and} \quad \tau_{\max} := \|\tau\|_{L^\infty(\partial\mathcal{T}_h)}.$$

The discrete equations are as follows: find $\mathbf{q}_h \in \mathbf{V}_h$, $u_h \in W_h$, $\widehat{u}_h \in M_h$, $\lambda_h \in X_h$, $\varphi_h \in Y_h$ such that

$$(\mathbf{q}_h, \mathbf{r})_{\mathcal{T}_h} - (u_h, \nabla\cdot\mathbf{r})_{\mathcal{T}_h} + \langle \widehat{u}_h, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0 \qquad \forall \mathbf{r}\in\mathbf{V}_h, \tag{4.1a}$$

$$(\nabla\cdot\mathbf{q}_h, w)_{\mathcal{T}_h} + \langle \tau(u_h - \widehat{u}_h), w\rangle_{\partial\mathcal{T}_h} = (f, w)_{\mathcal{T}_h} \qquad \forall w\in W_h, \tag{4.1b}$$

$$\langle \tau(u_h - \widehat{u}_h) + \mathbf{q}_h\cdot\mathbf{n}, \mu\rangle_{\partial\mathcal{T}_h} + \langle \lambda_h + \tau_{\mathrm{B}}(\varphi_h - \widehat{u}_h), \mu\rangle_\Gamma = 0 \qquad \forall \mu\in M_h, \tag{4.1c}$$

$$\langle \widehat{u}_h, \eta\rangle_\Gamma + \langle \mathcal{V}\lambda_h, \eta\rangle_\Gamma - \langle \widetilde{\mathcal{K}}\varphi_h, \eta\rangle_\Gamma = 0 \qquad \forall \eta\in X_h, \tag{4.1d}$$

$$\langle \widetilde{\mathcal{K}}^t\lambda_h, \psi\rangle_\Gamma + \langle \mathcal{W}\varphi_h, \psi\rangle_\Gamma + \langle \tau_{\mathrm{B}}(\varphi_h - \widehat{u}_h), \psi\rangle_\Gamma = 0 \qquad \forall \psi\in Y_h. \tag{4.1e}$$

A solution of (4.1) satisfies the energy identity

$$\begin{aligned}\|\mathbf{q}_h\|_\Omega^2 &+ \langle \mathcal{V}\lambda_h, \lambda_h\rangle_\Gamma + \langle \mathcal{W}\varphi_h, \varphi_h\rangle_\Gamma \\ &+ \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h\rangle_{\partial\mathcal{T}_h} + \langle \tau_{\mathrm{B}}(\varphi_h - \widehat{u}_h), \varphi_h - \widehat{u}_h\rangle_\Gamma = (f, w_h).\end{aligned} \tag{4.2}$$

This can be easily proved testing (4.1) with $\mathbf{q}_h$, u_h, $-\widehat{u}_h$, λ_h, and φ_h, respectively, and finally adding the result.

PROPOSITION 4.1. *Equations* (4.1) *are uniquely solvable.*

Proof. We have to prove that if $f = 0$, then the only solution to (4.1) is the trivial one. By (4.2), it is clear that $\mathbf{q}_h = \mathbf{0}$, $\lambda_h = 0$, $\varphi_h = 0$, and

$$\tau(u_h - \widehat{u}_h) = 0 \quad \text{on } \partial\mathcal{T}_h \quad \text{and} \quad \tau_{\mathrm{B}}(\varphi_h - \widehat{u}_h) = 0 \quad \text{on } \Gamma.$$

Testing (4.1d) with $\eta = \widehat{u}_h$ we prove that $\widehat{u}_h = 0$ on Γ. Because $\tau > 0$, then $u_h = \widehat{u}_h$ on $\partial\mathcal{T}_h$, which means that $u_h \in \mathcal{C}^0(\Omega)$ and that $u_h = 0$ on Γ.

Using Green's theorem in (4.1a) and noticing that $\widehat{u}_h = u_h$ on ∂K, it follows that $(\nabla u_h, \mathbf{r})_K = 0$ for all $\mathbf{r}\in\mathbf{V}_h$ and for all $K\in\mathcal{T}_h$. Then taking $\mathbf{r} = \nabla u_h$ we prove that u_h is constant and because it vanishes on Γ, then $u_h = 0$ and therefore $\widehat{u}_h = 0$. This finishes the proof. ∎

We consider the *HDG projection* defined in [8]: to $(\mathbf{q}, u)$ we associate a pair $(\Pi\mathbf{q}, \Pi u) \in \mathbf{V}_h \times W_h$ defined by solving the equations

$$(\Pi\mathbf{q} - \mathbf{q}, \mathbf{r})_K = 0 \qquad \forall \mathbf{r}\in\mathbf{P}_{k-1}(K), \tag{4.3a}$$

$$(\Pi u - u, v)_K = 0 \qquad \forall v\in\mathbb{P}_{k-1}(K), \tag{4.3b}$$

$$\langle (\Pi\mathbf{q} - \mathbf{q})\cdot\mathbf{n} + \tau(\Pi u - u), \mu\rangle_F = 0 \qquad \forall \mu\in P_k(F) \quad \forall F\in\mathcal{E}(K), \tag{4.3c}$$

element by element. We will study the approximation properties of $\boldsymbol{\varepsilon}_h^q := \Pi\mathbf{q} - \mathbf{q}_h$ with ε_h^u, $\widehat{\varepsilon}_h^u$, ε_h^λ, and ε_h^φ defined as before. In the forthcoming arguments we will make use of the boundary bilinear form $\mathcal{A}$ defined in (3.7).

PROPOSITION 4.2 (error equations). *It holds that*

$$(\boldsymbol{\varepsilon}_h^q, \mathbf{r})_{\mathcal{T}_h} - (\varepsilon_h^u, \nabla\cdot\mathbf{r})_{\mathcal{T}_h} + \langle \widehat{\varepsilon}_h^u, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} = (\Pi\mathbf{q} - \mathbf{q}, \mathbf{r})_{\mathcal{T}_h}, \tag{4.4a}$$

$$(\nabla\cdot\boldsymbol{\varepsilon}_h^q, w)_{\mathcal{T}_h} + \langle \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u), w\rangle_{\partial\mathcal{T}_h} = 0, \tag{4.4b}$$

(4.4c)

$$\langle \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u) + \boldsymbol{\varepsilon}_h^q\cdot\mathbf{n}, \mu\rangle_{\partial\mathcal{T}_h} + \langle \varepsilon_h^\lambda + \tau_{\mathrm{B}}(\varepsilon_h^\varphi - \widehat{\varepsilon}_h^u), \mu\rangle_\Gamma = \langle \tau_{\mathrm{B}}(\mathrm{I}^\Gamma\varphi - \varphi), \mu\rangle_\Gamma$$

$$\begin{aligned}\mathcal{A}\big((\varepsilon_h^\lambda, \varepsilon_h^\varphi), (\eta, \psi)\big) + \langle \widehat{\varepsilon}_h^u, \eta\rangle_\Gamma + \langle \tau_{\mathrm{B}}(\varepsilon_h^\varphi - \widehat{\varepsilon}_h^u), \psi\rangle_\Gamma &= \mathcal{A}\big((\mathrm{P}^\Gamma\lambda - \lambda, \mathrm{I}^\Gamma\varphi - \varphi), (\eta, \psi)\big) \\ &\quad + \langle \tau_{\mathrm{B}}(\mathrm{I}^\Gamma\varphi - \varphi), \psi\rangle_\Gamma\end{aligned} \tag{4.4d}$$

for all $(\mathbf{r}, w, \mu, \eta, \psi) \in \mathbf{V}_h \times W_h \times M_h \times X_h \times Y_h$.

Proof. These equations are a direct consequence of the definition of the projections and of the equations for the HDG-BEM scheme. ∎

PROPOSITION 4.3 (energy estimate). *We have the generic error estimate*

$$\|\boldsymbol{\varepsilon}_h^q\|_\Omega+\|\varepsilon_h^\lambda\|_{-1/2,\Gamma}+\|\varepsilon_h^\varphi\|_{1/2,\Gamma}+\|\tau^{1/2}(\varepsilon_h^u-\widehat{\varepsilon}_h^u)\|_{L^2(\partial\mathcal{T}_h)}+\|\tau_{\mathrm{B}}^{1/2}(\varepsilon_h^\varphi-\widehat{\varepsilon}_h^u)\|_{0,\Gamma}\lesssim \mathrm{App}_h,$$

where

$$\mathrm{App}_h:=\|\Pi\mathbf{q}-\mathbf{q}\|_\Omega+\|\mathrm{P}^\Gamma\lambda-\lambda\|_{-1/2,\Gamma}+\|\mathrm{I}^\Gamma\varphi-\varphi\|_{1/2,\Gamma}+\|\tau_{\mathrm{B}}^{1/2}(\mathrm{I}^\Gamma\varphi-\varphi)\|_{0,\Gamma}.$$

Proof. Testing the error equations (4.4) with $\boldsymbol{\varepsilon}_h^q,\varepsilon_h^u,-\widehat{\varepsilon}_h^u$, and $(\varepsilon_h^\lambda,\varepsilon_h^\varphi)$, respectively, and adding them we obtain that

$$\begin{aligned}\|\boldsymbol{\varepsilon}_h^q\|_\Omega^2&+\langle\mathcal{V}\varepsilon_h^\lambda,\varepsilon_h^\lambda\rangle_\Gamma+\langle\mathcal{W}\varepsilon_h^\varphi,\varepsilon_h^\varphi\rangle_\Gamma+\langle\tau(\varepsilon_h^u-\widehat{\varepsilon}_h^u),\varepsilon_h^u-\widehat{\varepsilon}_h^u\rangle_{\partial\mathcal{T}_h}+\langle\tau_{\mathrm{B}}(\varepsilon_h^\varphi-\widehat{\varepsilon}_h^u),\varepsilon_h^\varphi-\widehat{\varepsilon}_h^u\rangle_\Gamma\\&=(\Pi\mathbf{q}-\mathbf{q},\boldsymbol{\varepsilon}_h^q)_{\mathcal{T}_h}+\mathcal{A}\big((\mathrm{P}^\Gamma\lambda-\lambda,\mathrm{I}^\Gamma\varphi-\varphi),(\varepsilon_h^\lambda,\varepsilon_h^\varphi)\big)+\langle\tau_{\mathrm{B}}(\mathrm{I}^\Gamma\varphi-\varphi),\varepsilon_h^\varphi-\widehat{\varepsilon}_h^u\rangle_\Gamma.\end{aligned}$$

Using now the boundedness of $\mathcal{A}$ and the ellipticity of $\mathcal{V}$ and $\mathcal{W}$, that is, (3.9) and (3.10), the result follows readily. ∎

Using [8, Theorem 2.1] and the boundary approximation results collected in section 5, we can bound

$$\begin{aligned}\mathrm{App}_h\lesssim h^{k+1}|\mathbf{q}|_{k+1,\Omega}&+h^{k+1}\tau_{\max}|u|_{k+1,\Omega}\\&+h^{k+1+s}\Big(\|\lambda\|_{X^{k+1/2+s}(\Gamma)}+(1+T_{\max}^{1/2}h^{1/2})\|\varphi\|_{X^{k+3/2+s}(\Gamma)}\Big),\qquad s\in[0,1/2].\end{aligned}$$

Other estimates. Recall the notation $\underline{k}:=\min\{k,1\}$, which we are going to use to separate the case $k\geq 1$ (when there is superconvergence) from the case $k=0$ (when there is not). The principal estimate by duality is the one of ε_h^u.

PROPOSITION 4.4.

$$\|\varepsilon_h^u\|_\Omega\lesssim h^{\underline{k}}\Big(h^{1/2}\|\mathrm{P}^\Gamma\lambda-\lambda\|_{0,\Gamma}+\mathrm{App}_h\times\big(1+\tau_{\max}h+T_{\max}^{1/2}h^{1/2}+T_{\max}h\big)\Big).$$

Proof. Consider the solution to the dual problem (3.12). Testing the error equations (4.4) with $\Pi\mathbf{d}$, $-\Pi\Theta$, $\mathrm{P}^\partial\Theta$, and $(\mathrm{P}^\Gamma\xi,-\alpha_h)$, respectively, we obtain

$$(\mathbf{q}-\mathbf{q}_h,\Pi\mathbf{d})_{\mathcal{T}_h}-(\varepsilon_h^u,\nabla\cdot\Pi\mathbf{d})_{\mathcal{T}_h}+\langle\widehat{\varepsilon}_h^u,(\Pi\mathbf{d})\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}=0,\tag{4.5a}$$

$$-(\nabla\cdot\boldsymbol{\varepsilon}_h^q,\Theta)_{\mathcal{T}_h}-\langle\tau(\varepsilon_h^u-\widehat{\varepsilon}_h^u),\Pi\Theta\rangle_{\partial\mathcal{T}_h}=0,\tag{4.5b}$$

$$\langle\tau(\varepsilon_h^u-\widehat{\varepsilon}_h^u)+\boldsymbol{\varepsilon}_h^q\cdot\mathbf{n},\Theta\rangle_{\partial\mathcal{T}_h}+\langle\varepsilon_h^\lambda,\Theta\rangle_\Gamma+\langle\tau_{\mathrm{B}}(\varphi-\varphi_h-\widehat{\varepsilon}_h^u),\mathrm{P}^\partial\Theta\rangle_\Gamma=0,\tag{4.5c}$$

$$\begin{aligned}\langle\widehat{\varepsilon}_h^u,-\mathbf{d}\cdot\mathbf{n}\rangle_\Gamma&+\mathcal{A}\big((\lambda-\lambda_h,\varphi-\varphi_h),(\mathrm{P}^\Gamma\xi,-\alpha_h)\big)\\&-\langle\tau_{\mathrm{B}}(\varphi-\varphi_h-\widehat{\varepsilon}_h^u),\alpha_h\rangle_\Gamma=0.\end{aligned}\tag{4.5d}$$

Note that we have already applied some simplifications given by the definitions of the operators and the relations between the variables and that, as in Proposition 3.5, we momentarily consider any $\alpha_h\in Y_h$. By (3.17) we also have

$$\mathcal{A}((\lambda-\lambda_h,\varphi-\varphi_h),(-\xi,\alpha))-\langle\lambda-\lambda_h,\Theta\rangle_\Gamma=0.\tag{4.6}$$

In addition,

$$\begin{aligned}-(\varepsilon_h^u,\nabla\cdot\Pi\mathbf{d})_{\mathcal{T}_h}&=(\nabla\varepsilon_h^u,\Pi\mathbf{d})_{\mathcal{T}_h}-\langle\varepsilon_h^u,(\Pi\mathbf{d})\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\&=(\nabla\varepsilon_h^u,\mathbf{d})_{\mathcal{T}_h}-\langle\varepsilon_h^u,(\Pi\mathbf{d})\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\&=-\|\varepsilon_h^u\|_\Omega^2-\langle\varepsilon_h^u,(\Pi\mathbf{d}-\mathbf{d})\cdot\mathbf{n})\rangle_{\partial\mathcal{T}_h}\end{aligned}$$

and therefore (4.5a) is equivalent to

$$(\mathbf{q}-\mathbf{q}_h,\Pi\mathbf{d})_{\mathcal{T}_h}-\langle\varepsilon_h^u-\widehat{\varepsilon}_h^u,(\Pi\mathbf{d}-\mathbf{d})\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}+\langle\widehat{\varepsilon}_h^u,\mathbf{d}\cdot\mathbf{n}\rangle_\Gamma=\|\varepsilon_h^u\|_\Omega^2, \tag{4.7}$$

where we have applied that the normal component of $\mathbf{d}$ is continuous across element interfaces and that $\widehat{\varepsilon}_h^u$ is single-valued on interior faces. Note also that by (4.3c)

$$-\langle\varepsilon_h^u-\widehat{\varepsilon}_h^u,(\Pi\mathbf{d}-\mathbf{d})\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}-\langle\tau(\varepsilon_h^u-\widehat{\varepsilon}_h^u),\Pi\Theta\rangle_{\partial\mathcal{T}_h}+\langle\tau(\varepsilon_h^u-\widehat{\varepsilon}_h^u),\Theta\rangle_{\partial\mathcal{T}_h}=0.$$

Integration by parts yields

$$-(\nabla\cdot\boldsymbol{\varepsilon}_h^q,\Theta)_{\mathcal{T}_h}+\langle\boldsymbol{\varepsilon}_h^q\cdot\mathbf{n},\Theta\rangle_{\partial\mathcal{T}_h}=-(\boldsymbol{\varepsilon}_h^q,\mathbf{d})_{\mathcal{T}_h}.$$

Adding (4.7), (4.5b), (4.5c), (4.5d), and (4.6) and using these two last identities to simplify, we obtain

$$\begin{aligned}\|\varepsilon_h^u\|_\Omega^2=&\,(\mathbf{q}-\mathbf{q}_h,\Pi\mathbf{d})_{\mathcal{T}_h}-(\boldsymbol{\varepsilon}_h^q,\mathbf{d})_{\mathcal{T}_h}\\&+\langle\mathrm{P}^\Gamma\lambda-\lambda,\Theta\rangle_\Gamma+\mathcal{A}((\lambda-\lambda_h,\varphi-\varphi_h),(\mathrm{P}^\Gamma\xi-\xi,\alpha-\alpha_h))\\&+\langle\tau_{\mathrm{B}}(\varphi-\varphi_h-\widehat{\varepsilon}_h^u),\mathrm{P}^\Gamma\alpha-\alpha_h\rangle_\Gamma.\end{aligned}$$

Let us write P_{k-1} to denote the $L^2(\Omega)$ orthogonal projection onto the space of discontinuous piecewise polynomial functions of degree $k-1$ if $k\geq 1$ with P_{-1} equal to the null operator. To end with this chain of identities, we just note that

$$\begin{aligned}(\mathbf{q}-\mathbf{q}_h,\Pi\mathbf{d})_{\mathcal{T}_h}-(\boldsymbol{\varepsilon}_h^q,\mathbf{d})_{\mathcal{T}_h}&=(\mathbf{q}-\mathbf{q}_h,\Pi\mathbf{d}-\mathbf{d})_{\mathcal{T}_h}+(\mathbf{q}-\Pi\mathbf{q},\mathbf{d}-\mathrm{P}_{k-1}\mathbf{d})_{\mathcal{T}_h},\\ \langle\mathrm{P}^\Gamma\lambda-\lambda,\Theta\rangle_\Gamma&=\langle\mathrm{P}^\Gamma\lambda-\lambda,\alpha-\mathrm{P}^\Gamma\alpha\rangle_\Gamma,\\ \langle\tau_{\mathrm{B}}(\varphi-\varphi_h-\widehat{\varepsilon}_h^u),\mathrm{P}^\Gamma\alpha-\alpha_h\rangle_\Gamma&=\langle\tau_{\mathrm{B}}(\varphi-\mathrm{I}^\Gamma\varphi),\mathrm{P}^\Gamma\alpha-\alpha_h\rangle_\Gamma+\langle\varepsilon_h^\varphi,\tau_{\mathrm{B}}(\mathrm{P}^\Gamma\alpha-\alpha)\rangle_\Gamma\\&\quad+\langle\tau_{\mathrm{B}}(\varepsilon_h^\varphi-\widehat{\varepsilon}_h^u),\alpha-\alpha_h\rangle_\Gamma.\end{aligned}$$

At this point we choose $\alpha_h:=\mathrm{C}^\Gamma\Theta$ (see section 5). Inserting these equalities in the expression for $\|\varepsilon_h^u\|_\Omega^2$ and applying Proposition 4.3 and the bound (3.19), we obtain

$$\begin{aligned}\|\varepsilon_h^u\|_\Omega^2\lesssim&\,\|\mathrm{P}^\Gamma\lambda-\lambda\|_{0,\Gamma}\|\mathrm{P}^\Gamma\alpha-\alpha\|_{0,\Gamma}\\&+\mathrm{App}_h\Big(\|\Pi\mathbf{d}-\mathbf{d}\|_\Omega+\|\mathrm{P}_{k-1}\mathbf{d}-\mathbf{d}\|_\Omega+h\|\varepsilon_h^u\|_\Omega\\&+\|\tau_{\mathrm{B}}^{1/2}(\mathrm{P}^\Gamma\alpha-\alpha)\|_{0,\Gamma}+\|\tau_{\mathrm{B}}(\alpha-\alpha_h)\|_{0,\Gamma}+\|\tau_{\mathrm{B}}(\mathrm{P}^\Gamma\alpha-\alpha)\|_{-1/2,\Gamma}\Big).\end{aligned} \tag{4.8}$$

This is a list of bounds we next apply:

$$\begin{aligned}\|\Pi\mathbf{d}-\mathbf{d}\|&\lesssim h\|\mathbf{d}\|_{1,\Omega}+\tau_{\max}h^2\|\Theta\|_{2,\Omega}\lesssim h(1+\tau_{\max}h)\|\varepsilon_h^u\|_\Omega,\\ \|\mathrm{P}_{k-1}\mathbf{d}-\mathbf{d}\|_\Omega&\lesssim h^{\underline{k}}\|\mathbf{d}\|_{\underline{k}}\lesssim h^{\underline{k}}\|\varepsilon_h^u\|_\Omega,\\ \|\mathrm{P}^\Gamma\alpha-\alpha\|_{0,\Gamma}&\lesssim h^{\underline{k}+1/2}\|\alpha\|_{X^{\underline{k}+1/2}(\Gamma)}\lesssim h^{\underline{k}+1/2}\|\Theta\|_{\underline{k}+1,\Omega}\lesssim h^{\underline{k}+1/2}\|\varepsilon_h^u\|_\Omega,\\ \|\tau_{\mathrm{B}}^{1/2}(\mathrm{P}^\Gamma\alpha-\alpha)\|_{0,\Gamma}&\lesssim T_{\max}^{1/2}h^{\underline{k}+1/2}\|\varepsilon_h^u\|_\Omega,\\ \|\tau_{\mathrm{B}}(\mathrm{P}^\Gamma\alpha-\alpha)\|_{-1/2,\Gamma}&=\|\mathrm{P}^\Gamma(\tau_{\mathrm{B}}\alpha)-\tau_{\mathrm{B}}\alpha\|_{-1/2,\Gamma}\lesssim h^{1/2}\|\mathrm{P}^\Gamma(\tau_{\mathrm{B}}\alpha)-\tau_{\mathrm{B}}\alpha\|_{0,\Gamma}\\&\lesssim T_{\max}h^{\underline{k}+1}\|\varepsilon_h^u\|_\Omega,\\ \|\tau_{\mathrm{B}}^{1/2}(\alpha-\alpha_h)\|_{0,\Gamma}&\lesssim T_{\max}^{1/2}h^{3/2}\|\varepsilon_h^u\|_\Omega.\end{aligned}$$

The first bound appears in [8, Theorem 2.1] and the last one is given in Proposition 5.2. Direct insertion of this list of approximation properties in (4.8) proves the result. ∎

PROPOSITION 4.5. *For* $k \geq 1$,

$$\|\widehat{\varepsilon}_h^u\|_h \lesssim h\|\mathbf{q} - \mathbf{q}_h\|_\Omega + \|\varepsilon_h^u\|_\Omega,$$

whereas for $k = 0$, *assuming that* $\tau \gtrsim \mathfrak{h}$,

$$\|\widehat{\varepsilon}_h^u\|_h \lesssim \mathrm{App}_h + \|\varepsilon_h^u\|_\Omega.$$

Proof. Note that the superconvergence estimate for $\|\widehat{\varepsilon}_h^u\|_h$ of Proposition 3.7 is still valid for $k \geq 1$ because the structure of the first error equation for the HDG-BEM coupling is the same as that for the RT-BEM coupling. One, however, needs to use the degrees of freedom of the BDM space [3, 4] in the proof. When $k = 0$ and assuming that $\tau \gtrsim \mathfrak{h}$, we can easily show that

$$\|\widehat{\varepsilon}_h^u\|_h \leq \|\widehat{\varepsilon}_h^u - \varepsilon_h^u\|_h + \|\varepsilon_h^u\|_h \lesssim \|\tau^{1/2}(\widehat{\varepsilon}_h^u - \varepsilon_h^u)\|_{L^2(\partial\mathcal{T}_h)} + \|\varepsilon_h^u\|_\Omega.$$

The result then follows from Proposition 4.3. ∎

PROPOSITION 4.6. *If* $T_{\max} \lesssim h_\Gamma$, *then*

$$\|\lambda - \lambda_h\|_{-1,\Gamma} + \|\varphi - \varphi_h\|_{0,\Gamma} \lesssim h_\Gamma^{1/2}\,\mathrm{App}_h + h_{\min,\Gamma}^{-1/2}\|\widehat{\varepsilon}_h^u\|_h.$$

Proof. Using the same argument as in the proof of Proposition 3.8, we only need to bound

$$\sup_{\mathbf{0}\neq(\eta,\psi)\in H^0(\Gamma)\times H^1(\Gamma)} \frac{\mathcal{A}((\lambda-\lambda_h,\varphi-\varphi_h),(\mathrm{P}^\Gamma\eta,\mathrm{Q}^\Gamma\psi))}{\|\eta\|_{0,\Gamma}+\|\psi\|_{1,\Gamma}},$$

$\mathrm{Q}^\Gamma : H^{1/2}(\Gamma) \to Y_h$ being the best approximation operator on Y_h. The error equation (4.4d) shows that

$$\begin{aligned}\mathcal{A}((\lambda-\lambda_h,\varphi-\varphi_h),(\mathrm{P}^\Gamma\eta,\mathrm{Q}^\Gamma\psi)) &= -\langle\widehat{\varepsilon}_h^u,\mathrm{P}^\Gamma\eta\rangle_\Gamma + \langle\tau_{\mathrm{B}}(\mathrm{I}^\Gamma\varphi-\varphi),\mathrm{Q}^\Gamma\psi\rangle_\Gamma \\ &\quad - \langle\tau_{\mathrm{B}}(\varepsilon_h^\varphi - \widehat{\varepsilon}_h^u),\mathrm{Q}^\Gamma\psi\rangle_\Gamma \\ &\lesssim h_{\min,\Gamma}^{-1/2}\|\widehat{\varepsilon}_h^u\|_h\|\eta\|_{0,\Gamma} + T_{\max}\|\mathrm{I}^\Gamma\varphi-\varphi\|_{0,\Gamma}\|\psi\|_{1,\Gamma} \\ &\quad + T_{\max}^{1/2}\|\tau_{\mathrm{B}}(\varepsilon_h^\varphi-\widehat{\varepsilon}_h^u)\|_{0,\Gamma}\|\psi\|_{1,\Gamma},\end{aligned}$$

where we have bounded $\|\mathrm{Q}^\Gamma\psi\|_{0,\Gamma} \leq \|\mathrm{Q}^\Gamma\psi\|_{1/2,\Gamma} \leq \|\psi\|_{1/2,\Gamma} \leq \|\psi\|_{1,\Gamma}$. The result is now a simple consequence of Proposition 4.3. ∎

As in the RT-BEM coupling, if $\mathcal{T}_h$ is quasi-uniform near Γ and the largest elements of $\mathcal{T}_h$ are located near Γ, then the above estimate gives $h^{1/2}$ superconvergence. Note that in this case, the hypothesis on $T_{\max}$ can be equivalently written as $\tau_{\mathrm{B}} \lesssim \mathfrak{h}_\Gamma$.

The energy estimate of Proposition 4.3 can be used in a straightforward way to obtain a bound

$$\|\tau_{\mathrm{B}}^{1/2}\varepsilon_h^\varphi\|_{0,\Gamma} \lesssim \mathrm{App}_h + h_{\min,\Gamma}^{-1/2}T_{\max}^{1/2}\|\widehat{\varepsilon}_h^u\|_h.$$

The rest of this section will be devoted to obtaining bounds for the approximation of λ and φ when the penalization parameter τ_{B} is present and large. In particular, we are going to assume that

$$\tau_{\mathrm{B}} \approx \mathfrak{h}_\Gamma^{-1} \qquad \text{and} \qquad h \approx h_{\min,\Gamma}. \tag{4.9}$$

This means that the boundary grid Γ_h is quasi-uniform, that the largest elements of $\mathcal{T}_h$ are near Γ (the grid $\mathcal{T}_h$ can contain much smaller elements inside), and that we penalize the difference between $\widehat{u}_h$ (which is discontinuous) and φ_h (which is continuous) as h^{-1}. Recall that our interest lies in finding weaker norm estimates for the boundary variables (which are used in the computation of the approximate exterior solution) as well as in a strong functional estimate for $\lambda - \lambda_h$.

PROPOSITION 4.7. *If* (4.9) *holds, then*

$$\|\varepsilon_h^\varphi\|_{0,\Gamma} \lesssim h^{1/2}\mathrm{App}_h + h^{-1/2}\|\widehat{\varepsilon}_h^u\|_h.$$

Proof. It is a direct consequence of Proposition 4.3. □

PROPOSITION 4.8. *For all choices of* τ_{B} *and all grids, we can bound*

$$\|\lambda - \lambda_h\|_{-1,\Gamma} \lesssim h^{1/2}\|\lambda - \lambda_h\|_{-1/2,\Gamma} + \|\varphi - \varphi_h\|_{0,\Gamma} + h_{\min,\Gamma}^{-1/2}\|\widehat{\varepsilon}_h^u\|_h. \tag{4.10}$$

Therefore, if (4.9) *holds,*

$$\|\lambda - \lambda_h\|_{-1,\Gamma} \lesssim h^{1/2}\mathrm{App}_h + \|\varphi - \mathrm{I}^\Gamma\varphi\|_{0,\Gamma} + h^{-1/2}\|\widehat{\varepsilon}_h^u\|_h. \tag{4.11}$$

Proof. Because $\mathcal{V} : H^0(\Gamma) \to H^1(\Gamma)$ is an isomorphism and its adjoint $\mathcal{V}' : H^{-1}(\Gamma) \to H^0(\Gamma)$ is just the extension of $\mathcal{V}$ to $H^{-1}(\Gamma)$, we can prove (denoting $e_h^\lambda := \lambda - \lambda_h$ for simplicity) that

$$\begin{aligned}\|e_h^\lambda\|_{-1,\Gamma} &\lesssim \sup_{0\neq\eta\in H^0(\Gamma)} \frac{\langle \mathcal{V}e_h^\lambda, \eta\rangle_\Gamma}{\|\eta\|_{0,\Gamma}} \\ &\lesssim \sup_{0\neq\eta\in H^0(\Gamma)} \frac{\langle \mathcal{V}e_h^\lambda, \mathrm{P}^\Gamma\eta\rangle_\Gamma}{\|\eta\|_{0,\Gamma}} + \|e_h^\lambda\|_{-1/2,\Gamma} \sup_{0\neq\eta\in H^0(\Gamma)} \frac{\|\eta - \mathrm{P}^\Gamma\eta\|_{-1/2,\Gamma}}{\|\eta\|_{0,\Gamma}}.\end{aligned}$$

The last term is bounded using (5.1) (with $t = -1/2$ and $s = 0$). To bound the first term, we use the error equation (4.4d) (with test function $(\mathrm{P}^\Gamma\eta, 0)$) and see that

$$\langle \mathcal{V}e_h^\lambda, \mathrm{P}^\Gamma\eta\rangle_\Gamma = \langle \widetilde{\mathcal{K}}e_h^\varphi, \mathrm{P}^\Gamma\eta\rangle_\Gamma - \langle \widehat{\varepsilon}_h^u, \mathrm{P}^\Gamma\eta\rangle_\Gamma \lesssim \Big(\|e_h^\varphi\|_{0,\Gamma} + \|\widehat{\varepsilon}_h^u\|_{0,\Gamma}\Big)\|\eta\|_{0,\Gamma}.$$

(We have written $e_h^\varphi := \varphi - \varphi_h$.) Inequality (4.10) is a direct consequence of this argument. Finally, (4.11) follows from (4.10), Proposition 4.7, and the hypothesis (4.9). □

Since (4.9) implies that the boundary mesh Γ_h is quasi-uniform, we can use the inverse inequality (3.21) and obtain an $H^0(\Gamma)$ estimate for the approximation of λ. Using the error equation (4.4c) we have the explicit expression

$$\varepsilon_h^\lambda = \boldsymbol{\varepsilon}_h^q \cdot \mathbf{n} + \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u) + \tau_{\mathrm{B}}\mathrm{P}^\Gamma(\varphi - \varphi_h - \widehat{\varepsilon}_h^u).$$

A simple computation then proves that

$$\begin{aligned}\|\mathfrak{h}^{1/2}\varepsilon_h^\lambda\|_{0,\Gamma} &\lesssim \|\boldsymbol{\varepsilon}_h^q\|_\Omega + h_\Gamma^{1/2}\tau_{\max}^{1/2}\|\tau^{1/2}(\varepsilon_h^u - \widehat{\varepsilon}_h^u)\|_{L^2(\partial\mathcal{T}_h)} \\ &\qquad + h^{1/2}T_{\max}^{1/2}\Big(\|\tau_{\mathrm{B}}^{1/2}(\varphi - \mathrm{I}^\Gamma\varphi)\|_{0,\Gamma} + \|\tau_{\mathrm{B}}^{1/2}(\varepsilon_h^\varphi - \widehat{\varepsilon}_h^u)\|_{0,\Gamma}\Big) \\ &\lesssim \mathrm{App}_h\,(1 + h^{1/2}\tau_{\max}^{1/2} + h^{1/2}T_{\max}^{1/2}) + h^{1/2}T_{\max}\|\varphi - \mathrm{I}^\Gamma\varphi\|_{0,\Gamma},\end{aligned}$$

which, if (4.9) holds, gives a similar error estimate to the one provided by the inverse inequality (3.21).

The error estimates of sections 3 and 4, in the case of highest regularity, are gathered in Table 4.1.

Table 4.1

Error estimates for RT-BEM and HDG-BEM in the case of highest regularity. All the estimates assume that $0 < \tau \lesssim 1$ and that $0 \le \tau_{\rm B} \lesssim \mathfrak{h}_\Gamma^{-1}$. We have denoted $k^* = k + \min\{k-1, 0\}$.

		RT	HDG	Hypotheses: (A) $h \approx h_{\min,\Gamma}$ (B) $\tau_{\rm B} \lesssim \mathfrak{h}_\Gamma$ or $\tau_{\rm B} \approx \mathfrak{h}_\Gamma^{-1}$ (C) $\tau \gtrsim \mathfrak{h}$ if $k=0$
$\mathbf{q}-\mathbf{q}_h$	$(L^2(\Omega))^d$	$k+1$	$k+1$	
$u-u_h$	$L^2(\Omega)$	$k+1$	$k+1$	
ε_h^u	$L^2(\Omega)$	$k+2$	k^*+2	
$\lambda-\lambda_h$	$H^{-1/2}(\Gamma)$	$k+1$	$k+1$	
	$H^{-1}(\Gamma)$	$k+3/2$	$k^*+3/2$	(A), (B), and (C)
	$L^2(\Gamma)$	$k+1/2$	$k+1/2$	(A)
$\varphi-\varphi_h$	$H^{1/2}(\Gamma)$	$k+1$	$k+1$	
	$L^2(\Gamma)$	$k+3/2$	$k^*+3/2$	(A), (B), and (C)
$\widehat{\varepsilon}_h^u$	$L^2(\partial T_h)$	$k+2$	k^*+2	(C)

5. Some technicalities on the boundary. The polygonal (resp., polyhedral) boundary Γ is composed by a finite set of edges (resp., faces), which we list as $\{\Gamma_1, \ldots, \Gamma_m\}$. Sobolev spaces $H^s(\Gamma)$ are univocally defined for any $s \in [-1,1]$. Beyond these limits, there are several possible extensions. Sobolev spaces $H^s(\Gamma_j)$ on the straight edges (resp., flat faces) can be defined in a natural way for any $s \ge 0$: for instance, for s positive integer by mapping Γ_j from an $(d-1)$-dimensional domain and for intermediate values by interpolation. We then consider the spaces

$$X^s(\Gamma) := \prod_{j=1}^m H^s(\Gamma_j),$$

endowed with their product norms. It is simple to prove that the trace operator $H^m(\Omega) \to X^{m-1/2}(\Gamma)$ is bounded for all $m \ge 1$. Also, the normal derivative operator $\partial_\nu : H^m(\Omega) \to X^{m-3/2}(\Gamma)$ is bounded for any $m \ge 2$. Note that neither of these operators is due to be surjective. Recognizing the range of these operators requires a level of technical detail that is not needed for this paper (see [14] for an introduction). Note that for $s \ge d-1$, $X^s(\Gamma) \subset \prod_{j=1}^m \mathcal{C}(\Gamma_j)$, by the Sobolev embedding theorem. However, for the trace of sufficiently smooth functions, we actually go to $\mathcal{C}(\Gamma) \cap X^s(\Gamma)$. This is not the case for normal derivatives, since the normal vector field is discontinuous at corner points (resp., edges)

Let $\mathrm{P}^\Gamma : L^2(\Gamma) \to X_h$ be the orthogonal projection, $\mathrm{I}^\Gamma : \mathcal{C}(\Gamma) \to Y_h$ be the Lagrange interpolation operator, and $\mathrm{Q}^\Gamma : H^{1/2}(\Gamma) \to Y_h$ be the orthogonal projection onto Y_h.

Proposition 5.1. *For $-1 \le t \le 0 \le s \le k+1$*

$$\|\mathrm{P}^\Gamma\lambda - \lambda\|_{t,\Gamma} \lesssim h^{s-t}\|\lambda\|_{X^s(\Gamma)} \qquad \forall \lambda \in X^s(\Gamma). \tag{5.1}$$

For $d-1 \le s \le k+2$

$$\|\mathrm{I}^\Gamma\varphi - \varphi\|_{0,\Gamma} + h^{1/2}\|\mathrm{I}^\Gamma\varphi - \varphi\|_{1/2,\Gamma} \lesssim h^s\|\varphi\|_{X^s(\Gamma)} \qquad \forall \varphi \in X^s(\Gamma) \cap \mathcal{C}(\Gamma). \tag{5.2}$$

Finally

$$\|\mathrm{Q}^\Gamma\varphi - \varphi\|_{1/2,\Gamma} \lesssim h^{1/2}\|\varphi\|_{1,\Gamma} \qquad \forall \varphi \in H^1(\Gamma). \tag{5.3}$$

Proof. In a first step we can prove that

$$\|\mathrm{P}^\Gamma\lambda - \lambda\|_{t,\Gamma} \lesssim h^{-t}\|\lambda\|_{0,\Gamma}, \qquad -1 \le t \le 0,$$

using a duality argument, classical properties of the L^2 projection onto spaces of piecewise polynomials (to prove that case $t = -1$) and an interpolation argument. The remainder of the proof for (5.1) and (5.2) is based on the fact that P^Γ and I^Γ are operators that are defined element by element and that the local approximation properties for these operators are well understood. The result for Q^Γ is also classical. ∎

Consider the following approximation of the trace operator defined from $H^1(\Omega)$ to Y_h. Given $u \in H^1(\Omega)$, let $C_h u$ be the Clément approximation on the space of continuous piecewise $\mathbb{P}_{k+1}$ polynomial functions on the mesh $\mathcal{T}_h$ (see [6]). Then $\mathrm{C}^\Gamma u := (C_h u)|_\Gamma$.

PROPOSITION 5.2. *For all* $u \in H^2(\Omega)$,

$$\|\mathrm{C}^\Gamma u - u\|_{1/2,\Gamma} + h^{-1/2}\|\mathrm{C}^\Gamma u - u\|_{0,\Gamma} \lesssim h\|u\|_{2,\Omega}.$$

Proof. The $H^{1/2}(\Gamma)$ bound follows from the trace theorem and the approximation properties of the Clément operator C_h. To prove the $L^2(\Gamma)$ bound we act in an element-by-element fashion using the local version of the trace inequality

$$\sum_{e\in\Gamma_h}\|\mathrm{C}^\Gamma u - u\|_{0,e}^2 \lesssim \sum_{K\in\mathcal{T}_h}\Big(h^{-1}\|C_h u - u\|_{0,K}^2 + h|C_h u - u|_{1,K}^2\Big) \lesssim h^3\|u\|_{2,\Omega}^2. \qquad ∎$$

6. Modifications for the exterior Laplace equation. In this section we address the modifications needed to adapt the method for the case of the exterior Laplace operator, which is of more physical and practical interest. As will be explained below, the first novelty is the occurrence of energy-free solutions (constant functions), which create a one-dimensional kernel in two of the relevant integral operators. Having or not a stabilization term τ_{B} will be important in our approach to the HDG-BEM method. The second novelty happens only in the two-dimensional case and is related to the unboundedness at infinity of the fundamental solution of the Laplace operator.

In the model problem (2.1) we substitute (2.1c) by the Laplace equation plus a radiation condition at infinity:

$$-\Delta v = 0 \quad \text{in } \Omega_+, \qquad v = \mathcal{O}(1/r) \quad \text{at infinity}.$$

The last condition holds uniformly in all directions. For the exterior Yukawa equation (that is, the original (2.1c)), usually the radiation condition is not written, because it can be simply substituted by a finite energy condition $\|v\|_{1,\Omega_+} < \infty$. In the case of the Laplace operator, $v \notin L^2(\Omega_+)$ and the consideration of energy conditions requires the use of weighted Sobolev spaces. The fundamental solution is changed to

$$E(\mathbf{x},\mathbf{y}) := \begin{cases} -\dfrac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}| & \text{if } d = 2, \\ \dfrac{1}{4\pi|\mathbf{x}-\mathbf{y}|} & \text{if } d = 3, \end{cases}$$

and all the layer operators and integral equations are consequently redefined. The integral identities and the representation theorem are still valid.

The first novelty is related to the existence of nontrivial (constant) solutions for the interior Neumann problem. They affect some of the integral operators and now we have

$$\ker \mathcal{W} = \ker \widetilde{\mathcal{K}} = \mathbb{P}_0(\Gamma) \tag{6.1}$$

as well as vanishing layer potentials with nonzero densities:

$$\mathcal{D}1 \equiv 0 \qquad \text{in } \Omega_+. \tag{6.2}$$

Part of the ellipticity of $\mathcal{W}$ is lost. The operator $\mathcal{W}$ is now elliptic in

$$H_0^{1/2}(\Gamma) := \left\{ \varphi \in H^{1/2}(\Gamma) \,:\, \int_\Gamma \varphi = 0 \right\}.$$

This new ellipticity property can also be written as

$$\langle \mathcal{W}\varphi, \varphi\rangle_\Gamma \geq C\|\varphi - \mathcal{J}\varphi\|^2_{1/2,\Gamma} =: |\varphi|^2_{1/2,\Gamma} \qquad \forall \varphi \in H^{1/2}(\Gamma),$$

where $\mathcal{J}\varphi := \frac{1}{\text{meas}(\Gamma)} \int_\Gamma \varphi$ is the averaging operator.

6.1. The three-dimensional case. There are several possible adaptations of the method (4.1). The simplest solution requires that the nonnegative stabilization parameter $\tau_{\rm B}$ does not vanish identically, so $\int_\Gamma \tau_{\rm B} > 0$. In this case, (4.1) can be taken as they come.

If $\tau_{\rm B} \equiv 0$, then addition of a constant to φ_h is not perceived by (4.1), which compensates the fact that (4.1e) with $\psi = 1$ is void by (6.1). In this case, we take

$$\varphi_h \in Y_h^0 := Y_h \cap H_0^{1/2}(\Gamma), \qquad \varphi_h \approx u - \mathcal{J}u \quad \text{on } \Gamma.$$

Having the exterior trace approximated up to a constant does not affect the representation formula (we would take $v_h := \mathcal{D}\varphi_h - \mathcal{S}\lambda_h \approx v$ as usual), because (6.2) holds. The restriction introduced in φ_h is compensated by testing (4.1e) with Y_h^0 too.

Both cases can be analyzed with exactly the same techniques that we have detailed in previous sections, with the additional need to start working with the seminorm $|\cdot|_{1/2,\Gamma}$ (see (6.2)) for the error in φ.

6.2. The two-dimensional case. Existence of decaying solutions to the transmission problems requires the compatibility condition

$$\int_\Omega f = 0. \tag{6.3}$$

If this hypothesis does not hold, there are solutions that are logarithmically increasing at infinity, but once we accept unbounded solutions, we are automatically introducing the kernel of constant solutions to the transmission problem. The compatibility condition (6.3) gives some a priori information for $\lambda = \partial_\nu v$, since

$$\int_\Gamma \lambda = -\int_\Omega \text{div}\, \mathbf{q} = -\int_\Omega f = 0.$$

Therefore λ is in the space

$$H_0^{-1/2}(\Gamma) := \{\lambda \in H^{-1/2}(\Gamma) \,:\, \langle \lambda, 1\rangle_\Gamma = 0\},$$

where the integral operator $\mathcal{V}$ is elliptic too.

We can again offer two options. If $\tau_{\rm B} \neq 0$, then we look for $\lambda_h \in X_h^0 := X_h \cap H_0^{-1/2}(\Gamma)$ and for $\varphi_h \in Y_h$, but we do not modify (4.1), that is, the test in (4.1d) is taken in the entire X_h. Actually, what happens in this situation is that (4.1b) with $w = 1$, (4.1c) with $\mu = 1$, and (4.1e) with $\psi = 1$ are linearly dependent

$$\begin{aligned} \langle \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h), 1\rangle_{\partial\mathcal{T}_h} &= (f,1)_\Omega = 0 \\ \langle \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h), 1\rangle_{\partial\mathcal{T}_h} + \langle \tau_{\rm B}(\varphi_h - \widehat{u}_h), 1\rangle_1 &= 0, \\ \langle \tau_{\rm B}(\varphi_h - \widehat{u}_h), 1\rangle_\Gamma &= 0 \end{aligned}$$

as a consequence of the data condition (6.3) and the kernel property (6.1).

If $\tau_{\rm B} \equiv 0$, then we look for $\lambda_h \in X_h^0$, $\varphi_h \in Y_h^0$ and test (4.1d) with the entire X_h and (4.1e) with the restricted space Y_h^0. The apparent excess of equations is again compensated by the fact that condition (6.3) implies that (4.1b) with $w = 1$ and (4.1c) with $\mu = 1$ are the same.

7. Numerical experiments. In this section we show the performance of the HDG-BEM coupling for a simple two-dimensional model problem:

$$\begin{aligned} \kappa^{-1}\mathbf{q} + \nabla u &= 0 && \text{in } \Omega, \\ \nabla \cdot \mathbf{q} &= f && \text{in } \Omega, \\ -\Delta v &= 0 && \text{in } \Omega_+, \\ u &= v + \beta_0 && \text{on } \Gamma, \\ -\mathbf{q}\cdot\mathbf{n} &= \partial_\nu v + \beta_1 && \text{on } \Gamma, \\ v &= \mathcal{O}(1/r) && \text{at infinity,} \end{aligned}$$

where κ^{-1} is a strictly positive bounded function (the inverse of the diffusivity parameter for an equation $\nabla \cdot (\kappa \nabla u) + f = 0$) and $\beta_0, \beta_1 : \Gamma \to \mathbb{R}$ are functions determining the jumps across the interface Γ. A necessary and sufficient condition for uniqueness of solution is the compatibility condition

$$\int_\Omega f + \int_\Gamma \beta_1 = 0. \tag{7.1}$$

The domain is the rectangle $\Omega = (0, \frac{3}{2}) \times (0, 1)$. The exterior solution is

$$v(\mathbf{x}) := 10^3 \log \frac{|\mathbf{x} - (0.5, 0.6)|}{|\mathbf{x} - (1, 0.4)|}.$$

We take $\kappa(\mathbf{x}) := (x+2)(y+2)$ and $u(\mathbf{x}) := \exp((x+2)(y+2))$. Data are defined so that the equations are satisfied. An exterior observation point $\mathbf{x}_\circ = (1.7, 0.8) \in \Omega_+$ is chosen for comparison of the discrete and continuous exterior solutions.

When $\tau_{\rm B} > 0$ we consider the discrete equations

$$\begin{aligned} (\kappa^{-1}\mathbf{q}_h, \mathbf{r})_{\mathcal{T}_h} - (u_h, \nabla\cdot\mathbf{r})_{\mathcal{T}_h} + \langle \widehat{u}_h, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} &= 0 && \forall \mathbf{r} \in \mathbf{V}_h, \\ (\nabla\cdot\mathbf{q}_h, w)_{\mathcal{T}_h} + \langle \tau(u_h - \widehat{u}_h), w\rangle_{\partial\mathcal{T}_h} &= (f, w)_{\mathcal{T}_h} && \forall w \in W_h, \\ \langle \tau(u_h - \widehat{u}_h) + \mathbf{q}_h\cdot\mathbf{n}, \mu\rangle_{\partial\mathcal{T}_h} + \langle \lambda_h + \tau_{\rm B}(\varphi_h - \widehat{u}_h), \mu\rangle_\Gamma &= \langle \beta_1 + \tau_{\rm B}\beta_0, \mu\rangle_\Gamma && \forall \mu \in M_h, \\ \langle \widehat{u}_h, \eta\rangle_\Gamma + \langle \mathcal{V}\lambda_h, \eta\rangle_\Gamma - \langle \widetilde{\mathcal{K}}\varphi_h, \eta\rangle_\Gamma &= \langle \beta_0, \eta\rangle_\Gamma && \forall \eta \in X_h, \\ \langle \widetilde{\mathcal{K}}^t\lambda_h, \psi\rangle_\Gamma + \langle \mathcal{W}\varphi_h, \psi\rangle_\Gamma + \langle \tau_{\rm B}(\varphi_h - \widehat{u}_h), \psi\rangle_\Gamma &= -\langle \tau_{\rm B}\beta_0, \psi_h\rangle_\Gamma && \forall \psi \in Y_h. \end{aligned}$$

The compatibility condition (7.1) proves then that $\lambda_h \in X_h^0$ (see a similar argument in section 6.2), which ensures that the reconstruction formula

$$v_h = \mathcal{D}\varphi_h - \mathcal{S}\lambda_h$$

yields a solution that decays at infinity. When $\tau_B \equiv 0$, we substitute the bilinear form associated to the hypersingular operator $\mathcal{W}$ by

$$\langle \mathcal{W}\varphi_h, \psi\rangle_\Gamma + \langle \varphi_h, 1\rangle_\Gamma \langle \psi, 1\rangle_\Gamma.$$

This change is enough to make the resulting equations uniquely solvable and provides $\varphi_h \in Y_h^0$. An approximation of the exterior trace can be achieved with the simple postprocessed formula

$$\varphi_h + \frac{1}{\text{meas}(\Gamma)} \int_\Gamma (\widehat{u}_h - \beta_0). \tag{7.2}$$

We will tabulate the following errors:

$$\mathrm{E}_q := \frac{\|\mathbf{q}_h - \mathbf{q}\|_\Omega}{\|\mathbf{q}\|_\Omega}, \qquad \mathrm{E}_u := \frac{\|u_h - u\|_\Omega}{\|u\|_\Omega},$$

$$\mathrm{E}_\lambda := \frac{\|\lambda_h - \partial_\nu v\|_\Gamma}{\|\partial_\nu v\|_\Gamma}, \qquad \mathrm{E}_\varphi := \frac{\|\varphi_h - v\|_\Gamma}{\|v\|_\Gamma} \qquad \mathrm{E}_{\text{ext}} := \frac{|v_h(\mathbf{x}_\circ) - v(\mathbf{x}_\circ)|}{|v(\mathbf{x}_\circ)|}.$$

When $\tau_B \equiv 0$, the correction (7.2) is applied before computing E_φ. (This correction does not affect the exterior solution.)

For discretization we use a first nonuniform grid with 227 elements, generated using MATLAB's PDE toolbox. This triangulation produces 35 elements on the boundary. We next apply four consecutive uniform (red) refinements to this grid. Computation of the three boundary element matrices (those corresponding to $\mathcal{V}$, $\mathcal{K}$, and $\mathcal{W}$) is done off-line. BEM matrices are stored and uploaded as needed. The BEM part of this simulation is a FORTRAN implementation due to Norbert Heuer (Pontificia Universidad Católica of Chile). The HDG code is a MATLAB implementation of the method by Zhixing Fu (University of Delaware) and the third author of this paper. It includes discretization of $L^2(\Gamma)$ inner products for the pairs $X_h \times Y_h$, $X_h \times M_h$, and $Y_h \times M_h$ and testing of the boundary functions β_0 and β_1 with the discrete spaces on the boundary. The assembly leads to a hybridized form of the HDG equations so that the entire coupled system is only solved in the variables $(\widehat{u}_h, \lambda_h, \varphi_h)$. The interior variables $(\mathbf{q}_h, u_h)$ are obtained using element-by-element postprocessing. In Tables 7.1 through 7.6, we show errors and convergence rates with different values of the polynomial degree k and of the boundary penalization parameter τ_B. As we can see from the tables, we observe the predicted rates of convergence for E_q and E_u for $k = 0, 1, 2$. However, for E_λ and E_φ the orders of convergence observed for the above example seems to be $k+1$ and $k+2$, respectively. This suggests that our error analysis is not sharp for the boundary variables. At the moment we do not see how to improve our error analysis.

Comparison with other methods. An RT-BEM implementation based on the formulation (3.1), after hybridization (so that $\mathbf{q}_h$ and u_h are eliminated from the system), has essentially the same cost as the tested HDG-BEM scheme: the size of the final system is the same and the assembly process for the hybridized matrix for the RT system is of the same order of complexity. (Some more degrees of freedom are needed

TABLE 7.1
Relative errors and estimated convergence rates (e.c.r.) for the interior fields ($\mathbf{q}$ and u) with $k = 0$, $\tau \equiv 1$, and $\tau_B \equiv 0$.

	E_q	e.c.r.	E_u	e.c.r
h_0	0.176748207	–	3.543197272	–
$h_0/2$	0.089324685	0.9846	1.789018396	0.9859
$h_0/4$	0.044785117	0.9960	0.897052870	0.9959
$h_0/8$	0.022406347	0.9991	0.448942399	0.9987
$h_0/16$	0.011204309	0.9999	0.224547589	0.9995

TABLE 7.2
Relative errors and estimated convergence rates for the boundary fields ($\lambda = \partial_\nu v$ and $\varphi = \gamma v$) and the exterior solution at an observation point with $k = 0$, $\tau \equiv 1$, and $\tau_B \equiv 0$.

	E_λ	e.c.r.	E_φ	e.c.r.	E_{ext}	e.c.r.
h_0	7.658254342	–	1.067412575	–	1.581717609	–
$h_0/2$	3.382540862	1.1789	0.345891065	1.6257	0.527078813	1.5854
$h_0/4$	1.206965012	1.4867	0.124774869	1.4710	0.198919302	1.4058
$h_0/8$	0.441832194	1.4498	0.051048247	1.2894	0.083134471	1.2587
$h_0/16$	0.164879759	1.4221	0.022929088	1.1547	0.037478016	1.1494

TABLE 7.3
Relative errors and estimated convergence rates for the interior fields ($\mathbf{q}$ and u) with $k = 1$, $\tau \equiv 1$, and $\tau_B \equiv 1$.

	E_q	e.c.r.	E_u	e.c.r
h_0	0.012893173	–	0.261556417	–
$h_0/2$	0.003219988	2.0015	0.065346639	2.0009
$h_0/4$	0.000804992	2.0000	0.016337124	2.0000
$h_0/8$	0.000201311	1.9995	0.004085171	1.9997
$h_0/16$	0.000050340	1.9996	0.001021463	1.9998
$h_0/32$	0.000012587	1.9998	0.000255391	1.9999

TABLE 7.4
Relative errors and estimated convergence rates for the boundary fields ($\lambda = \partial_\nu v$ and $\varphi = \gamma v$) and the exterior solution at an observation point with $k = 1$, $\tau \equiv 1$, and $\tau_B \equiv 1$.

	E_λ	e.c.r.	E_φ	e.c.r.	E_{ext}	e.c.r.
h_0	0.889544206	–	0.046439700	–	0.003415007	–
$h_0/2$	0.300690334	1.5648	0.006112008	2.9256	0.000057955	5.8808
$h_0/4$	0.096396924	1.6412	0.000725244	3.0751	0.000009778	2.5673
$h_0/8$	0.027419128	1.8138	0.000084127	3.1078	0.000001790	2.4496
$h_0/16$	0.007294633	1.9103	0.000009926	3.0832	0.000000222	3.0054
$h_0/32$	0.001879924	1.9562	0.000001196	3.0526	0.000000024	3.1648

TABLE 7.5
Relative errors and estimated convergence rates for the interior fields ($\mathbf{q}$ and u) with $k = 2$, $\tau \equiv 1$, and $\tau_B \equiv 100$.

	E_q	e.c.r.	E_u	e.c.r
h_0	0.000700920	–	0.013832499	–
$h_0/2$	0.000086196	3.0236	0.001721509	3.0063
$h_0/4$	0.000010663	3.0149	0.000214982	3.0014
$h_0/8$	0.000001324	3.0087	0.000026872	3.0000
$h_0/16$	0.000000165	3.0049	0.000003359	2.9998

Table 7.6
Relative errors and estimated convergence rates for the boundary fields ($\lambda = \partial_\nu v$ and $\varphi = \gamma v$) and the exterior solution at an observation point with $k = 2$, $\tau \equiv 1$, and $\tau_B \equiv 100$.

	E_λ	e.c.r.	E_φ	e.c.r.	E_{ext}	e.c.r.
h_0	0.285056528	–	0.020209250	–	0.000402598	–
$h_0/2$	0.065714209	2.1170	0.001773231	3.5106	0.000022005	4.1934
$h_0/4$	0.011914757	2.4635	0.000141144	3.6511	0.000001125	4.2891
$h_0/8$	0.001868856	2.6725	0.000010320	3.7736	0.000000051	4.4586
$h_0/16$	0.000266199	2.8116	0.000000707	3.8665	0.000000001	4.8941

in RT for stability, but this might be compensated by having fewer inner products on the skeleton associated to the penalized terms.) Numerical experiments for the RT-BEM scheme based on the reduced formulation (3.4) have already been shown in [5]. This formulation involves many more degrees of freedom, especially for high values of k. The LDG-BEM formulations in [12] and [13] cannot be hybridized and require working with two fields in the interior domain $(\mathbf{q}_h, u_h)$ instead of only one field on the skeleton of the triangulation $(\widehat{u}_h)$. A recent paper [18] contains numerical experiments for coupling of interior penalty DG methods with BEM. (One of the methods proposed in [18] is related but not equal to methods proposed in [12] and [13].) In this case, only the field u_h shows up in the interior part of the final system, which, in general will have more degrees of freedom than the hybridized case.

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