A UNIFIED ANALYSIS OF SEVERAL MIXED METHODS FOR ELASTICITY WITH WEAK STRESS SYMMETRY

J. GUZMÁN

ABSTRACT. We give a unified error analysis of several mixed methods for linear elasticity which impose stress symmetry weakly. We consider methods where the rotations are approximated by discontinuous polynomials. The methods we consider are such that the approximate stress spaces contain standard mixed finite element spaces for the Laplace equation and also contain divergence free spaces that use bubble functions.

1. INTRODUCTION

We consider the linear elasticity equation in three dimensions

$$\operatorname{div} \underline{\boldsymbol{\sigma}} = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\mathcal{A}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\epsilon}}(\boldsymbol{u}) = \mathbf{0} \quad \text{in } \Omega, \tag{1.1b}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{1.1c}$$

where $\mathcal{A}$ is a bounded symmetric positive definite tensor, $\boldsymbol{f}$ is a given load function, and Ω is a polyhedral domain. Also, $\underline{\boldsymbol{\epsilon}}(\boldsymbol{u}) = (\operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^t)/2$. For notational convenience, we only consider the more difficult three dimensional case in this article. The two dimensional case is similar.

There are several mixed formulations for the above problem. The formulation we consider introduces the rotation $\underline{\boldsymbol{\rho}} = (\operatorname{grad} \boldsymbol{u} - (\operatorname{grad} \boldsymbol{u})^t)/2$ and then $(\underline{\boldsymbol{\sigma}}, \boldsymbol{u}, \underline{\boldsymbol{\rho}})$ solves

$$(\mathcal{A}\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{v}})_\Omega + (\boldsymbol{u}, \operatorname{div} \underline{\boldsymbol{v}})_\Omega + (\underline{\boldsymbol{\rho}}, \underline{\boldsymbol{v}})_\Omega = 0 \tag{1.2a}$$

$$(\operatorname{div} \underline{\boldsymbol{\sigma}}, \boldsymbol{\omega})_\Omega = (\boldsymbol{f}, \boldsymbol{\omega})_\Omega \tag{1.2b}$$

$$(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\eta}})_\Omega = 0, \tag{1.2c}$$

for all $(\underline{\boldsymbol{v}}, \boldsymbol{\omega}, \underline{\boldsymbol{\eta}}) \in \underline{\boldsymbol{H}}(\operatorname{div}; \Omega) \times \boldsymbol{L}^2(\Omega) \times \underline{\boldsymbol{AS}}(\Omega)$. Here the inner-product for matrix valued functions is defined by $(\underline{\boldsymbol{v}}, \underline{\boldsymbol{\sigma}})_\Omega := \int_\Omega \underline{\boldsymbol{v}} : \underline{\boldsymbol{\sigma}}$ where : is the Frobenius inner product and for vector-vauled functions $(\boldsymbol{v}, \boldsymbol{u})_\Omega := \int_\Omega \boldsymbol{v} \cdot \boldsymbol{u}$. Also, $\underline{\boldsymbol{H}}(\operatorname{div}; \Omega)$ is the space of matrix valued function such that each of its rows belongs to the space $\boldsymbol{H}(\operatorname{div}; \Omega)$, $\boldsymbol{L}^2(\Omega)$ are vector-valued functions with components in $L^2(\Omega)$, and the anti-symmetric space is defined as $\underline{\boldsymbol{AS}}(\Omega) := \{\underline{\boldsymbol{v}} \in [L^2(\Omega)]^{3\times 3} : \underline{\boldsymbol{v}} + \underline{\boldsymbol{v}}^t = 0\}$.

2000 *Mathematics Subject Classification.* 65M60,65N30,35L65.

Key words and phrases. finite elements, elasticity, weakly stress symmetry, mixed methods, bubble function.

This work was supported by the National Science Foundation (grant DMS-0914596).

A mixed method approximation will find $(\underline{\boldsymbol{\sigma}}^h, \boldsymbol{u}^h, \underline{\boldsymbol{\rho}}^h) \in \underline{\boldsymbol{V}}^h \times \boldsymbol{W}^h \times \underline{\boldsymbol{A}}^h$

$$(\mathcal{A}\underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{v}})_\Omega + (\boldsymbol{u}^h, \operatorname{div} \underline{\boldsymbol{v}})_\Omega + (\underline{\boldsymbol{\rho}}^h, \underline{\boldsymbol{v}})_\Omega = 0 \tag{1.3a}$$

$$(\operatorname{div} \underline{\boldsymbol{\sigma}}^h, \boldsymbol{\omega})_\Omega = (\boldsymbol{f}, \boldsymbol{\omega})_\Omega \tag{1.3b}$$

$$(\underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\eta}})_\Omega = 0, \tag{1.3c}$$

for all $(\underline{\boldsymbol{v}}, \boldsymbol{\omega}, \underline{\boldsymbol{\eta}}) \in \underline{\boldsymbol{V}}^h \times \boldsymbol{W}^h \times \underline{\boldsymbol{A}}^h$. The finite elements spaces are given by

$$\underline{\boldsymbol{V}}^h := \{\underline{\boldsymbol{v}} \in \underline{\boldsymbol{H}}(\operatorname{div}, \Omega) : \boldsymbol{v}|_K \in \underline{\boldsymbol{V}}(K), \text{ for all } K \in \mathcal{T}_h\}, \tag{1.4a}$$

$$\boldsymbol{W}^h := \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{w}|_K \in \boldsymbol{W}(K), \text{ for all } K \in \mathcal{T}_h\}, \tag{1.4b}$$

$$\underline{\boldsymbol{A}}^h := \{\underline{\boldsymbol{v}} \in \underline{\boldsymbol{L}}^2(\Omega) : \underline{\boldsymbol{v}}|_K \in \underline{\boldsymbol{A}}(K), \text{ for all } K \in \mathcal{T}_h\}. \tag{1.4c}$$

Here $\mathcal{T}_h$ is a shape-regular, simplicial subdivision of Ω. Note that we are considering only those methods in which $\underline{\boldsymbol{A}}^h$ are composed of discontinuous functions across inter-element faces. Such methods allow for one to eliminate $\underline{\boldsymbol{\rho}}^h$ from the final coupled linear system. Throughout this paper we assume that $\underline{\boldsymbol{A}}(K)$ is a space of polynomials (which is the typical situation) in order to apply inverse estimates in our proofs.

Several choices of the local spaces have been given [1, 3, 11, 16, 5, 7, 12]. In particular, families of three dimensional elements have been given in [3, 16, 7, 12]. With our analysis we will be able to analyze these four family of methods in a unified way. Moreover, we will be able to simplify the elements given by Stenberg [16]. In [5, 11] a connection between mixed methods for linear elasticity and mixed methods for the Stokes problem was made. Using this theory, Boffi et al. [5] provided a different analysis for the elements introduced in [3]. They needed to construct a projection onto the space $\underline{\boldsymbol{V}}^h$, which itself is very interesting. In general, they can analyze any element with the use of Stokes elements provided they can construct an appropriate projection onto $\underline{\boldsymbol{V}}^h$; see Theorem 4.6 in [5]. In contrast, for the elements we consider here no new projection operator needs to be constructed. Instead, we only use projection operators for mixed methods applied to the Laplace equation. The reason we are able to analyze all four family of elements [3, 16, 7, 12] in a unified way is that they all have the property that

$$\operatorname{curl}(\operatorname{curl}(\underline{\boldsymbol{A}}(K))\underline{\boldsymbol{b}}_K) \subset \underline{\boldsymbol{V}}(K),$$

for all tetrahedra $K \in \mathcal{T}_h$. Here $\underline{\boldsymbol{b}}_K$ is a bubble matrix (or bubble scalar).

We also mention that the analysis of the elements considered [7, 12] depended on the construction of projection operators. Here, we show that the construction of such projection operators, although interesting, was not necessary for the error analysis.

The analysis provided here will allow to make connections between the different elements. It will also show what are the essential parts of the spaces that make them stable. For example, as mentioned above, we will be able to simplify the element provided in [16]. We also argue that the family of elements considered in [7] are in some sense a reduced version of the elements considered in [3].

2. Building Block Spaces

The methods we are going to analyze are the ones that contain certain spaces that we introduce below. Before doing this we define some local spaces. The space of polynomials of degree at most k defined on K is denoted by $\mathcal{P}^k(K)$. We then set $\boldsymbol{\mathcal{P}}^k(K) := [\mathcal{P}^k(K)]^3$ and $\underline{\boldsymbol{\mathcal{P}}}^k := [\mathcal{P}^k(K)]^{3\times 3}$. Finally, we let $\underline{\boldsymbol{\mathcal{P}}}^k(K) \subset \underline{\boldsymbol{RT}}^k(K) \subset \underline{\boldsymbol{\mathcal{P}}}^{k+1}(K)$ be the space of matrix-valued functions defined on K such that each row belong to the Raviart-Thomas space $\boldsymbol{\mathcal{P}}^k(K) + \boldsymbol{x}\mathcal{P}^k(K)$ of index k.

2.1. **Spaces for Laplace Equation.** We consider mixed finite element spaces for an uncoupled system of Laplace equations.

Definition 2.1. We say that the pair of spaces $\widetilde{\underline{\boldsymbol{V}}}^h \times \widetilde{\boldsymbol{W}}^h \subset \underline{\boldsymbol{H}}(\mathrm{div}\,;\Omega) \times \boldsymbol{L}^2(\Omega)$ are stable spaces for the Laplace equation if

1. $\mathrm{div}\,\widetilde{\underline{\boldsymbol{V}}}^h \subset \widetilde{\boldsymbol{W}}^h$
2. There exists a projection $\underline{\boldsymbol{\Pi}} : \underline{\boldsymbol{H}}(\mathrm{div}\,;\Omega) \cap \boldsymbol{L}^p(\Omega) :\rightarrow \widetilde{\underline{\boldsymbol{V}}}^h$ (for some $p > 2$) such that $\mathrm{div}\,(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{v}}) = \boldsymbol{P}\mathrm{div}\,\underline{\boldsymbol{v}}$ for all $\underline{\boldsymbol{v}} \in \underline{\boldsymbol{V}}^h$ where $\boldsymbol{P}$ is the L^2-projection onto $\widetilde{\boldsymbol{W}}^h$.

The two examples we have in mind are the Raviart-Thomas-Nedelec spaces and the Brezzi-Douglas-Duran-Fortin(BDDF) spaces; see [14, 13, 15, 4].

Example 2.2. The first example is the Raviart-Thomas-Nedelec space. The choice is $\widetilde{\underline{\boldsymbol{V}}}^h = \{\underline{\boldsymbol{v}} \in \underline{\boldsymbol{H}}(\mathrm{div}\,;\Omega) : \underline{\boldsymbol{v}} \in \underline{\boldsymbol{RT}}^k(K), \text{ for all } K \in \mathcal{T}_h\}$, and $\widetilde{\boldsymbol{W}}^h = \{\boldsymbol{\omega} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\omega} \in \boldsymbol{\mathcal{P}}^k(K), \text{ for all } K \in \mathcal{T}_h\}$. In this case, the projection $\underline{\boldsymbol{\Pi}}$ would be defined by applying the Raviart-Thomas-Nedelec projection row-wise; see [13].

Example 2.3. We can take $\widetilde{\underline{\boldsymbol{V}}}^h = \{\underline{\boldsymbol{v}} \in \underline{\boldsymbol{H}}(\mathrm{div}\,;\Omega) : \underline{\boldsymbol{v}} \in \underline{\boldsymbol{\mathcal{P}}}^k, \text{ for all } K \in \mathcal{T}_h\}$, and $\widetilde{\boldsymbol{W}}^h = \{\boldsymbol{\omega} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\omega} \in \boldsymbol{\mathcal{P}}^{k-1}(K), \text{ for all } K \in \mathcal{T}_h\}$. In this case, the projection $\underline{\boldsymbol{\Pi}}$ would be defined by applying the BDDF projection or the Nedelec projection row-wise; see [4, 14].

2.2. **Divergence free space using bubbles.** In this section we define divergence-free functions using bubble matrices (or scalars). We start by giving conditions for the bubble matrix.

Definition 2.4. A matrix-valued function $\underline{\boldsymbol{b}}$ defined on Ω is said to be an admissible bubble matrix if for each $K \in \mathcal{T}_h$ the matrix $\underline{\boldsymbol{b}}_K := \underline{\boldsymbol{b}}|_K$ is a matrix with polynomial entries that satisfies

1. The tangential components of each row of $\underline{\boldsymbol{b}}_K$ vanish on all the faces of K,
2. $C_1\,(\underline{\boldsymbol{v}}, \underline{\boldsymbol{v}})_K \leq (\underline{\boldsymbol{v}}\underline{\boldsymbol{b}}_K, \underline{\boldsymbol{v}})_K \quad$ for all $\underline{\boldsymbol{v}} \in [L^2(K)]^{3\times 3}$,
3. $\|\underline{\boldsymbol{b}}_K\|_{L^\infty(K)} \leq C_2$,

where the positive constants C_1 and C_2 only depend on the shape regularity of $\mathcal{T}_h$.

Example 2.5. The first example is $\underline{\boldsymbol{b}}_K = b_K\underline{\boldsymbol{I}}$ where $\underline{\boldsymbol{I}}$ is the identity matrix and $b_K = \lambda_0\lambda_1\lambda_2\lambda_3$ is the scalar bubble. Here λ_i $(i = 0, 1, 2, 3)$ denote the barycentric coordinates of K. Clearly, this is an admissible bubble matrix.

Example 2.6. The second example was used in [7, 12]

$$\underline{\boldsymbol{b}}_K := \frac{1}{h_K^2} \sum_{\ell=0}^{3} \lambda_{\ell-3}\lambda_{\ell-2}\lambda_{\ell-1}\,(\mathrm{grad}\,\lambda_\ell)^t\,\mathrm{grad}\,\lambda_\ell.$$

where the sub-indices are calculated mod 4. Also, $h_K = \mathrm{diam}(K)$. The fact this bubble matrix is admissible is proven in [7]. Note that here we multiply by $\frac{1}{h_K^2}$ in order for (2) and (3) in Definition 2.4 to be satisfied. Furthermore, note that $(\mathrm{grad}\,\lambda_\ell)^t\,\mathrm{grad}\,\lambda_\ell$ is a 3×3 matrix since using our convention $\mathrm{grad}\,\lambda_\ell$ is a row vector.

Definition 2.7. Given a finite element space $\underline{\boldsymbol{A}}^h \subset \underline{\boldsymbol{AS}}(\Omega)$ and an admissible bubble matrix $\underline{\boldsymbol{b}}$ we define the divergence-free space as follows

$$\underline{\boldsymbol{M}}^h(\underline{\boldsymbol{A}}^h;\underline{\boldsymbol{b}}) := \operatorname{curl}(\operatorname{curl}(\underline{\boldsymbol{A}}^h)\underline{\boldsymbol{b}}).$$

Next we prove an important result that exploits the structure of $\underline{\boldsymbol{M}}^h(\underline{\boldsymbol{A}}^h;\underline{\boldsymbol{b}})$. In order to do that, we define the subset of $\underline{\boldsymbol{A}}^h$ with average zero on each tetrahedra.

$$\underline{\boldsymbol{A}}_0^h := \{\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{A}}^h : (\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{v}})_K = 0 \text{ for all } \underline{\boldsymbol{v}} \in \underline{\boldsymbol{\mathcal{P}}}^0(K) \text{ and for all } K \in \mathcal{T}_h\}.$$

Lemma 2.8. *Given $\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{A}}_0^h$ there exists $\underline{\boldsymbol{v}} \in \underline{\boldsymbol{M}}^h(\underline{\boldsymbol{A}}^h;\underline{\boldsymbol{b}})$ such that*

$$\underline{\boldsymbol{P}}\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{\eta}}, \tag{2.5a}$$

$$\|\underline{\boldsymbol{v}}\|_{L^2(\Omega)} \le C\,\|\underline{\boldsymbol{\eta}}\|_{L^2(\Omega)}. \tag{2.5b}$$

Here $\underline{\boldsymbol{P}} : \underline{\boldsymbol{L}}^2(\Omega) \to \underline{\boldsymbol{A}}^h$ is the L^2 projection onto $\underline{\boldsymbol{A}}^h$.

Proof. We begin by defining $\underline{\boldsymbol{\psi}} \in \underline{\boldsymbol{A}}_0^h$ as the solution to

$$(\operatorname{curl}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{b}}, \operatorname{curl}\underline{\boldsymbol{w}})_K = (\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{w}})_K \text{ for all } \underline{\boldsymbol{w}} \in \underline{\boldsymbol{A}}_0^h \text{ and } K \in \mathcal{T}_h.$$

We show that $\underline{\boldsymbol{\psi}}$ is in fact well defined. Since the above equations define a square system for the degrees of freedom of $\underline{\boldsymbol{\psi}}$, we only need to prove uniqueness. Therefore, if we assume $\underline{\boldsymbol{\eta}} = 0$ we get that $\operatorname{curl}(\underline{\boldsymbol{\psi}}) = 0$ by (2) of Definition 2.4. Next, we write

$$\underline{\boldsymbol{\psi}} = \begin{pmatrix} 0 & z_3 & -z_2 \\ -z_3 & 0 & z_1 \\ z_2 & -z_1 & 0 \end{pmatrix}$$

for some $\boldsymbol{z} = (z_1, z_2, z_3)^t$ where the average of each z_i (for $i = 1, 2, 3$) on every $K \in \mathcal{T}_h$ are zero. On each $K \in \mathcal{T}_h$

$$\operatorname{curl}\underline{\boldsymbol{\psi}} = (\operatorname{grad}\boldsymbol{z})^t - (\operatorname{div}\boldsymbol{z})\underline{\boldsymbol{I}}, \tag{2.6}$$

so we have

$$0 = ((\operatorname{grad}\boldsymbol{z})^t - (\operatorname{div}\boldsymbol{z})\underline{\boldsymbol{I}}) : \underline{\boldsymbol{I}} = -2\operatorname{div}\boldsymbol{z}.$$

This gives that $\operatorname{div}\boldsymbol{z} = 0$ on each $K \in \mathcal{T}_h$. Which in turn gives that $\operatorname{grad}\boldsymbol{z} = 0$ on each $K \in \mathcal{T}_h$, which implies that $\boldsymbol{z} = 0$ after we note that averages of each z_i are zero on $K \in \mathcal{T}_h$. In fact, we can use this result combined with a scaling argument to show the Poincare type inequality,

$$\|\underline{\boldsymbol{w}}\|_{L^2(K)} \le C\,h_K \|\operatorname{curl}(\underline{\boldsymbol{w}})\|_{L^2(K)} \tag{2.7}$$

for all $\underline{\boldsymbol{w}} \in \underline{\boldsymbol{A}}_0^h$ and all $K \in \mathcal{T}_h$.

We next show that

$$\|\operatorname{curl}(\underline{\boldsymbol{\psi}})\|_{L^2(K)} \le C\,h_K \|\underline{\boldsymbol{\eta}}\|_{L^2(K)}. \tag{2.8}$$

To see this, we use (2) of Definition 2.4 to get

$$\|\operatorname{curl}(\underline{\boldsymbol{\psi}})\|_{L^2(K)}^2 \le C\,(\operatorname{curl}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{b}}_K, \operatorname{curl}(\underline{\boldsymbol{\psi}})\,)_K = C\,(\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{\psi}})_K.$$

If we use (2.7) we get (2.8).

Now we set $\underline{\boldsymbol{v}} = \operatorname{curl}(\operatorname{curl}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{b}})$. Let us show that indeed $\underline{\boldsymbol{v}}$ has the desired properties. From property (1) of Definition 2.4 and using integration by parts we have

$$(\underline{\boldsymbol{v}}, \underline{\boldsymbol{w}})_K = (\operatorname{curl}(\operatorname{curl}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{b}}), \underline{\boldsymbol{w}})_K = (\operatorname{curl}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{b}}, \operatorname{curl}\underline{\boldsymbol{w}})_K = (\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{w}})_K,$$

for all $\underline{\boldsymbol{w}} \in \underline{\boldsymbol{A}}_0^h$ and $K \in \mathcal{T}_h$. This shows (2.5a). To show (2.5b) we first use an inverse estimate

$$\|\underline{\boldsymbol{v}}\|_{L^2(K)}^2 = \|\mathrm{curl}\,(\mathrm{curl}\,(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{b}})\|_{L^2(K)}^2 \leq \frac{C}{h_K^2}\|\mathrm{curl}\,(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{b}}\|_{L^2(K)}^2.$$

However, if we use (3) of Definition 2.4, we obtain

$$\|\mathrm{curl}\,(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{b}}\|_{L^2(K)}^2 \leq C\,\|\mathrm{curl}\,(\underline{\boldsymbol{\psi}})\|_{L^2(K)}^2.$$

Finally, by using (2.8), we get

$$\|\underline{\boldsymbol{v}}\|_{L^2(K)}^2 \leq C\|\underline{\boldsymbol{\eta}}\|_{L^2(K)}^2.$$

By taking the sum and then taking the square root proves (2.5b). □

2.3. **Lowest-Order Element.** We will need the lowest order element in [3] for the stress space.

$$\underline{\boldsymbol{V}}_1^h := \{\underline{\boldsymbol{v}} \in \underline{\boldsymbol{H}}(\mathrm{div}\,;\Omega) : \underline{\boldsymbol{v}}|_K \in \underline{\boldsymbol{\mathcal{P}}}^1(K),\ \text{for all } K \in \mathcal{T}_h\}.$$

The following result is contained in [3]; see also [5]. In order for this article to be sufficiently self contained, we give a proof of this proposition in the appendix.

Proposition 2.9. *Given $\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{A}}_c^h := \{\underline{\boldsymbol{w}} \in \underline{\boldsymbol{AS}}(\Omega) : \underline{\boldsymbol{v}}|_K \in \underline{\boldsymbol{\mathcal{P}}}^0(K),\ \text{for all } K \in \mathcal{T}_h\}$, there exists $\underline{\boldsymbol{v}} \in \underline{\boldsymbol{V}}_1^h$ such that*

$$\mathrm{div}\,\underline{\boldsymbol{v}} = 0, \tag{2.9a}$$

$$\underline{\boldsymbol{P}}^c\underline{\boldsymbol{v}} = \underline{\boldsymbol{\eta}}, \tag{2.9b}$$

$$\|\underline{\boldsymbol{v}}\|_{L^2(\Omega)} \leq C\,\|\underline{\boldsymbol{\eta}}\|_{L^2(\Omega)}, \tag{2.9c}$$

where $\underline{\boldsymbol{P}}^c$ is the L^2 projection onto $\underline{\boldsymbol{A}}_c^h$.

Finally, we make the following observation:

$$(\underline{\boldsymbol{v}}, \underline{\boldsymbol{\eta}})_\Omega = 0 \quad \text{for all } \underline{\boldsymbol{v}} \in \underline{\boldsymbol{M}}^h(\underline{\boldsymbol{A}}^h; \underline{\boldsymbol{b}}) \text{ and } \underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{A}}_c^h. \tag{2.10}$$

This follows from integration by parts, the fact that $(\mathrm{curl}\,\underline{\boldsymbol{\eta}})|_K = 0$ for every $K \in \mathcal{T}_h$, and (1) of Definition 2.4.

3. Error Analysis

3.1. **Error of $\underline{\boldsymbol{\rho}}$ and $\underline{\boldsymbol{\sigma}}$.** We begin with our main result.

Theorem 3.1. *Suppose the $(\underline{\boldsymbol{\sigma}}^h, \boldsymbol{u}^h, \underline{\boldsymbol{\rho}}^h) \in \underline{\boldsymbol{V}}^h \times \boldsymbol{W}^h \times \underline{\boldsymbol{A}}^h$ solves (1.3). Assume the spaces satisfy the following conditions*

$$\underline{\boldsymbol{V}}_1^h + \widetilde{\underline{\boldsymbol{V}}}^h + \underline{\boldsymbol{M}}^h(\underline{\boldsymbol{A}}^h; \underline{\boldsymbol{b}}) \subset \underline{\boldsymbol{V}}^h, \tag{3.11a}$$

$$\boldsymbol{W}^h = \widetilde{\boldsymbol{W}}^h, \tag{3.11b}$$

$$\mathrm{div}\,\underline{\boldsymbol{V}}^h \subset \boldsymbol{W}^h, \tag{3.11c}$$

where $(\widetilde{\underline{\boldsymbol{V}}}^h, \widetilde{\boldsymbol{W}}^h)$ is a stable pair of spaces for the Laplace equation with projection $\underline{\boldsymbol{\Pi}}$ (see Definition 2.1) and $\underline{\boldsymbol{b}}$ is an admissible bubble matrix. Then, we have

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}^h\|_{L^2(\Omega)} \leq C\,(\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\Pi}}\,\underline{\boldsymbol{\sigma}}\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)}). \tag{3.12}$$

Proof. Let us start by writing the error equations.

$$(\mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h), \underline{\boldsymbol{v}})_\Omega + (\boldsymbol{u} - \boldsymbol{u}^h, \operatorname{div} \underline{\boldsymbol{v}})_\Omega + (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}^h, \underline{\boldsymbol{v}})_\Omega = 0, \tag{3.13a}$$

$$(\operatorname{div} (\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h), \boldsymbol{\omega})_\Omega = 0, \tag{3.13b}$$

$$(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\eta}})_\Omega = 0, \tag{3.13c}$$

for all $(\underline{\boldsymbol{v}}, \boldsymbol{\omega}, \underline{\boldsymbol{\eta}}) \in \underline{\boldsymbol{V}}^h \times \boldsymbol{W}^h \times \underline{\boldsymbol{A}}^h$.

We first prove that the following estimate

$$\|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}^h\|_{L^2(\Omega)} \leq C\,(\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)}). \tag{3.14}$$

We let $\underline{\boldsymbol{\psi}} = \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}^h$ and we decompose it as

$$\underline{\boldsymbol{\psi}} = \underline{\boldsymbol{\psi}}_0 + \underline{\boldsymbol{\psi}}_c,$$

where $\underline{\boldsymbol{\psi}}_0 \in \underline{\boldsymbol{A}}_0^h$ and $\underline{\boldsymbol{\psi}}_c \in \underline{\boldsymbol{A}}_c^h$ where we define $\underline{\boldsymbol{\psi}}_c|_K := \frac{1}{|K|}\int_K \underline{\boldsymbol{\psi}}\, dx$ for all $K \in \mathcal{T}_h$. We first get an estimate for $\underline{\boldsymbol{\psi}}_0$. Let $\underline{\boldsymbol{v}} \in \underline{\boldsymbol{M}}^h(\underline{\boldsymbol{A}}^h; \underline{\boldsymbol{b}}_K)$ be from Lemma 2.8 with $\underline{\boldsymbol{\eta}} = \underline{\boldsymbol{\psi}}_0$.

$$\begin{aligned}
\|\underline{\boldsymbol{\psi}}_0\|^2_{L^2(\Omega)} =& (\underline{\boldsymbol{\psi}}_0, \underline{\boldsymbol{\psi}}_0)_\Omega \\
=& (\underline{\boldsymbol{\psi}}_0, \underline{\boldsymbol{v}})_\Omega && \text{by (2.5a)} \\
=& (\underline{\boldsymbol{\psi}}_0, \underline{\boldsymbol{v}})_\Omega + (\underline{\boldsymbol{\psi}}_c, \underline{\boldsymbol{v}})_\Omega && \text{by (2.10)} \\
=& (\underline{\boldsymbol{\psi}}, \underline{\boldsymbol{v}})_\Omega \\
=& (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}^h, \underline{\boldsymbol{v}})_\Omega + (\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}, \underline{\boldsymbol{v}})_\Omega \\
=& -(\mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h), \underline{\boldsymbol{v}})_\Omega + (\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}, \underline{\boldsymbol{v}})_\Omega && \text{by (3.13a) and (3.11a)}
\end{aligned}$$

Here we used that $\operatorname{div} \underline{\boldsymbol{v}} = 0$ since $\underline{\boldsymbol{v}} \in \underline{\boldsymbol{M}}^h(\underline{\boldsymbol{A}}^h, \underline{\boldsymbol{b}})$. Finally, using (2.5b) and the fact that $\mathcal{A}$ is bounded

$$\|\underline{\boldsymbol{\psi}}_0\|_{L^2(\Omega)} \leq C\,(\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)}). \tag{3.15}$$

Now we prove an estimate for $\underline{\boldsymbol{\psi}}_c$. We let $\underline{\boldsymbol{v}} \in \underline{\boldsymbol{V}}_1^h$ be from Proposition 2.9 with $\underline{\boldsymbol{\eta}} = \underline{\boldsymbol{\psi}}_c$. Then,

$$\begin{aligned}
\|\underline{\boldsymbol{\psi}}_c\|^2_{L^2(\Omega)} =& (\underline{\boldsymbol{\psi}}_c, \underline{\boldsymbol{\psi}}_c)_\Omega \\
=& (\underline{\boldsymbol{\psi}}_c, \underline{\boldsymbol{v}})_\Omega && \text{by (2.9b)} \\
=& (\underline{\boldsymbol{\psi}}, \underline{\boldsymbol{v}})_\Omega - (\underline{\boldsymbol{\psi}}_0, \underline{\boldsymbol{v}})_\Omega \\
=& (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}^h, \underline{\boldsymbol{v}})_\Omega + (\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}, \underline{\boldsymbol{v}})_\Omega - (\underline{\boldsymbol{\psi}}_0, \underline{\boldsymbol{v}})_\Omega \\
=& -(\mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h), \underline{\boldsymbol{v}})_\Omega + (\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}, \underline{\boldsymbol{v}})_\Omega - (\underline{\boldsymbol{\psi}}_0, \underline{\boldsymbol{v}})_\Omega && \text{by (3.13a), (2.9a) and (3.11a)}
\end{aligned}$$

Therefore, if we use (2.9c) and (3.15) we have

$$\|\underline{\boldsymbol{\psi}}_c\|_{L^2(\Omega)} \leq C\,(\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)}). \tag{3.16}$$

Hence, using (3.15), (3.16) and the triangle inequality we arrive at (3.14).

Now we get an estimate for $\underline{\boldsymbol{\sigma}}$. We first note that by (1) and (2) of Definition 2.1, (3.11b), (3.11c) and (3.13b) we have

$$\operatorname{div} (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h) = 0.$$

Therefore, by using this result, (3.13a) and (3.13c) we get

$$
\begin{aligned}
(\mathcal{A}(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h), (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h))_\Omega =& (\mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h), (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h))_\Omega + (\mathcal{A}(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h)_\Omega \\
=& -(\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}^h, (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h))_\Omega + (\mathcal{A}(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}), (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h))_\Omega \\
=& -(\underline{\boldsymbol{P}}\,\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}^h, (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h))_\Omega - (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\,\underline{\boldsymbol{\rho}}, (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h))_\Omega \\
&+ (\mathcal{A}(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}), (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h))_\Omega \\
=& -(\underline{\boldsymbol{P}}\,\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}^h, (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}))_\Omega - (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\,\underline{\boldsymbol{\rho}}, (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h))_\Omega \\
&+ (\mathcal{A}(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}), (\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h))_\Omega.
\end{aligned}
$$

If use (3.14), the triangle inequality and the fact that $\mathcal{A}$ is positive definite we get

$$
\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h\|_{L^2(\Omega)} \le C\,(\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\Pi}}\,\underline{\boldsymbol{\sigma}}\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)}).
$$

Finally, if we use (3.14) we arrive at (3.12). □

3.2. Error analysis for $\boldsymbol{u}$.

We start with the main result of this subsection.

Theorem 3.2. *Assuming the hypothesis of Theorem 3.1 we have*

$$
\|\boldsymbol{u} - \boldsymbol{u}^h\|_{L^2(\Omega)} \le C\,(\|\boldsymbol{u} - \boldsymbol{P}\,\boldsymbol{u}\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\Pi}}\,\underline{\boldsymbol{\sigma}}\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)}),
$$

where $\boldsymbol{P}$ is the L^2-projection operator onto $\boldsymbol{W}^h$.

Proof. It is well known that there exists $\underline{\boldsymbol{w}} \in H^1(\Omega)$ such that

$$
\operatorname{div}\underline{\boldsymbol{w}} = \boldsymbol{P}\boldsymbol{u} - \boldsymbol{u}^h, \tag{3.17a}
$$

$$
\|\underline{\boldsymbol{w}}\|_{H^1(\Omega)} \le C\,\|\boldsymbol{P}\boldsymbol{u} - \boldsymbol{u}^h\|_{L^2(\Omega)}. \tag{3.17b}
$$

Then, by the error equation (3.13a), property (2) of Definition 2.1, and (3.11a) we have

$$
\begin{aligned}
\|\boldsymbol{P}\boldsymbol{u} - \boldsymbol{u}^h\|^2_{L^2(\Omega)} =& (\operatorname{div}\underline{\boldsymbol{w}}, \boldsymbol{P}\boldsymbol{u} - \boldsymbol{u}^h)_\Omega \\
=& (\operatorname{div}(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{w}}), \boldsymbol{P}\boldsymbol{u} - \boldsymbol{u}^h)_\Omega \\
=& (\operatorname{div}(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{w}}), \boldsymbol{u} - \boldsymbol{u}^h)_\Omega \\
=& (\mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^h), \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{w}})_\Omega + (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}^h, \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{w}})_\Omega.
\end{aligned}
$$

The result now follows by using (3.12), the triangle inequality, and the fact that

$$
\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{w}}\|_{L^2(\Omega)} \le C\,\|\underline{\boldsymbol{w}}\|_{H^1(\Omega)} \le C\,\|\boldsymbol{P}\boldsymbol{u} - \boldsymbol{u}^h\|_{L^2(\Omega)}.
$$

□

We note that improved error estimates for $\boldsymbol{P}\boldsymbol{u} - \boldsymbol{u}^h$ can be obtained by using a duality argument, assuming elliptic regularity, and using that we now have estimates for both $\underline{\boldsymbol{\rho}}$ and $\underline{\boldsymbol{\sigma}}$ in Theorem 3.1. We leave the details to the reader.

4. Examples

In this section we give several examples.

Example 4.1. First we give the AFW [3] element. They chose as local spaces

$$\begin{aligned}\underline{\boldsymbol{A}}(K) =&\underline{\boldsymbol{A}}^k(K) := \underline{\boldsymbol{\mathcal{P}}}^k(K)\cap \underline{\boldsymbol{AS}}(K),\\ \boldsymbol{W}(K) =&\boldsymbol{\mathcal{P}}^k(K),\\ \underline{\boldsymbol{V}}(K) =&\underline{\boldsymbol{\mathcal{P}}}^{k+1}(K).\end{aligned}$$

We first note that

$$\begin{aligned}\widetilde{\underline{\boldsymbol{V}}}^h &= \{\underline{\boldsymbol{v}} \in \underline{\boldsymbol{H}}(\mathrm{div},\Omega) : \boldsymbol{v}|_K \in \underline{\boldsymbol{RT}}^k(K),\ \text{for all } K\in\mathcal{T}_h\},\\ \widetilde{\boldsymbol{W}}^h &= \{\boldsymbol{w}\in \boldsymbol{L}^2(\Omega) : \boldsymbol{w}|_K \in \boldsymbol{\mathcal{P}}^k(K),\ \text{for all } K\in\mathcal{T}_h\},\end{aligned}$$

are a stable pair of spaces for the Laplace equation; see Definition 2.1. Clearly we have the inclusion $\widetilde{\underline{\boldsymbol{V}}}^h \subset \underline{\boldsymbol{V}}^h$. Furthermore, we have the inclusion

$$\underline{\boldsymbol{M}}^h(\underline{\boldsymbol{A}}^h;\underline{\boldsymbol{b}}) \subset \underline{\boldsymbol{V}}^h,$$

where $\underline{\boldsymbol{b}}$ is the bubble matrix defined in Example 2.6. Therefore, we can apply Theorem 3.1.

The above analysis of the AFW element shows that one can reduce the space $\underline{\boldsymbol{V}}^h$ while keeping $\underline{\boldsymbol{A}}^h$ and $\boldsymbol{W}^h$ the same. This leads us to the following example which is the element contained in [7].

Example 4.2. The local spaces (for $k\geq 1$) are defined as follows:

$$\begin{aligned}\underline{\boldsymbol{A}}(K) =&\underline{\boldsymbol{A}}^k(K),\\ \boldsymbol{W}(K) =&\boldsymbol{\mathcal{P}}^k(K),\\ \underline{\boldsymbol{V}}(K) =&\underline{\boldsymbol{RT}}^k(K) + \mathrm{curl}\,(\mathrm{curl}\,(\underline{\boldsymbol{A}}(K))\,\underline{\boldsymbol{b}}_K),\end{aligned}$$

where $\underline{\boldsymbol{b}}_K$ is given in Example 2.6. The hypothesis of Theorem 3.1 hold with the stable pair of spaces for the Laplace equation again being

$$\begin{aligned}\widetilde{\underline{\boldsymbol{V}}}^h &= \{\underline{\boldsymbol{v}} \in \underline{\boldsymbol{H}}(\mathrm{div},\Omega) : \boldsymbol{v}|_K \in \underline{\boldsymbol{RT}}^k(K),\ \text{for all } K\in\mathcal{T}_h\},\\ \widetilde{\boldsymbol{W}}^h &= \{\boldsymbol{w}\in \boldsymbol{L}^2(\Omega) : \boldsymbol{w}|_K \in \boldsymbol{\mathcal{P}}^k(K),\ \text{for all } K\in\mathcal{T}_h\}.\end{aligned}$$

Note that the above example has same the convergence properties as the AFW element for the same k since $\|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)}$ in Theorem 3.1 would appear on the right-hand side for both elements.

If we instead use the BDDF [4] stable pair of spaces for the Laplace equation we get the element considered in [12].

Example 4.3.

$$\begin{aligned}\underline{\boldsymbol{A}}(K) =&\underline{\boldsymbol{A}}^k(K),\\ \boldsymbol{W}(K) =&\boldsymbol{\mathcal{P}}^{k-1}(K),\\ \underline{\boldsymbol{V}}(K) =&\underline{\boldsymbol{\mathcal{P}}}^k(K) + \mathrm{curl}\,(\mathrm{curl}\,(\underline{\boldsymbol{A}}(K))\,\underline{\boldsymbol{b}}_K),\end{aligned}$$

where $\underline{\boldsymbol{b}}_K$ is given in Example 2.6. In this case, the corresponding stable spaces for the Laplace equation are

$$\begin{aligned}\widetilde{\underline{\boldsymbol{V}}}^h &= \{\underline{\boldsymbol{v}} \in \underline{\boldsymbol{H}}(\mathrm{div},\Omega) : \boldsymbol{v}|_K \in \underline{\boldsymbol{\mathcal{P}}}^k(K),\ \text{for all } K\in\mathcal{T}_h\},\\ \widetilde{\boldsymbol{W}}^h &= \{\boldsymbol{w}\in \boldsymbol{L}^2(\Omega) : \boldsymbol{w}|_K \in \boldsymbol{\mathcal{P}}^{k-1}(K),\ \text{for all } K\in\mathcal{T}_h\}.\end{aligned}$$

Example 4.4. Here we consider the elements given by Stenberg [16] (for $k \geq 1$).

$$\begin{aligned}\underline{\boldsymbol{A}}(K) =&\underline{\boldsymbol{A}}^k(K),\\ \boldsymbol{W}(K) =&\boldsymbol{\mathcal{P}}^{k-1}(K),\\ \underline{\boldsymbol{V}}(K) =&\underline{\boldsymbol{\mathcal{P}}}^k(K) + \operatorname{curl}(\underline{\boldsymbol{\mathcal{P}}}^{k-1}(K)\,\underline{\boldsymbol{b}}_K),\end{aligned}$$

where now $\underline{\boldsymbol{b}}_K$ is given in Example 2.5. Since $\operatorname{curl}(\operatorname{curl}(\underline{\boldsymbol{A}}(K))\underline{\boldsymbol{b}}_K) \subset \operatorname{curl}(\underline{\boldsymbol{\mathcal{P}}}^{k-1}(K)\,\underline{\boldsymbol{b}}_K)$, we can easily apply Theorem 3.1 to get optimal error estimates. In fact, we see that the following reduced space for $\underline{\boldsymbol{V}}(K)$ would have the same convergence properties

$$\underline{\boldsymbol{V}}(K) = \underline{\boldsymbol{\mathcal{P}}}^k(K) + \operatorname{curl}(\operatorname{curl}(\underline{\boldsymbol{A}}(K))\,\underline{\boldsymbol{b}}_K).$$

We note here that Stenberg [16] modified his lowest order element, $k = 1$. Here, with our analysis, we see that his modification was not necessary. Falk [10] also proved that Stenberg's modification for the lowest order element is not necessary in the two dimensional case.

Acknowldegements. We would like to thank J. Gopalakrishnan for many fruitful discussions.

5. Appendix

Here we prove Proposition 2.9. The proof is in [3, 5]. However, we provide a proof here for completeness.

We start with a lemma concerning a lowest-order Stokes element. The techniques used here are now standard and the result is a trivial modification of a result given in [9].

Lemma 5.1. *For every* $m \in Q^h := \{w \in L^2(\Omega) : w|_K \in \mathcal{P}^0(K),$ *for all* $K \in \mathcal{T}_h\}$ *there exists* $\boldsymbol{v} \in \boldsymbol{S}^h := \{\boldsymbol{w} \in [H^1(\Omega)]^3 : \boldsymbol{w}|_K \in \boldsymbol{\mathcal{P}}^3(K)\}$ *such that*

$$P_Q \operatorname{div} \boldsymbol{v} = m, \tag{5.18a}$$

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C\|m\|_{L^2(\Omega)}. \tag{5.18b}$$

where P_Q *is the* L^2 *projection onto* Q^h*.*

Proof. It is well known that there exist $\boldsymbol{\omega} \in H^1(\Omega)$ such that

$$\operatorname{div} \boldsymbol{\omega} = m, \tag{5.19a}$$

$$\|\boldsymbol{\omega}\|_{H^1(\Omega)} \leq C\|m\|_{L^2(\Omega)}. \tag{5.19b}$$

We define $\boldsymbol{v}$ in the following way

$$\boldsymbol{v} := \mathsf{I}\boldsymbol{\omega} + \boldsymbol{\pi}(\boldsymbol{\omega} - \mathsf{I}\boldsymbol{\omega}),$$

where I is the Clément interpolant [8] onto continuous, piecewise linear functions. We now define the projection $\boldsymbol{\pi}$ by defining it component-wise (i.e. $(\boldsymbol{\pi u})_i = \pi u_i$ for $i = 1, 2, 3$). For a scalar function u define on each $K \in \mathcal{T}_h$, $\pi u \in \operatorname{span}\{\lambda_0\lambda_1\lambda_2, \lambda_0\lambda_1\lambda_3, \lambda_0\lambda_2\lambda_3, \lambda_1\lambda_2\lambda_3\}$ with the following degrees of freedom

$$\langle \pi u - u, 1\rangle_F = 0 \quad \text{for all faces } F \text{ of } K.$$

It is not difficult to see that π is well defined and hence $\boldsymbol{\pi}$ is well defined. Moreover, the following properties are not difficult to establish:

$$\begin{aligned}\boldsymbol{\pi} : [H^1(\Omega)]^3 \to & \{\boldsymbol{w} \in [H^1(\Omega)]^3 : \boldsymbol{w}|_K \in \boldsymbol{\mathcal{P}}^3(K)\},\\ \|\underline{\boldsymbol{\Pi}}\boldsymbol{u}\|_{H^1(K)} \leq & C\,(\|\boldsymbol{u}\|_{H^1(K)} + \frac{1}{h_K}\|\boldsymbol{u}\|_{L^2(K)}),\end{aligned}$$

for every $K \in \mathcal{T}_h$. The inequality follows from the trace theorem and a scaling argument.

Now we prove that $\boldsymbol{v}$ has the desired properties. Clearly, $\boldsymbol{v} \in \boldsymbol{S}^h$. If we let $w \in Q^h$ we get

$$\begin{aligned}(\operatorname{div}\boldsymbol{v}, w)_K =& -(\boldsymbol{v}, \operatorname{grad} w)_K + \langle \boldsymbol{v}\cdot\boldsymbol{n}, w\rangle_{\partial K} && \text{by integration by parts}\\ =& \langle \boldsymbol{v}\cdot\boldsymbol{n}, w\rangle_{\partial K} && \text{since } \operatorname{grad} w = 0,\\ =& \langle (\mathsf{I}\boldsymbol{\omega} + \boldsymbol{\pi}(\boldsymbol{\omega} - \mathsf{I}\boldsymbol{\omega}))\cdot\boldsymbol{n}, w\rangle_{\partial K} && \text{by the definition of } \boldsymbol{v}\\ =& \langle \boldsymbol{\omega}\cdot\boldsymbol{n}, w\rangle_{\partial K} && \text{by the definition of } \boldsymbol{\pi}\\ =& -(\boldsymbol{\omega}, \operatorname{grad} w)_K + \langle \boldsymbol{\omega}\cdot\boldsymbol{n}, w\rangle_{\partial K} && \text{since } \operatorname{grad} w = 0\\ =& (\operatorname{div}\boldsymbol{\omega}, w)_K && \text{by integration by parts}\\ =& (m, w)_K. && \text{by (5.19a)}\end{aligned}$$

This shows (5.18a). Now we prove (5.18b).

$$\begin{aligned}\|\boldsymbol{v}\|^2_{H^1(\Omega)} =& \sum_{K\in\mathcal{T}_h} \|\boldsymbol{v}\|^2_{H^1(K)}\\ \leq& 2\|\mathsf{I}\boldsymbol{\omega}\|^2_{H^1(\Omega)} + 2\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\pi}(\boldsymbol{\omega} - \mathsf{I}\boldsymbol{\omega})\|^2_{H^1(K)}\\ \leq& 2\|\mathsf{I}\boldsymbol{\omega}\|^2_{H^1(\Omega)} + C\sum_{K\in\mathcal{T}_h} (\|\boldsymbol{\omega} - \mathsf{I}\boldsymbol{\omega}\|^2_{H^1(K)} + \frac{1}{h_K}\|\boldsymbol{\omega} - \mathsf{I}\boldsymbol{\omega}\|_{L^2(K)})^2\\ \leq& C\|\boldsymbol{\omega}\|^2_{H^1(\Omega)} + C\sum_{K\in\mathcal{T}_h} \frac{1}{h_K^2}\|\boldsymbol{\omega} - \mathsf{I}\boldsymbol{\omega}\|^2_{L^2(K)}\\ \leq& C\|\boldsymbol{\omega}\|^2_{H^1(\Omega)}.\end{aligned}$$

Here we use standard properties of the Clément interpolant I and the fact that $\mathcal{T}_h$ is shape-regular. Therefore,

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq \|\boldsymbol{\omega}\|_{H^1(\Omega)}.$$

We obtain (5.18b) once we use (5.19b). □

Now we prove Proposition 2.9.

Proof. If $\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{A}}_c^h$ then,

$$\underline{\boldsymbol{\eta}} = \begin{pmatrix} 0 & z_3 & -z_2\\ -z_3 & 0 & z_1\\ z_2 & -z_1 & 0\end{pmatrix}$$

for $z_1, z_2, z_3 \in Q^h$. From Lemma 5.1 we can find $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \in \boldsymbol{S}^h$ such that

$$P_Q\operatorname{div}\boldsymbol{w}_i = -2\,z_i \tag{5.20a}$$

$$\|\boldsymbol{w}_i\|_{H^1(\Omega)} \leq C\,\|z_i\|_{L^2(\Omega)}. \tag{5.20b}$$

for $i = 1, 2, 3$.

Next we let $\underline{\boldsymbol{\psi}}$ be the matrix with i-th comlumn $\boldsymbol{w}_i$ and we use the following identity

$$-2\, \underline{\boldsymbol{skw}}\, \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}}) = \begin{pmatrix} 0 & \mathrm{div}\, \boldsymbol{w}_3 & -\mathrm{div}\, \boldsymbol{w}_2 \\ -\mathrm{div}\, \boldsymbol{w}_3 & 0 & \mathrm{div}\, \boldsymbol{w}_1 \\ \mathrm{div}\, \boldsymbol{w}_2 & -\mathrm{div}\, \boldsymbol{w}_1 & 0 \end{pmatrix}.$$

Here $\underline{\boldsymbol{skw}}\, \underline{\boldsymbol{v}} = (\underline{\boldsymbol{v}} - \underline{\boldsymbol{v}}^t)/2$ for any matrix $\underline{\boldsymbol{v}}$.

Hence,

$$\underline{\boldsymbol{P}}^c \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}}) = \frac{1}{2} \begin{pmatrix} 0 & -P_Q\, \mathrm{div}\, \boldsymbol{w}_3 & P_Q\, \mathrm{div}\, \boldsymbol{w}_2 \\ P_Q\, \mathrm{div}\, \boldsymbol{w}_3 & 0 & -P_Q\, \mathrm{div}\, \boldsymbol{w}_1 \\ -P_Q\, \mathrm{div}\, \boldsymbol{w}_2 & P_Q\, \mathrm{div}\, \boldsymbol{w}_1 & 0 \end{pmatrix},$$

and so

$$\underline{\boldsymbol{P}}^c \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}}) = \underline{\boldsymbol{\eta}}. \tag{5.21}$$

We let $\underline{\boldsymbol{v}} = \underline{\boldsymbol{\Pi}}_1 \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}})$, where $\underline{\boldsymbol{\Pi}}_1$ is the three copies of the lowest order BDDF projection (see [4]) which is defined as follows

$$\langle (\underline{\boldsymbol{\Pi}}_1 \underline{\boldsymbol{w}} - \underline{\boldsymbol{w}})\, \boldsymbol{n}, \boldsymbol{\mu} \rangle_F = 0 \quad \text{ for all } \boldsymbol{\mu} \in \mathcal{P}^1(F), \tag{5.22}$$

and for all faces F of K.

By the properties of the BDDF projection we have

$$\mathrm{div}\, \underline{\boldsymbol{v}} = 0, \tag{5.23}$$

since $\mathrm{div}\, \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}}) = 0$.

Let $\underline{\boldsymbol{\theta}} \in \underline{\boldsymbol{A}}_c^h$, then for each $K \in \mathcal{T}_h$ we have $\underline{\boldsymbol{\theta}}|_K = \mathrm{grad}\, \boldsymbol{\omega}$ for some linear function $\boldsymbol{\omega}$. Hence,

$$\begin{aligned}
(\underline{\boldsymbol{v}}, \underline{\boldsymbol{\theta}})_K =& (\underline{\boldsymbol{v}}, \mathrm{grad}\, \boldsymbol{\omega})_K \\
=& \langle \underline{\boldsymbol{v}}\, \boldsymbol{n}, \boldsymbol{\omega} \rangle_{\partial K} && \text{by (5.23)} \\
=& \langle \underline{\boldsymbol{\Pi}}_1 \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}})\, \boldsymbol{n}, \boldsymbol{\omega} \rangle_{\partial K} && \text{by the definition of } \underline{\boldsymbol{v}} \\
=& \langle \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}})\, \boldsymbol{n}, \boldsymbol{\omega} \rangle_{\partial K} && \text{by (5.22)} \\
=& (\mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}}), \mathrm{grad}\, \boldsymbol{\omega})_K \\
=& (\mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}}), \underline{\boldsymbol{\theta}})_K \\
=& (\underline{\boldsymbol{P}}^c \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}}), \underline{\boldsymbol{\theta}})_K \\
=& (\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{\theta}})_K. && \text{by (5.21)}
\end{aligned}$$

This shows that $\underline{\boldsymbol{P}}^c \underline{\boldsymbol{v}} = \underline{\boldsymbol{\eta}}$. Finally,

$$\begin{aligned}
\|\underline{\boldsymbol{v}}\|_{L^2(\Omega)}^2 =& \|\underline{\boldsymbol{\Pi}}_1 \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}})\|_{L^2(\Omega)}^2 \\
\le& 2\|\underline{\boldsymbol{\Pi}}_1 \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}}) - \mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}})\|_{L^2(\Omega)}^2 \\
&+ 2\|\mathrm{curl}\, (\underline{\boldsymbol{\psi}} - \mathrm{trace}(\underline{\boldsymbol{\psi}})\underline{\boldsymbol{I}})\|_{L^2(\Omega)}^2 \\
\le& C \sum_{K \in \mathcal{T}_h} h_K^2 \|\underline{\boldsymbol{\psi}}\|_{H^2(K)}^2 + C\, \|\underline{\boldsymbol{\psi}}\|_{H^1(\Omega)}^2 \\
\le& C\, \|\underline{\boldsymbol{\psi}}\|_{H^1(\Omega)}^2.
\end{aligned}$$

In the last inequality we used inverse estimates which we are allowed to do since $\underline{\boldsymbol{\psi}}$ is a polynomial on K. Finally, using the definition of $\underline{\boldsymbol{\psi}}$ and (5.20b) we have

$$\|\underline{\boldsymbol{v}}\|_{L^2(\Omega)} \leq C \, \|\underline{\boldsymbol{\eta}}\|_{L^2(\Omega)}.$$

This completes the proof.

□

References

[1] M. Amara and J. M. Thomas, *Equilibrium finite elements for the linear elastic problem*, Numer. Math., 33 (1979), pp. 367–383.

[2] D.N. Arnold, F. Brezzi and J. Douglas, *PEERS: a new mixed finite element for plane elasticity*, Japan J. Appl. Math. 1 (1984), no. 2, 347–367.

[3] D.N. Arnold, R. Falk and R. Winther, *Mixed finite element methods for linear elasticity with weakly imposed symmetry*, Math. Comp. 76 (2007), no. 260, 1699–1723.

[4] F. Brezzi, J. Douglas, R. Durán and M. Fortin, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math. 51 (1987), no. 2, 237–250.

[5] D. Boffi, F. Brezzi and M. Fortin, *Reduced symmetry elements in linear elasticity*, Commun. Pure Appl. Anal. 8 (2009), no. 1, 95–121.

[6] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland Publishing Company, Amsterdam, 1978.

[7] B. Cockburn, J. Gopalakrishnan, and J. Guzmàn, *A new elasticity element made for enforcing weak stress symmetry*, Math. Comp, to appear.

[8] Ph. Clément, *Approximation by finite element functions using local regularization*, RAIRO Anal. Numer. 9 (1975), no. R-2, 77–84.

[9] M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations*. I. Rev. Franaise Automat. Informat. Recherche Opérationnelle Sér. Rouge 7 (1973), no. R-3, 33–75.

[10] R. Falk, *Finite elements for linear elasticity*, in Mixed Finite Elements: Compatibility Conditions (eds. D. Boffi and L. 5Gastaldi), Lecture Notes in Math., 1939 Springer-Verlag, Heilderberg, 2008.

[11] M. Farhloul and M. Fortin, *Dual hybrid methods for the elasticity and the Stokes problems: a unified approach*, Numer. Math., 76 (1997), pp. 419–440.

[12] J. Gopalakrishnan, and J. Guzmán, *A second elasticity element using the matrix bubble with tightened stress symmetry*, submitted.

[13] J.-C. Nédélec, *Mixed Finite Elements in* $\mathbb{R}^3$, Numer. Math., 35 (1980), pp. 315–341.

[14] J.-C. Nédélec, *A new family of mixed finite elements in* $\mathbb{R}^3$ Numer. Math. 50 (1986), no. 1, 57–81.

[15] P.-A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Springer, Berlin, 1977, pp. 292–315. Lecture Notes in Math., Vol. 606.

[16] R. Stenberg, *A family of mixed finite elements for the elasticity problem*, Numer. Math. 53 (1988), no. 5, 513–538.

Division of Applied Mathematics, Brown University, Providence, RI 02912
E-mail address: johnny_guzman@brown.edu