LOCAL AND POINTWISE ERROR ESTIMATES OF THE LOCAL DISCONTINUOUS GALERKIN METHOD APPLIED TO STOKES PROBLEM

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Abstract. We prove local and pointwise error estimates for the local discontinuous Galerkin method applied to Stokes problem in two and three dimensions. By using techniques originally developed by A. Schatz [Math. Comp., 67 (1998), 877-899] to prove pointwise estimates for the Laplace equation, we prove optimal weighted pointwise estimates for both the velocity and the pressure for domains with smooth boundaries.

1. Introduction

In this paper, we study the local and pointwise behavior of the Local Discontinuous Galerkin (LDG) method for the following problem

$$\begin{aligned} -\triangle \vec{u} + \nabla p &= \vec{f} \quad \text{in } \Omega, \\ \nabla \cdot \vec{u} &= g \quad \text{in } \Omega, \\ \vec{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset R^N$ $(N = 2, 3)$ is bounded and has a smooth boundary. Here $\vec{u} = (u_1, \cdots, u_N)$ represents the velocity of the fluid, $p \in L^2_0(\Omega)$ the pressure, $\vec{f} = (f_1, \cdots, f_N)$ is a smooth external force and $g \in L^2_0(\Omega)$ is a smooth function (for Stokes problem we take $g \equiv 0$). The space $L^2_0(\Omega)$ consist of functions in $L^2(\Omega)$ with mean zero.

The LDG method for Stokes problem was introduced by Cockburn et al. [10]; see the review [8]. The LDG finite dimensional spaces for the both the velocity and pressure are discontinuous across interelement boundaries. Therefore, the LDG method allows meshes with hanging nodes and allows flexibility when choosing the local finite element spaces. Cockburn et al. [6] generalized this method to Oseen equations. Finally, in [7] the LDG method was extended to the stationary incompressible Navier-Stokes equation; see also the follow up note [9]. Although the LDG method considered in [10] satisfies the incompressibility condition only weakly, it is shown in [7] that one can enforce exact incompressibility by a simple element by element post-processing technique.

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Global L^2 error analysis was performed in [10] for the LDG method applied to (1.1). In this paper we prove *local* L^2 error estimates along with pointwise error estimates. Roughly speaking, the local L^2 analysis shows that the error for both the pressure and the gradient of the velocity measured by the $L^2(D_0) - norm$ for a subdomain $D_0 \subset \Omega$ is bounded by the best approximation error in the $L^2(D_1) - norm$ for a slightly larger subdomain D_1 plus the error in a weaker norm. These estimates are very similar to the local error estimates obtained by Arnold and Liu [2] for *conforming* mixed methods applied to (1.1). However, the results in [2] are for interior subdomains D_0 whereas in this paper we allow D_0 to touch $\partial\Omega$. Many of the techniques to prove local error estimates presented in this paper and in [2] are borrowed from the techniques developed by Nitsche and Schatz [18] for proving local estimates of conforming finite element methods for the Laplace equation. However, the pressure term and the incompressibility equation adds extra difficulties when analyzing the Stokes problem. Moreover, the fact that the LDG spaces are discontinuous and that the primal formulation of the LDG method does not satisfy the Galerkin orthogonality property adds even more challenges when analyzing the LDG method for (1.1). Local error estimates for the LDG method applied to *Laplace's* equation were carried out by Chen [5]. Later Guzmán [17] proved similar results for three DG methods, including the LDG method, in primal form.

We use the local L^2 error estimates to prove weighted pointwise estimates. These pointwise estimates are optimal and describe how the error at a point x depends on the behavior of the exact solution in regions away from x. Recently, Chen [3] used the local estimates derived in [2] to prove pointwise estimates of *conforming* mixed methods for (1.1) on a domain Ω with a smooth boundary. Chen makes use of techniques originally developed by Schatz [19] to prove pointwise estimates for the Laplace equation. In this paper we also use the techniques found in [19] and our results are very similar to the results contained in [3]. However, in order to prove pointwise estimates Chen *assumed* local error estimates for subdomains that touch $\partial\Omega$ which are *not* contained in [2]. As mentioned above, in this paper we prove local estimates for subdomains that touch $\partial\Omega$ for the LDG method. Furthermore, Chen *assumed* that functions in the finite element subspace for the velocity are zero on $\partial\Omega$, but such spaces are difficult to construct for curved edges. Since we are analyzing the LDG method there is no need to choose subspaces that agree with the boundary data.

To further put our work in perspective, we describe previous work concerning pointwise error estimates for Stokes problem. Pointwise error estimates for conforming mixed methods applied to Stokes problem was first carried out by Durán et al. [12]. For a stabilized Petrov-Galerkin mixed method the analysis was carried out in [14]. The drawback of these articles is that the analysis is two dimensional and the estimates are sub-optimal by a logarithmic factor for higher order elements. Recently, Girault et al. [16] removed the logarithmic factor and extended the results to three dimensions. In this paper and in [3] the logarithmic factor is also not present for higher order elements. The proof in [16] uses techniques for maximum-norm estimates for finite element approximations of the Laplace equation [23] whereas in this paper and in [3] techniques from [19] were used. This allows us to establish a more local dependence of the error on the exact solution as compared to the results in [16]. However, our results are restricted to domains with smooth boundaries whereas the results in [16] hold for polygonal/polyhedral domains. We use an integral representation of solutions to (1.1) and sharp bounds for the kernels whereas in [16] an integral representation for the the inverse of the divergence operator and sharp bounds for that kernel are used; see [15].

Instead of discretizing the viscosity term $-\triangle\vec{u}$ with the LDG method one can discretize this term using methods in [1] to come up with different DG methods for (1.1); see [21]. If we use the methods in [1] that are *consistent*, *adjoint consistent* and have bilinear forms that our *coercive* to discretize the viscosity term of (1.1), then we can easily prove similar results for the resulting methods for (1.1).

The rest of the paper is organized as follows: In the next section we define the LDG method and present our main results. Section 3 contains the proofs of the theorems.

2. The Main Results

2.1. **The LDG Method.** We assume we have a family of triangulations $\mathcal{T}_h$ which fit the boundary of Ω exactly, where $\Omega = \cup_{T\in\mathcal{T}_h} T$. We allow hanging nodes, but we assume our family of meshes are quasi-uniform and that the elements are shape-regular. The collection of edges/faces will be denoted by $\mathcal{E}_h = \mathcal{E}_h^{\mathcal{I}} \cup \mathcal{E}_h^{\mathcal{B}}$ where $\mathcal{E}_h^{\mathcal{I}}$ is the set of interior edges/faces and $\mathcal{E}_h^{\mathcal{B}}$ is the set of boundary edges/faces.

The LDG approximations belong to the following spaces:

$$\begin{aligned}
\vec{V}_h^k &= \{\vec{v} \in [L^2(\Omega)]^N : \vec{v}|_T \in [P_k(T)]^N, \ \ \forall T \in \mathcal{T}_h\},\\
\underline{\Sigma}_h^k &= \{\underline{\sigma} \in [L^2(\Omega)]^{N\times N} : \underline{\sigma}|_T \in [P_k(T)]^{N\times N}, \ \ \forall T \in \mathcal{T}_h\},\\
Q_h^k &= \{q \in L_0^2(\Omega) : \ q|_T \in P_{k-1}(T) \ \ \forall T \in \mathcal{T}_h\},\\
\tilde{Q}_h^k &= \{q \in L^2(\Omega) : \ q|_K \in P_{k-1}(T) \ \ \forall T \in \mathcal{T}_h\}.
\end{aligned}$$

Here $P_l(T)$ are the set of polynomials of degree less than or equal to l defined on T. An arrow above a function means that the function is vector-valued and a line under the function means that the function is matrix-valued.

To write a compact form of the method we will need to define the jump and average operators. The jump operator is given by

$$[\![(\phi \odot \vec{n})]\!] = \begin{cases} (\phi \odot \vec{n}) \quad \text{on boundary edges in } \mathcal{E}_h^{\mathcal{B}} \\ (\phi^+ \odot \vec{n}_{K^+}) + (\phi^- \odot \vec{n}_{K^-}) \quad \text{on interior edges in } \mathcal{E}_h^{\mathcal{I}}, \end{cases}$$

where $\phi^{\pm}$ denote traces of ϕ on the edge $e = \partial K^+ \cap \partial K^-$ taken from within the interior of $K^{\pm}$. The vector $\vec{n}_K$ is the outward unit vector normal to K. The symbol $\odot$ denotes a multiplication operator. The average operator is defined as

$$\{\!\{\phi\}\!\} = \begin{cases} \phi \quad \text{on boundary edges in } \mathcal{E}_h^{\mathcal{B}} \\ \frac{1}{2}(\phi^+ + \phi^-) \quad \text{on interior edges in } \mathcal{E}_h^{\mathcal{I}}. \end{cases}$$

We can now define the LDG approximation. To simplify notations we take the stabilization parameters to be 1 (i.e. $c_{11} = d_{11} = 1$ in (2.21) [10]). Since we are working with quasi-uniform meshes we use h everywhere instead of the local mesh size.

Find $(\vec{u}_h, p_h) \in \vec{V}_h^k \times Q_h^k$ such that

$$\begin{aligned} A_h(\vec{u}_h, \vec{v}) + B_h(\vec{v}, p_h) &= \int_\Omega \vec{f} \cdot \vec{v}\, dx \\ -B_h(\vec{u}_h, q) + D_h(p_h, q) &= \int_\Omega gq dx \qquad \forall (v,q) \in \vec{V}_h^k \times Q_h^k, \end{aligned} \tag{2.1}$$

where

$$A_h(\vec{u}, \vec{v}) = \int_\Omega (\nabla_h \vec{u} - \underline{\mathcal{L}}(\vec{u})) : (\nabla_h \vec{v} - \underline{\mathcal{L}}(\vec{v}))\, dx + h^{-1} \sum_{e \in \mathcal{E}_h} \int_e [\![\vec{u} \otimes \vec{n}]\!] : [\![\vec{v} \otimes \vec{n}]\!]\, ds,$$

$$B_h(\vec{v}, q) = -\int_\Omega q \nabla_h \cdot \vec{v}\, dx + \sum_{e \in \mathcal{E}_h^{\mathcal{I}}} \int_e \{\!\{q\}\!\} [\![\vec{v} \cdot \vec{n}]\!]\, ds + \sum_{e \in \mathcal{E}_h^{\mathcal{B}}} \int_e q \vec{v} \cdot \vec{n} ds,$$

$$D_h(p, q) = h \sum_{e \in \mathcal{E}_h^{\mathcal{I}}} \int_e [\![p\vec{n}]\!] \cdot [\![q\vec{n}]\!]\, ds.$$

For $\vec{u} \in [H_h^1(\Omega)]^N$ the lifting operator $\underline{\mathcal{L}}(\vec{u}) \in \underline{\Sigma}_h^k$ is defined by

$$\int_\Omega \underline{\mathcal{L}}(\vec{u}) : \underline{\sigma}\, dx = \sum_{e \in \mathcal{E}_h} \int_e [\![\vec{u} \otimes \vec{n}]\!] : \{\!\{\underline{\sigma}\}\!\}\, ds \;\; \forall \underline{\sigma} \in \underline{\Sigma}_h^k.$$

We used the standard notation $(\nabla \vec{v})_{ij} = \partial_j v_i$ and $(\nabla \cdot \underline{\sigma})_i = \sum_{i=1}^N \partial_j \sigma_{ij}$. We also have $\vec{v} \cdot \vec{n} = \sum_{i=1}^N v_i n_i$, $(\vec{v} \otimes \vec{n})_{ij} = v_i n_j$ and $\underline{\sigma} : \underline{\tau} = \sum_{i,j=1}^N \sigma_{ij} \tau_{ij}$. Here $\nabla_h \vec{u}$ is the piecewise defined function such that $\nabla_h \vec{u} = \nabla u$ on each element $T \in \mathcal{T}_h$.

By using the lifting operator $\underline{\mathcal{L}}$ we eliminated the unknown σ_h appearing the in the original LDG method [10]. As a result, the Galerkin orthogonality property is not satisfied. That is, if $(\vec{u}, p)$ solves (1.1), then we have

$$\begin{aligned} A_h(\vec{u}, \vec{v}) + B_h(\vec{v}, p) &= \int_\Omega \vec{f} \cdot \vec{v}\, dx + R(\vec{u}, \vec{v}) \\ -B_h(\vec{u}, q) + D_h(p, q) &= \int_\Omega g\, q\, dx \qquad \forall (v,q) \in H_h^1(\Omega) \times L_0^2(\Omega). \end{aligned} \tag{2.2}$$

The residual term $R(\vec{u}, \vec{v})$ is given by

$$R(\vec{u}, \vec{v}) = \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\underline{\Pi}(\nabla \vec{u}) - \nabla \vec{u}\}\!\} : [\![v \otimes n]\!] ds.$$

2.2. **Sobolev Norms.** In order to describe the main results we need to introduce some norms. If $\Omega_0 \subset \Omega$, we define our discontinuous Sobolev space as in [3]:

$$W_h^{r,p}(\Omega_0) = \{v : v \in W^{r,p}(T \cap \Omega_0),\ \forall\ T \in \mathcal{T}_h\}.$$

Let $\Omega_0 \subset \Omega$ then we define the broken norm for $r = 1$ and $1 \le p < \infty$

$$||\vec{v}||^p_{W_h^{1,p}(\Omega_0)} = \sum_{T \in \mathcal{T}_h} ||\nabla \vec{v}||^p_{L^p(T \cap \Omega_0)} + h^{1-p} \sum_{e \in \mathcal{E}_h} ||[\![\vec{v} \otimes \vec{n}]\!]||^p_{L^p(e \cap \Omega_0)}.$$

If $p = \infty$, we define

$$||\vec{v}||_{W_h^{1,\infty}(\Omega_0)} = \sup_{T \in \mathcal{T}_h} ||\nabla \vec{v}||_{L^p(T \cap \Omega_0)} + h^{-1} \sup_{e \in \mathcal{E}_h} ||[\![\vec{v} \otimes \vec{n}]\!]||_{L^\infty(e \cap \Omega_0)}.$$

For the pressure we use the following norm for $1 \leq p < \infty$

$$||q||^p_{L^p_h(\Omega_0)} = ||q||^p_{L^p(\Omega_0)} + h \sum_{e \in \mathcal{E}_h^{\mathcal{I}}} ||[\![q\vec{n}]\!]||^p_{L^p(e\cap\Omega_0)} + h \sum_{e\in\mathcal{E}_h} ||\{\!\{q\vec{n}\}\!\}||^p_{L^p(e\cap\Omega_0)}.$$

For $r > 1$ and $1 \leq p \leq \infty$, we define

$$||\vec{v}||^p_{W_h^{r,p}(\Omega_0)} = \sum_{T\in\mathcal{T}_h} ||\vec{v}||^p_{W^{r,p}(T\cap\Omega_0)}$$

The case $p = \infty$ can be defined similarly. We write $H^r_h = W^{r,2}_h$ for any $r \geq 1$. We will also need to define negative-order Sobolev norms. Let $D \subset \Omega$ and $q \in L^2(D)$ then we define the $H^{-1}(D)$ norm as follows

$$||q||_{H^{-1}(D)} = \sup_{\substack{r\in C_c^\infty(D)\\ ||r||_{H^1(D)}=1}} \int_D qr dx.$$

We present a function space, as in [25], that will let us define a slightly different negative-order norm. If $S \subset D \subset \Omega$, let $\partial_<(S,D) = dist(\partial S \setminus \partial\Omega, \partial D \setminus \partial\Omega)$. The space is defined as follows:

$$C^\infty_<(D) = \{v \in C^\infty : \partial_<(supp(v), D) > 0\}.$$

The $H^{-1}_<(D)$ norm is defined by

$$||q||_{H^{-1}_<(D)} = \sup_{\substack{r\in C_<^\infty(D)\\ ||r||_{H^1(D)}=1}} \int_D qr dx.$$

Notice that $H^{-1}_<(D)$ and $H^{-1}(D)$ norms coincide if $D \subset\subset \Omega$.

2.3. **Local Estimates.** For the rest of this paper Π will denote the L^2 projection into $\tilde{Q}^k_h$, $\vec{\Pi}$ will denote the L^2 projection into $\vec{V}^k_h$ and $\underline{\Pi}$ the L^2 projection into $\underline{\Sigma}^k_h$

Theorem 2.1. *Suppose that* $(\vec{u}_h, p_h) \in \vec{V}^k_h \times Q^k_h$ *and* $(\vec{u}, p) \in [H^1_h(\Omega)]^N \times L^2_0(\Omega)$ *satisfy*

$$\begin{aligned} A_h(\vec{u}-\vec{u}_h, \vec{v}) + B_h(\vec{v}, p-p_h) &= R(\vec{u},\vec{v}) \\ -B_h(\vec{u}-\vec{u}_h, q) + D_h(p-p_h, q) &= 0 \qquad \forall(\vec{v},q) \in \vec{V}^k_h \times Q^k_h, \end{aligned} \tag{2.3}$$

then for $D_0 \subset D_d \subset \Omega$ *with* $\partial_<(D_0, D_d) = d \geq 2\,h$

$$\begin{aligned} &||\vec{u}-\vec{u}_h||_{H^1_h(D_0)} + ||p-p_h||_{L^2(D_0)} + (h \sum_{e\in\mathcal{E}_h^{\mathcal{I}}} ||[\![(p-p_h)\vec{n}]\!]||^2_{L^2(e\cap D_0)})^{1/2} \\ &\leq C(||\vec{u}-\vec{\Pi}\vec{u}||_{H^1_h(D_d)} + h||\vec{u}-\vec{\Pi}(\vec{u})||_{H^2_h(D_d)} + ||p-\Pi p||_{L^2_h(D_d)}) \\ &+Cd^{-1}(||\vec{u}-\vec{\Pi}\vec{u}||_{L^2(D_d)} + ||p-\Pi p||_{H^{-1}_<(D_d)}) \\ &+Cd^{-1}(||\vec{u}-\vec{u}_h||_{L^2(D_d)} + ||p-p_h||_{H^{-1}_<(D_d)}). \end{aligned}$$

In these estimates we have the norm $H^{-1}_<$ norm of the pressure appearing in the right hand side instead of the H^{-1} norm. The H^{-1} norm appears in the estimates found in [2] since in their analysis only interior subdomains were considered.

2.4. **Pointwise Estimates.** We need to define weighted norms in order to describe the results of this section. We will use the weight used in [19], $\sigma_x(y) = \frac{h}{h+|x-y|}$. The weighted norms for $1 \le p < \infty$ are given by

$$||\vec{v}||^p_{W_h^{1,p}(\Omega_0),x,s} = \sum_{T\in\mathcal{T}_h} ||\sigma_x^s \nabla \vec{v}||^p_{L^p(T\cap\Omega_0)} + h^{1-p} \sum_{e\in\mathcal{E}_h} ||\sigma_x^s [\![\vec{v}\otimes\vec{n}]\!]||^p_{L^p(e\cap\Omega_0)},$$

and for $p = \infty$

$$||\vec{v}||_{W_h^{1,\infty}(\Omega_0),x,s} = \sup_{T\in\mathcal{T}_h} ||\sigma_x^s \nabla \vec{v}||_{L^p(T\cap\Omega_0)} + \sup_{e\in\mathcal{E}_h} h_e^{-1} ||\sigma_x^s [\![\vec{v}\otimes\vec{n}]\!]||_{L^\infty(e\cap\Omega_0)}.$$

Also,

$$\begin{aligned} ||q||^p_{L_h^p(\Omega_0),x,s} &= ||\sigma_x^s q||^p_{L^p(e\cap\Omega_0)} \\ &\quad + h \sum_{e\in\mathcal{E}_h^{\mathcal{I}}} ||\sigma_x^s [\![q\vec{n}]\!]||^p_{L^p(e\cap\Omega_0)} + h \sum_{e\in\mathcal{E}_h} ||\sigma_x^s \{\!\{q\}\!\}||^p_{L^p(e\cap\Omega_0)}, \end{aligned}$$

and for $p = \infty$

$$||q||_{L^\infty(\Omega),x,s} = ||\sigma_x^s q||_{L^\infty(\Omega_0)}$$

Now we can state the pointwise error estimates for the velocity.

Theorem 2.2. *Suppose* $(\vec{u}, p) \in [W^{1,\infty}(\Omega)]^N \times L^\infty(\Omega) \cap L_0^2(\Omega)$ *and* $(\vec{u}_h, p_h) \in \vec{V}_h^k \times Q_h^k$ *satisfy (2.3). Let* $x \in \bar{\Omega}$ *and* s *satisfy* $0 \le s \le k-1$*. Then, there exists a constant* C *independent of* $x, (\vec{u}, p), (\vec{u}_h, p_h)$ *and* h *such that*

$$\begin{aligned} |(\vec{u}-\vec{u}_h)(x)| &\le Ch\log(1/h)^{\bar{s}}(||\vec{u}-\vec{\Pi}(\vec{u})||_{W_h^{1,\infty}(\Omega),x,s} + ||p-\Pi(p)||_{L_h^\infty(\Omega),x,s} \\ &\quad + ||\sigma_x^s(\nabla_h\vec{u} - \underline{\Pi}(\nabla_h\vec{u}))||_{L^\infty(\Omega)}) \end{aligned}$$

where $\bar{s} = 0$ *if* $0 \le s < k-1$*, and* $\bar{s} = 1$ *if* $s = k-1$*.*

Notice that if $k \ge 2$ we can take $s = 0 < k-1$ and we get the optimal $L^\infty(\Omega)$ found in [16] for the velocity. But, if we take $s > 0$ then the error at x depends much more on the behavior of the exact solution in regions close to x rather than the behavior of the exact solution in regions far from x. In fact, one can prove error expansion inequalities; see Theorem 4.1 in [19] for the corresponding result for the Laplace equation. If $k = 1$ then we are forced to take $s = 0$ in Theorem 2.2. In this case, the logarithmic factor does appear. Also, we see that the estimates is no longer local. That is, the error of the velocity at the point x depends on the exact solution equally on all of Ω; see [11] for the sharpness of this result for the Laplace equation.

The pointwise estimate for the pressure is given in the next theorem.

Theorem 2.3. *Suppose* $(\vec{u}, p) \in [W^{1,\infty}(\Omega)]^N \times L^\infty(\Omega) \cap L_0^2(\Omega)$ *and* $(\vec{u}_h, p_h) \in \vec{V}_h^k \times Q_h^k$ *satisfy (2.3). Let* $x \in \bar{\Omega}$ *and* s *satisfy* $0 \le s \le k$*. Then, there exists a constant* C *independent of* $x, (\vec{u}, p), (\vec{u}_h, p_h)$ *and* h *such that*

$$\begin{aligned} |(p-p_h)(x)| &\le C\log(1/h)^{\bar{\bar{s}}}(||\vec{u}-\vec{\Pi}(\vec{u})||_{W_h^{1,\infty}(\Omega),x,s} + ||p-\Pi(p)||_{L_h^\infty(\Omega),x,s} \\ &\quad + ||\sigma_x^s(\nabla\vec{u} - \underline{\Pi}(\nabla\vec{u}))||_{L^\infty(\Omega)}) \end{aligned}$$

where $\bar{\bar{s}} = 0$ *if* $0 \le s < k$*, and* $\bar{\bar{s}} = 1$ *if* $s = k$*.*

The logarithmic will not appear in the estimates for the pressure as long as we take $0 \le s < k$ which can always be done since $k \ge 1$. Since we can always choose $0 < s < k$, we see that error of the pressure at the x has more of a dependence on the behavior of the exact solution near x rather than the behavior of the exact solution far from x.

3. Proofs

Before we prove Theorems 2.1, 2.2 and 2.3, we state some preliminaries results.

3.1. **Preliminary results.**

3.1.1. *Continuity Of Bilinear Forms.* We can easily prove the following bound for our lifting operator. For any $\vec{v} \in [W_h^{1,p}(\Omega)]^N$ we have

$$||\underline{\mathcal{L}}(\vec{v})||_{L^p(\Omega)} \le Ch^{1/p-1}(\sum_{e\in\mathcal{E}_h} ||[\![\vec{v}\otimes\vec{n}]\!]||^p_{L^p(e)})^{1/p}$$

Now it easily follows that

$$A_h(\vec{u},\vec{v}) \le C||\vec{u}||_{W_h^{1,l}(\Omega)}||\vec{v}||_{W_h^{1,r}(\Omega)}$$

where $\frac{1}{l}+\frac{1}{r}=1$. In fact, one has

$$A_h(\vec{u},\vec{v}) \le C||\vec{u}||_{W_h^{1,l}(\Omega),x,-s}||\vec{v}||_{W_h^{1,r}(\Omega),x,s}.$$

We also have

$$B(\vec{v},q) \le C||\vec{u}||_{W_h^{1,l}(\Omega),x,-s}||q||_{L_h^r(\Omega),x,s}.$$

3.1.2. *Regularity and Global Error Estimates.* The following result is standard; see [24].

Proposition 3.1. *If $\vec{f} \in [H^l(\Omega)]^N$ and $g \in H^{l+1}(\Omega)\cap L_0^2(\Omega)$ with for $l \ge -1$, then there exists a unique solution $(\vec{u},p) \in H^{l+2}\times H^{l+1}(\Omega)$ of (1.1). Furthermore, the following bound holds*

$$||\vec{u}||_{H^{l+2}(\Omega)} + ||p||_{H^{l+1}(\Omega)} \le C(||f||_{H^l(\Omega)} + ||g||_{H^{l+1}(\Omega)}).$$

Global error estimates were obtained in [10]. Here we state the result in a slightly different form.

Proposition 3.2. *Let $(\vec{u},p)$ solve (1.1) and let $(\vec{u}_h,p_h)$ be the LDG approximation defined by (2.1), then*

$$\begin{aligned}
&||\nabla(\vec{u}-\vec{u}_h)||_{L^2(\Omega)} + ||p-p_h||_{L^2(\Omega)} \\
&\le C(||\nabla(\vec{u}-\vec{\Pi}(\vec{u}))||_{L^2(\Omega)} + ||p-\Pi(p)||_{L^2(\Omega)}) \\
&\quad + C(||\nabla\vec{u}-\underline{\Pi}(\nabla\vec{u})||_{L^2(\Omega)} + h||\nabla\vec{u}-\underline{\Pi}(\nabla\vec{u})||_{H^1(\Omega)}).
\end{aligned}$$

3.1.3. *Approximation.* We start by stating well-known trace inequalities. Let e be an face of $T \in \mathcal{T}_h$. Then, for $1 \leq p \leq \infty$, we have

$$||\phi||_{L^p(e)} \leq C(h^{-\frac{1}{p}}||\phi||_{L^p(T)} + h^{1-\frac{1}{p}}|\phi|_{W^{1,p}(T)}). \tag{3.1}$$

If we restrict ϕ to $P_k(T)$, then

$$||\phi||_{W^{l,t}(T)} \leq Ch^{[\frac{N}{s}-\frac{N}{t}]+l-q}||\phi||_{W^{q,s}(T)}, \tag{3.2}$$

$$||\phi||_{L_p(e)} \leq Ch^{-\frac{1}{p}}||\phi||_{L_p(T)}, \tag{3.3}$$

where C does not depend on ϕ, h, e, or T. Here $1 \leq t \leq s \leq \infty$ and $0 \leq l \leq q$ The following is a standard elementwise approximation result. Let $v \in W_h^{j,p}(\Omega)$ with $0 \leq i \leq j \leq k-1$. Then,

$$||v - \Pi(v)||_{W^{i,p}(T)} \leq Ch^{j-i}|v|_{W^{j,p}(T)}, \quad \forall T \in \mathcal{T}_h, \tag{3.4}$$

where C does not depend on v, h, or T.

The same result holds for $\vec{\Pi}$ and $\underline{\Pi}$, but in these cases $0 \leq i \leq j \leq k$. We now define a standard negative-order Sobolev norm. The following inequalities are similar to Lemma 2.2 in [17].

Lemma 3.3. *Let $\chi \in P_{r-1}(T)$, for and let ω be a smooth function. Suppose there exist constants $C > 0$ and $d \geq h$, $||D^l\omega||_{L^\infty(\Omega)} \leq Cd^{-l}$ for $l = 0, 1, \cdots, r+1$. Then, for $r \geq 2$*

$$|\omega^2\chi|_{H^r(T)} \leq \frac{C}{h^{r-2}}(d^{-1}||\omega\chi||_{H^1(T)} + d^{-2}||\chi||_{L^2(T)}), \tag{3.5a}$$

$$|\omega\chi|_{H^r(T)} \leq \frac{C}{h^{r-2}}(d^{-1}||\chi||_{H^1(T)} + d^{-2}||\chi||^2_{L^2(T)}), \tag{3.5b}$$

and for $r = 1$

$$|\omega^2\chi|_{H^1(T)} \leq Cd^{-1}||\omega\chi||_{L^2(T)}, \tag{3.5c}$$

$$|\omega\chi|_{H^1(T)} \leq Cd^{-1}||\chi||_{L^2(T)}. \tag{3.5d}$$

Here C is independent of ω, χ, T, and h.

Now we state a super-approximation result (see [17]) which easily follows from (3.4) and (3.5a) if we set $r - 1 = k$.

Lemma 3.4. *Let $\partial_<(D_0, D_d) = d > 2h$, where $\omega \in C^\infty_<(D_0)$. Suppose $||D^l\omega||_{L^\infty(S_0)} \leq Cd^{-l}$ for $l = 0, 1, \cdots, k+2$. Then, for all $\vec{v} \in \vec{V}_h^k$*

$$\begin{aligned}&\frac{1}{h}||\omega^2\vec{v} - \vec{\Pi}(\omega^2\vec{v})||_{L^2(D_0)} + ||\omega^2\vec{v} - \vec{\Pi}(\omega^2\vec{v})||_{H^1_h(D_0)}\\ &\leq Ch(d^{-1}||\omega\vec{v}||_{H^1_h(D_d)} + d^{-2}||v||_{L_2(D_d)}),\end{aligned}$$

where C is independent of $\vec{v}$ and ω.

We will also need the following superapproximation result.

Lemma 3.5. *Let ω be as in Lemma 3.4. Then, for all $p \in Q_h$*

$$\begin{aligned}&\frac{1}{h}||\omega^2p - \Pi(\omega^2p)||_{L^2(D_0)} + ||\nabla_h(\omega^2p - \Pi(\omega^2p))||_{L^2(D_0)}\\ &\leq Cd^{-1}(||\omega p||_{L^2(D_d)} + \frac{h}{d}||p||_{L^2(D_d)}).\end{aligned}$$

where C is independent of h, p and ω.

3.2. **Proof of Theorem 2.1.** With a covering argument, as was used in [20], it is enough to show Theorem 2.1 with D_0 and D_d replaced with S_d and S_{2d}, respectively. Here $S_d = B_d \cap \Omega$ and $S_{2d} = B_{2d} \cap \Omega$ and $B_d \subset B_{2d}$ are concentric balls with common center in Ω and of radius d and $2d$, respectively. We prove this result in several steps.

3.2.1. *Step1: Reduce to weighted stability estimates.*

Lemma 3.6. *Let $\omega \in C^\infty_<(S_{3d/2})$ with $\omega \equiv 1$ on S_d and $|D^l\omega|_{L^\infty} \le Cd^{-l}$ for $l = 1, 2, \dots, k+2$, then Theorem 2.1 is implied by the following inequality*

$$\begin{aligned}&||\omega\vec{u}_h||_{H^1_h(\Omega)} + ||\omega p_h||_{L^2(\Omega)} + D_h(\omega p_h, \omega p_h)\\ &\le C(||\vec{u}||_{H^1_h(S_{2d})} + h||\vec{u}||_{H^2_h(S_{2d})} + ||p||_{L^2_h(S_{2d})})\\ &\quad +Cd^{-1}(||\vec{u}_h||_{L^2(S_{2d})} + ||p_h||_{H^{-1}_<(S_{2d})}).\end{aligned} \tag{3.6}$$

Proof. Since $\vec{u} - \vec{u}_h = (\vec{u} - \vec{\Pi}(\vec{u})) - (\vec{u}_h - \vec{\Pi}(\vec{u}))$, Theorem 2.1 follows from

$$\begin{aligned}&||\vec{u} - \vec{u}_h||_{H^1_h(S_d)} + ||p - p_h||_{L^2(S_d)} + (h \sum_{e\in\mathcal{E}^{\mathcal{I}}_h} ||[\![(p - p_h)\vec{n}]\!]||_{L^2(e\cap S_d)})^{1/2}\\ &\le C(||\vec{u}||_{H^1_h(S_{2d})} + h||\vec{u}||_{H^2_h(S_{2d})} + ||p||_{L^2_h(S_{2d})})\\ &\quad +Cd^{-1}(||\vec{u}||_{L^2(S_{2d})} + ||p||_{H^{-1}_<(S_{2d})})\\ &\quad +Cd^{-1}(||\vec{u} - \vec{u}_h||_{L^2(S_{2d})} + ||p - p_h||_{H^{-1}_<(S_{2d})}).\end{aligned}$$

By the triangle inequality this in turn follows from

$$\begin{aligned}&||\vec{u}_h||_{H^1_h(S_d)} + ||p_h||_{L^2(S_d)} + (h \sum_{e\in\mathcal{E}^{\mathcal{I}}_h} ||[\![p_h\vec{n}]\!]||_{L^2(e\cap S_d)})^{1/2}\\ &\le C(||\vec{u}||_{H^1_h(S_{2d})} + h||\vec{u}||_{H^2_h(S_{2d})} + ||p||_{L^2_h(S_{2d})})\\ &\quad +Cd^{-1}(||\vec{u}_h||_{L^2(S_{2d})} + ||p_h||_{H^{-1}_<(S_{2d})}).\end{aligned}$$

Since $\omega \equiv 1$ on S_d we have

$$\begin{aligned}&||\vec{u}_h||_{H^1_h(S_d)} + ||p_h||_{L^2(S_d)} + (h \sum_{e\in\mathcal{E}^{\mathcal{I}}_h} ||[\![p_h\vec{n}]\!]||_{L^2(e\cap S_d)})^{1/2}\\ &\le ||\omega\vec{u}_h||_{H^1_h(S_d)} + ||\omega p_h||_{L^2(S_d)} + D_h(\omega p_h, \omega p_h).\end{aligned}$$

Lemma 3.6 now follows. □

3.2.2. *Step 2: Weighted Stability estimates for the Pressure.* We first estimate the term $||\omega p_h||_{L^2(S_d)}$ in terms of the other terms in the right-hand side of (3.6).

Lemma 3.7. *Let ω be as in Lemma 3.6, then*

$$\begin{aligned}||\omega p_h||^2_{L^2(S_d)} \le\ & C||\omega\vec{u}_h||^2_{H^1_h(S_d)} + CD_h(\omega p_h, \omega p_h)\\ &+C(||\vec{u}||^2_{H^1_h(S_{2d})} + h||\vec{u}||^2_{H^2_h(S_{2d})} + ||p||^2_{L^2_h(S_{2d})})\\ &+Cd^{-2}(||\vec{u}_h||^2_{L^2(S_{2d})} + ||p_h||^2_{H^{-1}_<(S_{2d})}).\end{aligned}$$

Proof. By the triangle inequality we have

$$||\omega p_h||_{L^2(S_{2d})} \le ||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||_{L^2(S_{2d})} + ||\text{avg}_{S_{2d}}(\omega p_h)||_{L^2(S_{2d})} \tag{3.7}$$

where $\text{avg}_{S_{2d}}(\omega p_h) = \frac{1}{|S_{2d}|}\int_{S_{2d}} \omega p_h dx$. It is easy to show that

$$||\text{avg}_{S_{2d}}(\omega p_h)||_{L^2(S_{2d})} \leq \frac{C}{d}||p_h||_{H^{-1}_{<}(S_{2d})}. \tag{3.8}$$

It is well known that one can find a function $\vec{v} \in [H^1(S_{2d})]^N$ (see [15]) that satisfies

$$\begin{aligned} -\nabla \cdot \vec{v} &= \omega p_h - \text{avg}_{S_{2d}}(\omega p_h), \quad &\text{in } S_{2d} \\ \vec{v} &= 0, &\text{on } \partial S_{2d}, \end{aligned}$$

and

$$||\nabla \vec{v}||_{L^2(S_{2d})} \leq C||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||_{L^2(S_{2d})}. \tag{3.9}$$

We define $\vec{v}$ on all of Ω by defining it to be zero outside of S_{2d}.

By the definition of B_h, we have

$$||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} = B_h(\vec{v}, \omega p_h - \text{avg}_{S_{2d}}(\omega p_h))$$

Using integration by parts we can rewrite $B(\cdot,\cdot)$ as

$$B_h(\vec{v}, q) \quad = \quad \int_\Omega \vec{v} \cdot \nabla_h q \; dx + \sum_{e \in \mathcal{E}_h^{\mathcal{I}}} \int_e [\![q\vec{n}]\!] \cdot \{\!\{\vec{v}\}\!\} \; ds.$$

Hence, $B_h(\vec{v}, c) = 0$ if c is a constant. Therefore,

$$\begin{aligned} ||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} &= B_h(\vec{v}, \omega p_h) \\ &= B_h(\vec{v} - \vec{\Pi}(\vec{v}), \omega p_h) + B_h(\vec{\Pi}(\vec{v}), \omega p_h). \end{aligned}$$

The first term is

$$\begin{aligned} B_h(\vec{v} - \vec{\Pi}(\vec{v}), \omega p_h) &= \int_\Omega (\vec{v} - \vec{\Pi}(\vec{v})) \cdot \nabla_h(\omega p_h) dx \\ &\quad - \sum_{e \in \mathcal{E}_h^{\mathcal{I}}} \int_e \{\!\{(\vec{v} - \vec{\Pi}(\vec{v}))\}\!\} \cdot [\![\omega p_h \vec{n}]\!] ds. \end{aligned}$$

It easily follows using (3.1), (3.4) and (3.9) that

$$\begin{aligned} - \sum_{e \in \mathcal{E}_h^{\mathcal{I}}} \int_e \{\!\{\vec{v} - \vec{\Pi}(\vec{v})\}\!\} \cdot [\![\omega p_h \vec{n}]\!] ds \quad &\leq \quad \epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} \\ &\quad + \frac{C}{\epsilon} D_h(\omega p_h, \omega p_h). \end{aligned}$$

Also, we have using (3.4) and (3.9)

$$\begin{aligned} &\int_\Omega (\vec{v} - \vec{\Pi}(\vec{v})) \cdot \nabla_h(\omega p_h) = \int_\Omega (\vec{v} - \vec{\Pi}(\vec{v})) \cdot \nabla_h(\omega p_h - \Pi(\omega p_h)) dx \\ &\leq \epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{Ch^2}{\epsilon d^2}||p_h||^2_{L^2(S_{2d})}, \end{aligned}$$

where we also used

$$||\nabla_h(\omega p_h - \Pi(\omega p_h))||_{L^2(S_{2d})} \leq \frac{C}{d}||p_h||_{L^2(S_{2d})}$$

which follows from (3.5d).

Therefore, after applying inverse estimates we see that

$$\begin{aligned}||\omega p_h - \mathrm{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} &\leq 2\epsilon||\omega p_h - \mathrm{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})}\\ &+\frac{C}{\epsilon}D_h(\omega p_h,\omega p_h) + \frac{C}{\epsilon d^2}||p_h||^2_{H^{-1}(S_{2d})}\\ &+B_h(\vec{\Pi}(\vec{v}),\omega p_h).\end{aligned}$$

A simple exercise shows that

$$B_h(\vec{\Pi}(\vec{v}),\omega p_h) = B(\omega\vec{\Pi}(\vec{v}),p_h) + \int_\Omega p_h\nabla(\omega)\cdot\vec{\Pi}(\vec{v}).$$

Clearly,

$$\int_\Omega p_h\nabla(\omega)\cdot\vec{\Pi}(\vec{v}) = \int_\Omega p_h\nabla(\omega)\cdot(\vec{\Pi}(\vec{v})-\vec{v})dx + \int_\Omega p_h\nabla(\omega)\cdot\vec{v}dx.$$

Using (3.4), (3.9) and inverse estimates we get that

$$\int_\Omega p_h\nabla(\omega)\cdot(\vec{\Pi}(\vec{v})-\vec{v})dx \leq \epsilon||\omega p_h - \mathrm{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{C}{\epsilon d^2}||p_h||^2_{H^{-1}(S_{2d})}.$$

Also, we have

$$\begin{aligned}\int_\Omega p_h\nabla(\omega)\cdot\vec{v}dx &\leq ||\nabla(\omega)\cdot\vec{v}||_{H^1(S_{2d})}||p_h||_{H^{-1}_{<}(S_{2d})}\\ &\leq \frac{1}{d}||\nabla\vec{v}||_{L^2(S_{2d})}||p_h||_{H^{-1}_{<}(S_{2d})}\\ &\leq \epsilon||\omega p_h - \mathrm{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{C}{\epsilon d^2}||p_h||^2_{H^{-1}(S_{2d})}.\end{aligned}$$

Here we used Poincare's inequality $||\vec{v}||_{L^2(S_{2d})} \leq Cd||\nabla\vec{v}||_{L^2(S_{2d})}$.

Therefore,

$$\begin{aligned}||\omega p_h - \mathrm{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} &\leq 4\epsilon||\omega p_h - \mathrm{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} \qquad (3.10)\\ &+\frac{C}{\epsilon d^2}||p_h||^2_{H^{-1}_{<}(S_{2d})}\\ &+\frac{C}{\epsilon}D_h(\omega p_h,\omega p_h) + B_h(\omega\vec{\Pi}(\vec{v}),p_h). \qquad (3.11)\end{aligned}$$

We are left to bound $B_h(\omega\vec{\Pi}(\vec{v}),p_h)$. To this end, we have by (2.3)

$$\begin{aligned}B_h(\omega\vec{\Pi}(\vec{v}),p_h) &= B_h(\vec{\Pi}(\omega\vec{\Pi}(\vec{v})),p_h) + B_h(\omega\vec{\Pi}(\vec{v}) - \vec{\Pi}(\omega\vec{\Pi}(\vec{v})),p_h)\\ &= A_h(\vec{u},\vec{\Pi}(\omega\vec{\Pi}(\vec{v}))) + B_h(\vec{\Pi}(\omega\vec{\Pi}(\vec{v})),p) +\\ &\quad -A_h(\vec{u}_h,\vec{\Pi}(\omega\vec{\Pi}(\vec{v}))) - R(\vec{u},\vec{\Pi}(\omega\vec{\Pi}(\vec{v})))\\ &\quad +B_h(\omega\vec{\Pi}(\vec{v}) - \vec{\Pi}(\omega\vec{\Pi}(\vec{v})),p_h)\\ &= \sum_{i=1}^{6} J_i, \qquad (3.12)\end{aligned}$$

where

$$\begin{aligned}&J_1 = A_h(\vec{u},\vec{\Pi}(\omega\vec{\Pi}(\vec{v}))), && J_2 = B_h(\vec{\Pi}(\omega\vec{\Pi}(\vec{v})),p)\\ &J_3 = -A_h(\vec{u}_h,\omega\vec{\Pi}(\vec{v}) - \vec{\Pi}(\omega\vec{\Pi}(\vec{v}))), && J_4 = B_h(\omega\vec{\Pi}(\vec{v}) - \vec{\Pi}(\omega\vec{\Pi}(\vec{v})),p_h)\\ &J_5 = A_h(\vec{u}_h,\omega\vec{\Pi}(\vec{v})), && J_6 = -R(\vec{u},\vec{\Pi}(\omega\vec{\Pi}(\vec{v}))).\end{aligned}$$

By the continuity of A_h and B_h we have

$$|J_1| + |J_2| \leq C||\vec{\Pi}(\omega\vec{\Pi}(\vec{v}))||_{H_h^1(S_{2d})}(||\vec{u}||_{H_h^1(S_{2d})} + ||p||_{L_h^2(S_{2d})}).$$

Using the triangle inequality we get

$$\begin{aligned} ||\vec{\Pi}(\omega\vec{\Pi}(\vec{v}))||_{H_h^1(S_{2d})} &\leq ||\vec{\Pi}(\omega\vec{\Pi}(\vec{v})) - \omega\vec{\Pi}(\vec{v})||_{H_h^1(S_{2d})} \\ &\quad + ||\omega\vec{\Pi}(\vec{v}) - \omega\vec{v}||_{H_h^1(S_{2d})} + ||\omega\vec{v}||_{H_h^1(S_{2d})}. \end{aligned}$$

It is not difficult to show using approximation properties of $\vec{\Pi}$, Poincare's inequality and the fact that $||D^1\omega||_{L^\infty(S_{2d})} \leq Cd^{-1}$ that

$$||\vec{\Pi}(\omega\vec{\Pi}(\vec{v})) - \omega\vec{\Pi}(\vec{v})||_{H_h^1(S_{2d})} + ||\omega\vec{\Pi}(\vec{v}) - \omega\vec{v}||_{H_h^1(S_{2d})} \leq C||\nabla\vec{v}||_{L^2(S_{2d})}.$$

Moreover, using that the jumps of $\vec{v}$ are zero and Poincare's inequality we have

$$||\omega\vec{v}||_{H_h^1(S_{2d})} = ||\nabla(\omega\vec{v})||_{L^2(S_{2d})} \leq C||\nabla\vec{v}||_{L^2(S_{2d})}.$$

Therefore, after using (3.9) we get

$$||\vec{\Pi}(\omega\vec{\Pi}(\vec{v}))||_{H_h^1(S_{2d})} \leq C||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||_{L^2(S_{2d})}.$$

Hence,

$$|J_1| + |J_2| \leq \epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{C}{\epsilon}(||\vec{u}||^2_{H_h^1(S_{2d})} + ||p||^2_{L_h^2(S_{2d})}).$$

By the continuity of A_h

$$|J_3| \leq ||u_h||_{H_h^1(S_{2d})}||\omega\vec{\Pi}(\vec{v}) - \vec{\Pi}(\omega\vec{\Pi}(\vec{v}))||_{H_h^1(S_{2d})}$$

Using (3.4) and (3.5b) we have

$$\begin{aligned} ||\omega\vec{\Pi}(\vec{v}) - \vec{\Pi}(\omega\vec{\Pi}(\vec{v}))||_{H_h^1(S_{2d})} &\leq Ch(d^{-1}||\nabla_h\vec{\Pi}(\vec{v})||_{L^2(S_{2d})} + d^{-2}||\vec{\Pi}(\vec{v})||_{L^2(S_{2d})}) \\ &\leq \frac{Ch}{d}||\nabla\vec{v}||_{L^2(S_{2d})}, \end{aligned} \tag{3.13}$$

where in the last inequality we used the stability of $\vec{\Pi}$ and Poincare's inequality. Hence, using inverse estimates and Young's inequality we have

$$|J_3| \leq \epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{C}{\epsilon d^2}||\vec{u}_h||^2_{L^2(S_{2d})}.$$

One can easily show using the Cauchy-Schwarz inequality, (3.1) and (3.13) that

$$\begin{aligned} J_4 &= -\sum_{e\in\mathcal{E}_h^{\mathcal{I}}} \int_e \{\!\{\omega\vec{\Pi}(\vec{v}) - \vec{\Pi}(\omega\vec{\Pi}(\vec{v})\}\!\} \cdot [\![p_h\vec{n}]\!] ds \\ &\leq \epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{C}{\epsilon d^2}||p_h||^2_{H^{-1}(S_{2d})}. \end{aligned}$$

To handle the next term we first use integration by parts to get

$$\begin{aligned} J_5 &= A(\omega\vec{u}_h, \vec{\Pi}(\vec{v})) + \int_\Omega \nabla(\omega) \otimes \vec{u}_h : (\nabla\vec{\Pi}(\vec{v}) - \underline{\mathcal{L}}(\vec{\Pi}(\vec{v}))dx \\ &\quad + \int_\Omega \nabla_h u_h : (\underline{\mathcal{L}}(\omega\vec{\Pi}(\vec{v})) - \omega\underline{\mathcal{L}}(\vec{\Pi}(\vec{v}))dx \\ &\quad + \int_\Omega \nabla_h\vec{\Pi}(\vec{v}) : (\underline{\mathcal{L}}(\omega\vec{u}_h) - \omega\underline{\mathcal{L}}(\vec{u}_h))dx. \end{aligned}$$

It simple to see using the continuity of A_h, properties of $\vec{\Pi}$ and (3.9) that

$$A(\omega\vec{u}_h, \vec{\Pi}(\vec{v})) \leq \epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{C}{\epsilon}||\omega\vec{u}_h||^2_{H^1_h(S_{2d})}.$$

Similarly, one has

$$\int_\Omega \nabla(\omega) \otimes \vec{u}_h : (\nabla\vec{\Pi}(\vec{v}) - \underline{\mathcal{L}}(\vec{\Pi}(\vec{v}))dx \leq \epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{C}{\epsilon d^2}||\vec{u}_h||^2_{L^2(\Omega)}.$$

Using the definition of the $\underline{\Pi}$ and $\underline{\mathcal{L}}$ we have

$$\int_\Omega \nabla_h u_h : (\underline{\mathcal{L}}(\omega\vec{\Pi}(\vec{v})) - \omega\underline{\mathcal{L}}(\vec{\Pi}(\vec{v}))dx = \sum_{e\in\mathcal{E}_h} \int_e [\![\vec{\Pi}(\vec{v}) \otimes \vec{n}]\!] : \{\!\{\omega\nabla_h u_h - \underline{\Pi}(\omega\nabla_h u_h)\}\!\}dx.$$

Therefore, using Lemma (3.5d) and (3.9) we get

$$\begin{aligned}&\int_\Omega \nabla_h u_h : (\underline{\mathcal{L}}(\omega\vec{\Pi}(\vec{v})) - \omega\underline{\mathcal{L}}(\vec{\Pi}(\vec{v}))dx \\ &\leq C||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||_{L^2(S_{2d})}\frac{h}{d}||u_h||_{H^1_h(S_{2d})} \\ &\leq \epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{C}{\epsilon d^2}||u_h||^2_{L^2(S_{2d})}.\end{aligned}$$

In a similar fashion we can show

$$\begin{aligned}&\int_\Omega \nabla_h\vec{\Pi}(\vec{v}) : (\underline{\mathcal{L}}(\omega\vec{u}_h) - \omega\underline{\mathcal{L}}(\vec{u}_h))dx \\ &\leq \epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{C}{\epsilon d^2}||u_h||^2_{L^2(S_{2d})}.\end{aligned}$$

Hence,

$$|J_5| \leq 4\epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{C}{\epsilon d^2}||u_h||^2_{L^2(S_{2d})} + \frac{C}{\epsilon}||\omega\vec{u}_h||^2_{H^1_h(S_{2d})}.$$

One can show using (3.1) and approximation properties $\vec{\Pi}$ that

$$|J_6| \leq \epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} + \frac{Ch^2}{\epsilon}||\vec{u}||^2_{H^2_h(S_{2d})}.$$

Hence, combining the bounds for the $J_i's$ and using (3.12) we have that

$$\begin{aligned}B_h(\omega\vec{\Pi}(\vec{v}), p_h) \leq\ & 8\epsilon||\omega p_h - \text{avg}_{S_{2d}}(\omega p_h)||^2_{L^2(S_{2d})} \\ &+\frac{C}{\epsilon}(||\omega\vec{u}_h||^2_{H^1_h(S_{2d})} + D_h(\omega p_h, \omega p_h)) \\ &+\frac{C}{\epsilon}(||\vec{u}||^2_{H^1_h(S_{2d})} + ||p||^2_{L^2_h(S_{2d})} + h^2||\vec{u}||^2_{H^2_h(S_{2d})}) \\ &+\frac{C}{\epsilon d^2}(||u_h||^2_{L^2(S_{2d})} + ||p_h||^2_{H^{-1}(S_{2d})}).\end{aligned} \tag{3.14}$$

Finally, combining (3.14), (3.11), (3.8) and (3.7) and taking ϵ small enough proves Lemma 3.7. □

3.2.3. *Step 3: Weighted Stability Estimates For The Velocity.*

Lemma 3.8. *For every $1 > \delta > 0$ we have*

$$||\omega\vec{u}_h||^2_{H^1_h(S_d)} + D_h(\omega p_h, \omega p_h) \leq \delta||\omega p_h||^2_{L^2(S_d)}$$
$$+\frac{C}{\delta}(||\vec{u}||^2_{H^1_h(S_{2d})} + h||\vec{u}||^2_{H^2_h(S_{2d})} + ||p||^2_{L^2_h(S_{2d})})$$
$$+\frac{C}{\delta d^2}(||\vec{u}_h||^2_{L^2(S_{2d})} + ||p_h||^2_{H^{-1}(S_{2d})}).$$

Proof. In order to prove the result we use a stability result concerning only A_h bilinear form. The proof is almost identical to the proof of a similar inequality in [17] for the LDG method applied to Laplace's equation; see (3.3) in [17].

Proposition 3.9. *There exists a fixed number $C_1 > 1$ such that*

$$||\omega\vec{u}_h||^2_{H^1_h(S_{2d})} \leq C_1 A_h(\vec{u}_h, \omega^2\vec{u}_h) + C_1 d^{-2}||\vec{u}_h||^2_{L^2(S_{2d})}. \tag{3.15}$$

Therefore, we need only to find a bound for $A_h(\vec{u}_h, \omega^2\vec{u}_h)$. To this end, we use (2.3) to write

$$\begin{aligned} &A_h(\vec{u}_h, \omega^2\vec{u}_h) \\ &= A_h(\vec{u}_h, \omega^2\vec{u}_h - \vec{\Pi}(\omega^2\vec{u}_h)) + A_h(u_h, \vec{\Pi}(\omega^2\vec{u}_h)) \\ &= A_h(\vec{u}_h, \omega^2\vec{u}_h - \vec{\Pi}(\omega^2\vec{u}_h)) + A_h(\vec{u}, \vec{\Pi}(\omega^2\vec{u}_h)) + B_h(\vec{\Pi}(\omega^2\vec{u}_h), p) \\ &\quad -B_h(\vec{\Pi}(\omega^2\vec{u}_h), p_h) - R(\vec{u}, \vec{\Pi}(\omega^2\vec{u}_h)) \\ &= \sum_{j=1}^{6} I_j \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} I_1 &= A_h(\vec{u}_h, \omega^2\vec{u}_h - \vec{\Pi}(\omega^2\vec{u}_h)), & I_2 &= A_h(\vec{u}, \vec{\Pi}(\omega^2\vec{u}_h)), \\ I_3 &= B_h(\vec{\Pi}(\omega^2\vec{u}_h), p), & I_4 &= -B_h(\vec{\Pi}(\omega^2\vec{u}_h) - \omega^2\vec{u}_h, p_h), \\ I_5 &= -B_h(\omega^2\vec{u}_h, p_h), & I_6 &= -R(\vec{u}, \vec{\Pi}(\omega^2\vec{u}_h)). \end{aligned}$$

Here we used that (2.3) holds if Q_h^k is replaced with $\tilde{Q}_h^k$ since $B_h(\vec{v}, c) = 0$ for constant c.

By the continuity of A_h and Lemma 3.4 we see that

$$\begin{aligned} I_1 &\leq Ch||\vec{u}_h||_{H^1_h(S_{3d/2})}(\frac{1}{d}||\omega\vec{u}_h||_{H^1_h(S_{2d})} + \frac{1}{d^2}||\vec{u}_h||_{L^2(S_{2d})}) \\ &\leq \delta_1||\omega\vec{u}_h||^2_{H^1_h(S_{2d})} + \frac{1}{\delta_1 d^2}||\vec{u}_h||^2_{L^2(S_{2d})}. \end{aligned}$$

In the last step we used the inverse estimate $h||\vec{u}_h||_{H^1_h(S_{3/2d})} \leq ||\vec{u}_h||_{L^2(S_{2d})}$. Here δ_1 is a small positive number that will be chosen later.

By the continuity of A_h, the stability of $\vec{\Pi}$ and Young's inequality we have

$$I_2 \leq \delta_1||\omega\vec{u}_h||^2_{H^1_h(S_{2d})} + \frac{C}{\delta_1}||\vec{u}||^2_{H^1_h(S_{2d})} + \frac{C}{\delta_1 d^2}||\vec{u}_h||^2_{L^2(S_{2d})}.$$

By applying Hölder's inequality, the stability of $\vec{\Pi}$ and Lemma 3.4 we can easily show that

$$I_3 \leq \delta_1 ||\omega \vec{u}_h||^2_{H^1_h(S_{2d})} + \frac{C}{\delta_1} ||p||^2_{L^2_h(S_{2d})} + \frac{C}{\delta_1 d^2} ||\vec{u}_h||^2_{L^2(S_{2d})}.$$

The following can be written as

$$I_4 = -\sum_{e \in \mathcal{E}_h^{\mathcal{I}}} \int_e \{\!\{\vec{\Pi}(\omega^2 \vec{u}_h) - \omega^2 \vec{u}_h\}\!\} \cdot [\![p_h \vec{n}]\!].$$

By applying the Cauchy-Schwarz inequality, (3.1) and (3.2)

$$I_4 \leq \frac{C}{h} ||p_h||_{L^2(S_{2d})} (||\vec{\Pi}(\omega^2 \vec{u}_h) - \omega^2 \vec{u}_h||_{L^2(S_{2d})} + h ||\nabla_h (\vec{\Pi}(\omega^2 \vec{u}_h) - \omega^2 \vec{u}_h)||_{L^2(S_{2d})}).$$

By applying Lemma 3.4, inverse estimates and Young's inequality we get

$$\begin{aligned} I_4 \quad &\leq \quad \delta_1 ||\omega \vec{u}_h||^2_{H^1(S_{2d})} + C d^{-2} ||\vec{u}_h||^2_{L^2(S_{2d})} \\ &\quad + C(1 + \frac{1}{\delta_1}) d^{-2} ||p_h||^2_{H^{-1}(S_{2d})}. \end{aligned}$$

Using the product rule we get

$$\begin{aligned} I_5 \quad &= \quad -B_h(\vec{u}_h, \omega^2 p_h) - 2 \int_\Omega \omega p_h \vec{u}_h \cdot \nabla(\omega) \, dx \\ &= \quad -B_h(\vec{u}_h, \omega^2 p_h - \Pi(\omega^2 p_h)) - B_h(\vec{u}_h, \Pi(\omega^2 p_h)) \\ &\quad -2 \int_\Omega \omega p_h \vec{u}_h \cdot \nabla(\omega) \, dx. \end{aligned}$$

By using Lemma 3.5 and inverse estimates we get

$$\begin{aligned} -B_h(\vec{u}_h, \omega^2 p_h - \Pi(\omega^2 p_h)) \quad &\leq \quad \delta_1 ||\omega p_h||^2_{L^2(S_{2d})} + \frac{C}{d^2} ||p_h||^2_{H^{-1}(S_{2d})} \\ &\quad + \frac{C}{d^2} (1 + \frac{1}{\delta_1}) ||\vec{u}_h||^2_{L^2(S_{2d})}. \end{aligned}$$

It easily follows that

$$\int_\Omega \omega p_h \vec{u}_h \cdot \nabla(\omega) \, dx \leq \delta_1 ||\omega p_h||^2_{L^2(S_{2d})} + \frac{C}{\delta_1 d^2} ||\vec{u}_h||^2_{L^2(S_{2d})}.$$

Using (2.3) we have

$$\begin{aligned} -B_h(\vec{u}_h, \Pi(\omega^2 p_h)) \quad &= \quad B_h(\vec{u}, \Pi(\omega^2 p_h)) - D_h(p_h - p, \Pi(\omega^2 p_h)) \\ &= \quad -B_h(\vec{u}, \Pi(\omega^2 p_h)) - D_h(p_h - p, \Pi(\omega^2 p_h)) \\ &= \quad -B_h(\vec{u}, \Pi(\omega^2 p_h)) - D_h(p_h, \omega^2 p_h) \\ &\quad -D_h(p_h, \Pi(\omega^2 p_h) - \omega^2 p_h) + D_h(p, \Pi(\omega^2 p_h)). \end{aligned}$$

By using inverse estimates and stability of the L^2 projection Π, we have

$$-B_h(\vec{u}, \Pi(\omega^2 p_h)) \leq \delta_1 ||\omega p_h||^2_{L^2(S_{2d})} + \frac{C}{\delta_1} ||\vec{u}_h||^2_{H^1_h(S_{2d})},$$

and

$$D_h(p, \Pi(\omega^2 p_h)) \leq \delta_1 ||\omega p_h||^2_{L^2(S_{2d})} + \frac{C\,h}{\delta_1} \sum_{e \in \mathcal{E}_h^{\mathcal{I}}} ||[\![p\vec{n}]\!]||^2_{L^2(e \cap S_{2d})}.$$

If we use Lemma 3.5 and inverse estimates we get

$$-D_h(p_h, \Pi(\omega^2 p_h) - \omega^2 p_h) \leq \delta_1 ||\omega p_h||^2_{L^2(S_{2d})} + \frac{C}{d^2}(1+\delta_1^{-1})||p_h||_{H^{-1}(S_{2d})}.$$

Hence,

$$\begin{aligned} I_5 \quad \leq \quad & -D_h(\omega p_h, \omega p_h) + 4\delta_1 ||\omega p_h||^2_{L^2(S_{2d})} \\ & + \frac{C}{\delta_1}(||p||^2_{L^2_h(S_{2d})} + ||\vec{u}||^2_{H^1_h(S_{2d})}) \\ & + \frac{C}{\delta_1 d^2}(||p_h||^2_{H^{-1}(S_{2d})} + ||\vec{u}_h||^2_{L^2(S_{2d})}), \end{aligned}$$

where we used that $\delta_1 < 1$.

Finally, by applying Lemma 3.4 we can show

$$\begin{aligned} I_6 \quad \leq \quad & \delta_1 ||\omega \vec{u}_h||^2_{H^1_h(\Omega)} \\ & + \frac{C}{d^2}||\vec{u}_h||^2_{L^2(S_{2d})} + Ch^2(1+\delta_1^{-1})||\vec{u}||^2_{H^2_h(S_{2d})}. \end{aligned}$$

Therefore, by combining (3.15), (3.16), the bounds for I_j, $j = 1, \cdots, 6$ and choosing δ_1 so that $\delta_1 C_1 \leq 1/2$ proves Lemma 3.8 where we let $\delta = 20\,\delta_1\, C_1$. □

3.2.4. *Step 4: Completion of the proof.* By combining Lemmas 3.8 and 3.7 and taking δ sufficiently small we get

$$\begin{aligned} & ||\omega \vec{u}_h||^2_{H^1_h(S_d)} + D_h(\omega p_h, \omega p_h) \\ & \leq C(||\vec{u}||^2_{H^1_h(S_{2d})} + h||\vec{u}||^2_{H^2_h(S_{2d})} + ||p||^2_{L^2_h(S_{2d})}) \\ & \quad + \frac{C}{d^2}(||\vec{u}_h||^2_{L^2(S_{2d})} + ||p_h||^2_{H^{-1}_{<}(S_{2d})}). \end{aligned}$$

Finally, by combining this inequality with Lemma 3.7 gives us (3.6) and hence completes the proof of Theorem 2.1.

3.3. **Proof of Theorem 2.2.**

3.3.1. *Step 1: Reduce To Error Estimates For Approximate Greens Function.*

Lemma 3.10. *Let $x \in T_x$ where $T_x \in \mathcal{T}_h$ and let $\vec{\rho} \in [C^\infty_c(T_x)]^N$ with $||\vec{\rho}||_{L^2(T_x)} = h^{-N/2}$. Then, Theorem 2.2 follows from*

$$||\vec{g} - \vec{g}_h||_{W^{1,1}_h(\Omega),x,-s} + ||\lambda - \lambda_h||_{L^1(\Omega),x,-s} \leq \log(1/h)^{\bar{s}} h,$$

where $(\vec{g}, \lambda)$ (with $\int_\Omega \lambda \, dx = 0$) solve

$$\begin{aligned} -\triangle \vec{g} + \nabla \lambda &= \vec{\rho} \\ \nabla \cdot \vec{g} &= 0 \\ \vec{g} &= 0 \end{aligned} \tag{3.17}$$

and $(\vec{g}_h, \lambda_h) \in \vec{V}^k_h \times Q^k_h$ satisfy

$$\begin{aligned} A_h(\vec{g}_h, \vec{v}) + B_h(\vec{v}, \lambda_h) &= \int_\Omega \vec{\rho} \cdot \vec{v} dx \\ -B_h(\vec{g}_h, q) + D_h(\lambda_h, q) &= 0 \qquad \forall (\vec{v}, q) \in \vec{V}^k_h \times Q^k_h. \end{aligned} \tag{3.18}$$

Proof. By the triangle inequality and inverse estimates

$$|(\vec{u}-\vec{u}_h)(x)| \le |(\vec{u}-\vec{\Pi}(\vec{u}))(x)| + Ch^{-N/2}||\vec{\Pi}(\vec{u})-\vec{u}_h||_{L^2(T_x)}.$$

By the triangle inequality and Hölder's inequality, we have

$$|(\vec{u}-\vec{u}_h)(x)| \le C||\vec{u}-\vec{\Pi}(\vec{u})||_{L^\infty(T_x)} + Ch^{-N/2}||\vec{u}-\vec{u}_h||_{L^2(T_x)}.$$

Using the fact that $1/2 \le \sigma_x(y)$ for any $y \in T_x$, we have

$$|(\vec{u}-\vec{u}_h)(x)| \le Ch||\vec{u}||_{W_h^{1,\infty}(\Omega),x,s} + Ch^{-N/2}||u-u_h||_{L^2(T_x)}.$$

Since $\vec{u}-\vec{u}_h = \vec{u}-\vec{\Pi}(\vec{u}) - (\vec{u}_h - \vec{\Pi}(\vec{u}))$, we easily see that

$$|(\vec{u}-\vec{u}_h)(x)| \le Ch||(\vec{u}-\vec{\Pi}(\vec{u}))||_{W_h^{1,\infty}(\Omega),x,s} + Ch^{-N/2}||\vec{u}-\vec{u}_h||_{L_2(T_x)}.$$

We will use that

$$h^{-N/2}||\vec{u}-\vec{u}_h||_{L^2(T_x)} = \sup_{\substack{\rho\in C_c^\infty(T_x)\\ ||\vec{\rho}||_{L^2(T_x)}=h^{-N/2}}} \int_\Omega (\vec{u}-\vec{u}_h)\cdot\vec{\rho}\ dx.$$

For a fix $\vec{\rho}$ let $(\vec{g},\lambda)$ and $(\vec{g}_h,\lambda_h)$ be the solutions of (3.17) and (3.18), respectively. By using the consistency result for the LDG method and (2.3) we have

$$\begin{aligned}\int_\Omega \vec{\rho}\cdot(\vec{u}-\vec{u}_h)dx &= A_h(\vec{u}-\vec{u}_h,g) + B_h(\vec{u}-\vec{u}_h,\lambda) - R(\vec{g},\vec{u}-\vec{u}_h)\\
&= A_h(\vec{u}-\vec{u}_h,\vec{g}-\vec{g}_h) - B(\vec{g}_h,p-p_h) + R(\vec{u},\vec{g}_h)\\
&\quad +B_h(\vec{u}-\vec{u}_h,\lambda-\lambda_h) + D_h(\lambda_h,p-p_h) - R(\vec{g},\vec{u}-\vec{u}_h)\\
&= A_h(\vec{u}-\vec{u}_h,\vec{g}-\vec{g}_h) + B_h(\vec{u}-\vec{u}_h,\lambda-\lambda_h) - R(\vec{g},\vec{u}-\vec{u}_h)\\
&\quad +B(\vec{g}-\vec{g}_h,p-p_h) - D_h(\lambda-\lambda_h,p-p_h) + R(\vec{u},\vec{g}_h)\\
&= A_h(\vec{u}-\vec{\Pi}(\vec{u}),\vec{g}-\vec{g}_h) + B_h(\vec{u}-\vec{\Pi}(\vec{u}),\lambda-\lambda_h)\\
&\quad -R(\vec{g},\vec{u}-\vec{\Pi}(\vec{u})) + B(\vec{g}-\vec{g}_h,p-\Pi(p))\\
&\quad -D_h(\lambda-\lambda_h,p-\Pi(p)) + R(\vec{u},\vec{g}_h).\end{aligned}$$

Hence, by the continuity of our bilinear forms and the definition of $R(\cdot,\cdot)$ we have

$$\begin{aligned}&\int_\Omega \vec{\rho}\cdot(\vec{u}-\vec{u}_h)dx\\
&\le C(||\vec{u}-\vec{\Pi}(\vec{u})||_{W_h^{1,\infty}(\Omega),x,s} + ||\sigma_x^s(\nabla_h\vec{u}-\underline{\Pi}(\nabla_h\vec{u}))||_{L^\infty(\Omega)}\\
&\qquad +||p-\Pi(p)||_{L^\infty(\Omega),x,s})\times\\
&(||\vec{g}-\vec{g}_h||_{W_h^{1,1}(\Omega),x,-s} + h\sum_{e\in\mathcal{E}_h}||\sigma_x^{-s}\{\!\{\nabla\vec{g}-\underline{\Pi}(\nabla\vec{g})\}\!\}||_{L^1(e)} + ||\lambda-\lambda_h||_{L^1(\Omega),x,-s}).\end{aligned}$$

Here we also used that $R(\vec{u},\vec{g}_h) = R(\vec{u},\vec{g}_h-\vec{g})$ since the jumps of $\vec{g}$ are zero.

Theorem 2.2 will follow if we can show

$$\begin{aligned}&||\vec{g}-\vec{g}_h||_{W_h^{1,1}(\Omega),x,-s} + h\sum_{e\in\mathcal{E}_h}||\sigma_x^{-s}\{\!\{\nabla\vec{g}-\underline{\Pi}(\nabla\vec{g})\}\!\}||_{L^1(e)} + ||\lambda-\lambda_h||_{L^1(\Omega),x,-s}\\
&\quad\le C\log(1/h)^{\bar{s}}h.\end{aligned}$$

By using the approximation properties of $\underline{\Pi}$, global regularity bounds and Proposition 3.14 one can show

$$h \sum_{e \in \mathcal{E}_h} ||\sigma_x^{-s} \{\!\{ \nabla \vec{g} - \underline{\Pi}(\nabla \vec{g}) \}\!\} ||_{L^1(e)} \le C \log(1/h)^{\bar{s}} h. \tag{3.19}$$

We leave the details to the reader. This completes the proof of Lemma 3.10. □

3.3.2. *Step 2: Dyadic Decomposition And Error Estimates For Approximate Greens Functions.*

Lemma 3.11. *Let $(\vec{g}, \lambda)$ and $(\vec{g}_h, \lambda_h)$ be as in Lemma 3.10*

$$||\vec{g} - \vec{g}_h||_{W_h^{1,1}(\Omega),x,-s} + ||\lambda - \lambda_h||_{L^1(\Omega),x,-s} \le \log(1/h)^{\bar{s}} h,$$

Proof. Let

$$d_j = 2^{-j} \text{ for } j = 0, 1, 2, \dots$$

and set

$$\begin{aligned}
\Omega_j &= \{y \in \Omega : d_{j+1} < |y - x| < d_j\}, \\
\Omega_j^{(1)} &= \{y \in \Omega : d_{j+2} < |y - x| < d_{j-1}\}, \\
\Omega_j^{(2)} &= \{y \in \Omega : d_{j+3} < |y - x| < d_{j-2}\}, \\
\Omega_j^{(3)} &= \{y \in \Omega : d_{j+4} < |y - x| < d_{j-3}\}, \\
\Omega_j^{(4)} &= \{y \in \Omega : d_{j+5} < |y - x| < d_{j-4}\}.
\end{aligned}$$

We now state two important lemmas that we need. The proofs can be found in the next subsection.

Lemma 3.12. *If $d_j > 8h$, then*

$$\begin{aligned}
&||\vec{g} - \vec{g}_h||_{H_h^1(\Omega_j)} + ||\lambda - \lambda_h||_{L^2(\Omega_j)} \\
&\le C h^k d_j^{1-k-N/2} + d_j^{-1} (||\vec{g} - \vec{g}_h||_{L^2(\Omega_j^{(1)})} + C ||\lambda - \lambda_h||_{H_<^{-1}(\Omega_j^{(1)})}).
\end{aligned}$$

Lemma 3.13. *If $d_j > 8h$, then*

$$\begin{aligned}
&||\vec{g} - \vec{g}_h||_{L^2(\Omega_j^{(1)})} + ||\lambda - \lambda_h||_{H_<^{-1}(\Omega_j^{(1)})} \\
&\le C h^k d_j^{1-k-N/2} (||\vec{g} - \vec{g}_h||_{W_h^{1,1}(\Omega)} + ||\lambda - \lambda_h||_{L_h^1(\Omega)}) \\
&\quad + C h (||\vec{g} - \vec{g}_h||_{H_h^1(\Omega_j^{(4)})} + ||\lambda - \lambda_h||_{L_h^2(\Omega_j^{(4)})}) + C \log(\frac{1}{h})^{\bar{s}} h^{k+1} d_j^{1-N/2-k}.
\end{aligned}$$

Let M be a real number to be determined later and let J be an integer such that $d_J = Mh$. Set $\vec{E} = \vec{g} - \vec{g}_h$ and $r = \lambda - \lambda_h$. Notice that

$$\begin{aligned}
||\vec{E}||_{W_h^{1,1}(\Omega),x,-s} + ||r||_{L_h^1(\Omega),x,-s} &\le ||\vec{E}||_{W_h^{1,1}(S_{Mh}),x,-s} + ||r||_{L_h^1(S_{Mh}),x,-s} \\
&\quad + \sum_{j=0}^{J} (||\vec{E}||_{W_h^{1,1}(\Omega_j),x,-s} + ||r||_{L_h^1(\Omega_j),x,-s}).
\end{aligned}$$

Without loss of generality we have assumed that $diam(\Omega) \le 1$. Since $\sigma_x^{-s}(z) \le \frac{C d_j^s}{h^s}$ for $z \in \Omega_j$, using the fact that $\text{meas}(\Omega_j) \le C d_j^N$ and applying Hölder's inequality we can show

$$||\vec{E}||_{W_h^{1,1}(\Omega_j),x,-s} + ||r||_{L_h^1(\Omega_j),x,-s} \le C d_j^{N/2+s} h^{-s}(||E||_{H_h^1(\Omega_j)} + ||r||_{L_h^2(\Omega_j)}).$$

One also has

$$\begin{aligned} ||\vec{E}||_{W_h^{1,1}(S_{Mh}),x,-s} + ||r||_{L_h^1(S_{Mh})} &\le CM^{N/2+s}h^{N/2}(||\vec{E}||_{H_h^1(S_{Mh})} + ||r||_{L_h^2(S_{Mh})}) \\ &\le CM^{N/2+s}h^{N/2+1}(||\vec{g}||_{H^2(\Omega)} + ||\lambda||_{H^1(\Omega)}) \\ &\le CM^{N/2+s}hh^{N/2}||\vec{\rho}||_{L^2(\Omega)} \le ChM^{N/2+s}. \end{aligned}$$

Here we used global error estimates and regularity results.

Therefore, we have

$$||\vec{E}||_{W_h^{1,1}(\Omega),x,-s} + ||r||_{L_h^1(\Omega),x,-s} \le CM^{N/2+s}h + C\eta, \tag{3.20}$$

where

$$\eta = \sum_{j=0}^{J} d_j^{N/2+s} h^{-s}(||\vec{E}||_{H_h^1(\Omega_j)} + ||r||_{L_h^2(\Omega)}).$$

If we apply Lemma 3.12, we get

$$\eta \le Ch\Theta(k-1-s) + C\sum_{j=0}^{J} d_j^{N/2-1+s} h^{-s}(||\vec{E}||_{L^2(\Omega_j^{(1)})} + ||r||_{H_<^{-1}(\Omega_j^{(1)})}),$$

where

$$\Theta(\alpha) = \sum_{j=0}^{J} (h/d_j)^{\alpha}.$$

Now applying Lemma 3.13, we have

$$\begin{aligned} &\sum_{j=0}^{J} d_j^{N/2-1+s} h^{-s}(||\vec{E}||_{L_2(\Omega_j^{(1)})} + ||r||_{H_<^{-1}(\Omega_j^{(1)})}) \\ &\le C\Theta(k-s)(||\vec{E}||_{W_h^{1,1}(\Omega)} + ||r||_{L_h^1(\Omega)}) \\ &\quad + C\sum_{j=0}^{J} d_j^{N/2-1+s} h^{1-s}(||\vec{E}||_{H_h^1(\Omega_j^{(4)})} + ||r||_{L_h^2(\Omega_j^{(4)})}) \\ &\le C\Theta(k-s)(||\vec{E}||_{W_h^{1,1}(\Omega)} + ||r||_{L_h^1(\Omega)}) \\ &\quad + Cd_J^{N/2-1+s} h^{1-s}(||\vec{E}||_{H_h^1(S_{Mh})} + ||r||_{L_h^2(S_{Mh})}) + \frac{C}{M}\eta \\ &\le C\Theta(k-s)(||\vec{E}||_{W_h^{1,1}(\Omega)} + ||r||_{L_h^1(\Omega)}) \\ &\quad + ChM^{N/2-1+s} + \frac{C}{M}\eta. \end{aligned}$$

Therefore,

$$\begin{aligned} \eta &\le Ch\Theta(k-1-s) + C\Theta(k-s)(||\vec{E}||_{W_h^{1,1}(\Omega)} + ||r||_{L_h^1(\Omega)}) \\ &\quad + M^{N/2-1+s}h + \frac{C}{M}\eta. \end{aligned}$$

By Choosing M sufficiently large we have

$$\eta \le Ch\Theta(k-1-s) + C\Theta(k-s)(||\vec{E}||_{W_h^{1,1}(\Omega)} + ||r||_{L_h^1(\Omega)}) + M^{N/2-1+s}h.$$

Substituting this bound into (3.20), we have

$$\begin{aligned} ||\vec{E}||_{W_h^{1,1}(\Omega),x,-s} + ||r||_{L_h^1(\Omega),x,-s} &\leq CM^{N/2+s}h + Ch\Theta(k-1-s) \\ &+C\Theta(k-s)(||\vec{E}||_{W_h^{1,1}(\Omega)} + ||r||_{L_h^1(\Omega)}). \end{aligned} \tag{3.21}$$

In particular, we see that

$$\begin{aligned} ||\vec{E}||_{W_h^{1,1}(\Omega)} + ||r||_{L_h^1(\Omega)} &\leq CM^{N/2+s}h + Ch\Theta(k-1) \\ &+C\Theta(k-1)(||E|||_{W_h^{1,1}(\Omega)} + ||r||_{L_h^1(\Omega)}). \end{aligned}$$

Since $J \leq \log(\frac{1}{h})$, we obtain

$$\Theta(\alpha) \leq C \begin{cases} \log(\frac{1}{h}) & \text{if } \alpha = 0 \\ M^{-\alpha} \frac{1-(1/2)^{\alpha \log(\frac{1}{h})}}{1-(1/2)^{\alpha}} & \text{if } \alpha > 0. \end{cases}$$

Hence, by choosing M large enough so that $C\Theta(k) \leq 1/2$, we get

$$||\vec{E}||_{W_h^{1,1}(\Omega)} + ||r||_{L_h^1(\Omega)} \leq CM^{N/2+s}h + Ch\Theta(k-1).$$

By this inequality and (3.21), we have

$$\begin{aligned} &||\vec{E}||_{W_h^{1,1}(\Omega),x,-s} + ||r||_{L_h^1(\Omega),x,-s} \\ &\leq CM^{N/2+s}h + Ch\Theta(k-1-s) + Ch\Theta(k-s)\Theta(k-1) \\ &\leq Ch\log(\frac{1}{h})^{\bar{s}}. \end{aligned}$$

which proves Lemma 3.11 □

3.3.3. *Step 3: Proof of Lemmas 3.12 and 3.13.* In order to complete the proof of Theorem 2.2 it remains to prove Lemmas 3.12 and 3.13.

We first prove Lemma 3.12

Proof. From Theorem 2.1 and approximation properties we get

$$\begin{aligned} &||\vec{g} - \vec{g}_h||_{H_h^1(\Omega_j)} + ||\lambda - \lambda_h||_{L^2(\Omega_j)} \\ &\leq Ch^k(|\vec{g}|_{H^{k+1}(\Omega_j^{(1)})} + |\lambda|_{H^k(\Omega_j^{(1)})}) \\ &+Cd_j^{-1}(||\vec{g} - \vec{g}_h||_{L_2(\Omega_j^1)} + ||\lambda - \lambda_h||_{H_<^{-1}(\Omega_j^1)}). \end{aligned}$$

We need only to approximate $|\vec{g}|_{H^{k+1}(\Omega_j^{(2)})} + |\lambda|_{H^k(\Omega_j^{(2)})}$. In order to do so we will use a Greens's function representation of $\vec{g}$ and λ. The result is contained in ([22], Theorem 1.1).

Proposition 3.14. *Let* $(\vec{v}, q)$ *(with* $\int_\Omega q\, dx = 0$*)solve*

$$\begin{aligned} -\triangle \vec{v} + \nabla q &= \vec{m} \quad \text{in } \Omega, \\ \nabla \cdot \vec{v} &= r \quad \text{in } \Omega, \\ \vec{v} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

with $\vec{m} \in [L^2(\Omega)]^N$ *and* $r \in H^1(\Omega)$ *with* $\int_\Omega r dx = 0$*. Then, the pair* $(\vec{v}, q)$ *have the following representations*

$$\vec{v}(x) = \int_\Omega (\underline{G}(x,y)\vec{m}(y) + \vec{\Psi}(x,y)r(y) + \underline{\Gamma}(x,y)\nabla r(y))dy$$

and

$$q(x) \quad = \quad \int_{\Omega}(\vec{H}(x,y)\cdot\vec{m}(y)+\Theta(x,y)r(y)+\vec{\Xi}(x,y)\cdot\nabla r(y))dy.$$

Furthermore, for $1\leq i,j\leq N$

$$|D_x^{\beta}D_y^{\alpha}\underline{G}_{ij}(x,y)|\leq\frac{C}{|x-y|^{N-2+|\beta|+|\alpha|}},\qquad \textit{for } N-2+|\alpha|+|\beta|>0,$$
$$|\underline{G}_{ij}(x,y)|\leq C(1+\log(\frac{1}{|x-y|})),\qquad \textit{for } N-2+|\alpha|+|\beta|=0,$$
$$|D_x^{\beta}D_y^{\alpha}\vec{H}_i(x,y)|\leq\frac{C}{|x-y|^{N-1+|\beta|+|\alpha|}},\qquad \textit{for } N-2+|\alpha|+|\beta|\geq 0.$$

The components of $\vec{\Psi}$ *and* $\underline{\Gamma}$ *have the same bounds as the components of* $\underline{G}$ *and* Θ *and the components of* $\vec{\Xi}$ *have the same bounds as the components of* $\vec{H}$.

Applying Proposition 3.14 and using that $\vec{\rho}$ has support in T_x and that $d_j>8h$ we get for $z\in\Omega_j^{(1)}$ and $|\beta|=k+1$

$$|D_z^{|\beta|}g_1(z)|+|D_z^{|\beta|}g_2(z)|\leq\frac{C}{d_j^{N-1+k}}||\vec{\rho}||_{L^1(T_x)}\leq\frac{C}{d_j^{k+1}}.$$

Hence,

$$|g|_{H^{k+1}(\Omega_j^{(1)})}\leq\frac{C}{d_j^{-1+N/2+k}}.$$

Similarly, we can show that

$$|\lambda|_{H^{k}(\Omega_j^{(1)})}\leq\frac{C}{d_j^{-1+N/2+k}}.$$

This completes the proof of Lemma 3.12. □

Now we prove Lemma 3.13.

Proof. We first prove the bound for $||\vec{g}-\vec{g}_h||_{L^2(\Omega_j^{(1)})}$. To this end, set $\vec{E}=\vec{g}-\vec{g}_h$ and notice that

$$||\vec{E}||_{L^2(\Omega_j^{(1)})}=\sup_{\substack{\vec{\phi}\in[C_c^{\infty}(\Omega_j^{1})]^N\\ ||\vec{\phi}||_{L^2(\Omega)}=1}}\int_{\Omega_j^{(1)}}\vec{E}\,\vec{\phi}dx.$$

Let $\vec{\phi}\in[C_c^{\infty}(\Omega_j^1)]^N$ with $||\vec{\phi}||_{L^2(\Omega)}=1$ and $(\vec{\psi},\theta)$ solve

$$-\triangle\vec{\psi}+\nabla\theta=\vec{\phi}\quad\text{in }\Omega,$$
$$\nabla\cdot\vec{\psi}=0\quad\text{in }\Omega,$$
$$\vec{\psi}=0\quad\text{on }\partial\Omega.$$

We have

$$\begin{aligned}\int_\Omega \vec{E}\cdot\vec{\phi}\,dx &= A_h(\vec{\psi},\vec{E}) + B_h(\vec{E},\theta) - R(\vec{\psi},\vec{E})\\ &= A_h(\vec{\psi}-\vec{\Pi}(\vec{\psi}),\vec{E}) - B_h(\vec{\Pi}(\vec{\psi}),\lambda-\lambda_h)\\ &\quad +B_h(\vec{E},\theta-\Pi(\theta)) + D_h(\lambda-\lambda_h,\Pi(\theta))\\ &\quad -R(\vec{\psi},\vec{E}) + R(\vec{g},\vec{\Pi}(\vec{\psi}))\\ &= A_h(\vec{\psi}-\vec{\Pi}(\vec{\psi}),\vec{E}) + B_h(\vec{\psi}-\vec{\Pi}(\vec{\psi}),\lambda-\lambda_h)\\ &\quad +B_h(\vec{E},\theta-\Pi(\theta)) + D_h(\lambda-\lambda_h,\theta-\Pi(\theta))\\ &\quad -R(\vec{\psi},\vec{E}) + R(\vec{g},\vec{\Pi}(\vec{\psi}))\end{aligned}$$

If $S\subset\Omega$, we define $A_{h,S}$ to be the terms of A_h with integration restricted to S. In a similar fashion we define the restrictions of B_h, D_h and R.

Hence,

$$\int_\Omega \vec{E}\cdot\vec{\phi}\,dx = I_1 + I_2$$

where

$$\begin{aligned}I_1 &= A_{h,\Omega_j^{(3)}}(\vec{\psi}-\vec{\Pi}(\vec{\psi}),\vec{E}) + B_{h,\Omega_j^{(3)}}(\vec{\psi}-\vec{\Pi}(\vec{\psi}),\lambda-\lambda_h)\\ &\quad +B_{h,\Omega_j^{(3)}}(\vec{E},\theta-\Pi(\theta)) + D_{h,\Omega_j^{(3)}}(\lambda-\lambda_h,\theta-\Pi(\theta))\\ &\quad -R_{\Omega_j^{(3)}}(\vec{\psi},\vec{E}) + R_{\Omega_j^{(3)}}(\vec{g},\vec{\Pi}(\vec{\psi})),\end{aligned}$$

and

$$\begin{aligned}I_2 &= A_{h,\Omega\setminus\Omega_j^{(3)}}(\vec{\psi}-\vec{\Pi}(\vec{\psi}),\vec{E}) + B_{h,\Omega\setminus\Omega_j^{(3)}}(\vec{\psi}-\vec{\Pi}(\vec{\psi}),\lambda-\lambda_h)\\ &\quad +B_{h,\Omega\setminus\Omega_j^{(3)}}(\vec{E},\theta-\Pi(\theta)) + D_{h,\Omega\setminus\Omega_j^{(3)}}(\lambda-\lambda_h,\theta-\Pi(\theta))\\ &\quad -R_{\Omega\setminus\Omega_j^{(3)}}(\vec{\psi},\vec{E}) + R_{\Omega\setminus\Omega_j^{(3)}}(\vec{g},\vec{\Pi}(\vec{\psi})).\end{aligned}$$

From local continuity properties of our operators we have

$$\begin{aligned}I_1 &\le C\,Q_1(||\vec{\psi}-\vec{\Pi}(\vec{\psi})||_{H_h^1(\Omega_j^{(4)})} + (h\sum_{e\in\mathcal{E}_h}||\{\!\{\nabla\vec{\psi}-\underline{\Pi}(\nabla\vec{\psi})\}\!\}||^2_{L^2(e\cap\Omega_j^{(4)})})^{1/2}\\ &\quad +||\theta-\Pi(\theta)||_{L_h^2(\Omega_j^{(4)})})\\ &\le C\,h\,Q_1(||\vec{\psi}||_{H^2(\Omega)} + ||\lambda||_{H^1(\Omega)})\\ &\le ChQ_1,\end{aligned}$$

where

$$Q_1 = (||\vec{E}||_{H_h^1(\Omega_j^{(4)})} + (h\sum_{e\in\mathcal{E}_h}||\{\!\{\nabla\vec{g}-\underline{\Pi}(\nabla\vec{g})\}\!\}||^2_{L^2(e\cap\Omega_j^{(4)})})^{1/2} + ||\lambda-\lambda_h||_{L_h^2(\Omega_j^{(4)})}).$$

We can easily show using approximation properties of $\underline{\Pi}$ and Proposition 3.14 that

$$h^{1/2}(\sum_{e\in\mathcal{E}_h}||\{\!\{\nabla\vec{g}-\underline{\Pi}(\nabla\vec{g})\}\!\}||^2_{L^2(e\cap\Omega_j^{(4)})})^{1/2} \le Ch^k d^{1-N/2-k}.$$

Hence,

$$I_1 \le Ch^{k+1}d^{1-N/2-k} + Ch(||\vec{E}||_{H_h^1(\Omega_j^{(4)})} + ||\lambda-\lambda_h||_{L_h^2(\Omega_j^{(4)})}).$$

For I_2 we have

$$\begin{aligned} I_2 &\leq CQ_2(||\vec{\psi}-\vec{\Pi}(\vec{\psi})||_{W_h^{1,\infty}(\Omega\backslash\Omega_j^{(3)})}||\nabla\vec{\psi}-\underline{\Pi}(\nabla\vec{\psi})||_{L^\infty(\Omega\backslash\Omega_j^{(3)})} \\ &\qquad +||\theta-\Pi(\theta)||_{L_h^\infty(\Omega\backslash\Omega_j^{(3)})}) \\ &\leq Ch^kQ_2(||\vec{\psi}||_{W^{k+1,\infty}(\Omega\backslash\Omega_j^{(2)})}+||\theta||_{W^{k,\infty}(\Omega\backslash\Omega_j^{(2)})}), \end{aligned}$$

where

$$Q_2=||\vec{E}||_{W_h^{1,1}(\Omega\backslash\Omega_j^{(3)})}+||\lambda-\lambda_h||_{L_h^1(\Omega\backslash\Omega_j^{(3)})}+h\sum_{e\in\mathcal{E}_h}||\{\!\{\nabla\vec{g}-\underline{\Pi}(\nabla\vec{g})||_{L^1(e\cap\Omega\backslash\Omega_j^{(3)})}.$$

If we use (3.19) we get

$$Q_2\leq||\vec{E}||_{W_h^{1,1}(\Omega\backslash\Omega_j^{(3)})}+||\lambda-\lambda_h||_{L_h^1(\Omega\backslash\Omega_j^{(3)})}+\log(1/h)^{\bar{s}}.$$

Using Proposition 3.14 along with the fact that $\vec{\phi}$ has support in $\Omega_j^{(1)}$ we can easily show

$$||\vec{\psi}||_{W^{k+1,\infty}(\Omega\backslash\Omega_j^{(2)})}+||\theta||_{W^{k,\infty}(\Omega\backslash\Omega_j^{(2)})}\leq Cd_j^{1-N/2-k}||\vec{\phi}||_{L^2(\Omega)}\leq Cd_j^{1-N/2-k}.$$

Hence, we have shown that

$$\begin{aligned} ||\vec{E}||_{L^2(\Omega_j^{(1)})}\,dx &\leq Ch^kd_j^{1-N/2-k}(||\vec{g}-\vec{g_h}||_{W_h^{1,1}(\Omega)}+||\lambda-\lambda_h||_{L_h^1(\Omega)}) \\ &\quad +Ch(||\vec{g}-\vec{g_h}||_{H_h^1(\Omega_j^{(4)})}+||\lambda-\lambda_h||_{L_h^2(\Omega_j^{(4)})}) \\ &\quad +C\log(\frac{1}{h})^{\bar{s}}h^{k+1}d_j^{1-N/2-k}. \end{aligned} \tag{3.22}$$

Now we prove the bound for $||\lambda-\lambda_h||_{H_<^{-1}(\Omega_j^{(1)})}$. Let $r=\lambda-\lambda_h$ and notice that

$$||r||_{H_<^{-1}(\Omega_j^{(1)})}=\sup_{\substack{\gamma\in C_<^\infty(\Omega_j^{(1)})\\ ||\gamma||_{H^1(\Omega_j^{(1)})}=1}}\int_{\Omega_j^{(1)}}r\,\gamma dx.$$

Let $\gamma\in C_<^\infty(\Omega_j^{(1)})$ with $||\gamma||_{H^1(\Omega_j^{(1)})}=1$ and let $(\vec{w},q)$ with $(\int_\Omega q\,dx=0)$ solve

$$\begin{aligned} -\triangle\vec{w}+\nabla q=0 \quad &\text{in } \Omega, \\ \nabla\cdot\vec{w}=\gamma-\operatorname{avg}_\Omega(\gamma) \quad &\text{in } \Omega, \\ \vec{w}=0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{3.23}$$

By the consistency result for the LDG method we have

$$\begin{aligned} \int_\Omega r\gamma\,dx &= \int_\Omega r(\gamma-\operatorname{avg}_\Omega(\gamma))dx=B_h(\vec{w},r) \\ &= B_h(\vec{w}-\vec{\Pi}(\vec{w}),r)-A_h(\vec{\Pi}(\vec{w}),\vec{E})+R(\vec{g},\vec{\Pi}(\vec{w})) \\ &= B_h(\vec{w}-\vec{\Pi}(\vec{w}),r)+A_h(\vec{w}-\vec{\Pi}(\vec{w}),\vec{E}) \\ &\quad +B(\vec{E},q)+R(\vec{g},\vec{\Pi}(\vec{w}))-R(\vec{w},\vec{E}) \\ &= B_h(\vec{w}-\vec{\Pi}(\vec{w}),r)+A_h(\vec{w}-\vec{\Pi}(\vec{w}),\vec{E}) \\ &\quad +B_h(\vec{E},q-\Pi(q))+D_h(r,\Pi(q)-q) \\ &\quad +R(\vec{g},\vec{\Pi}(\vec{w}))-R(\vec{w},\vec{E}). \end{aligned}$$

Following a similar argument as was done to bound $||\vec{E}||_{L^2(\Omega_j^{(1)})}$ we can show

$$
\begin{aligned}
&\int_\Omega r\gamma \, dx \\
&\leq Ch(||\vec{E}||_{H_h^1(\Omega_j^{(4)})} + ||\lambda - \lambda_h||_{L_h^2(\Omega_j^{(4)})}) + Ch^{k+1} d_j^{1-N/2-k} \\
&\quad + Ch^k(||\vec{w}||_{W^{k+1,\infty}(\Omega\backslash\Omega_j^{(2)})} + ||q||_{W^{k,\infty}(\Omega\backslash\Omega_j^{(2)})}) \times \\
&\qquad (||\vec{E}||_{W_h^{1,1}(\Omega\backslash\Omega_j^{(3)})} + ||\lambda - \lambda_h||_{L_h^1(\Omega\backslash\Omega_j^{(3)})} + \log(\frac{1}{h})^{\bar{s}} h).
\end{aligned}
$$

The proof will be complete once we show that

$$
||\vec{w}||_{W^{k+1,\infty}(\Omega\backslash\Omega_j^{(2)})} + ||q||_{W^{k,\infty}(\Omega\backslash\Omega_j^{(2)})} \leq C d_j^{1-N/2-k}. \tag{3.24}
$$

We would easily be able to show this inequality using the Green's function representation of $(\vec{w}, q)$ if $\text{avg}_\Omega(\gamma)$ was not present in equation (3.23). We have to perform an intermediate step since $\text{avg}_\Omega(\gamma)$ does not have support in $\Omega_j^{(1)}$. In particular, we need the following Schauder estimate which follows from (1.5) and (4.5) in [22].

Proposition 3.15. *Let $0 < \alpha < 1$ and let $(\vec{w}, q)$ satisfy*

$$
\begin{aligned}
-\triangle \vec{v} + \nabla p &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \vec{v} &= \beta \quad \text{in } \Omega, \\
\vec{v} &= \vec{m} \quad \text{on } \partial\Omega.
\end{aligned}
$$

Then, for every $x \in \Omega$ and $d > 0$ we have the following bound

$$
\begin{aligned}
|D^{k+1}\vec{v}(x)| + |D^k p(x)| \quad &\leq \quad C d^\alpha([\beta]_{k,\alpha,B_d\cap\Omega} + [\vec{m}]_{k+1,\alpha,B_d\cap\partial\Omega}) \\
&\quad + C d^{-(k+1)} ||\vec{v}||_{L^\infty(B_d\cap\Omega)} + C d^{-k} ||p||_{L^\infty(B_d\cap\Omega)}.
\end{aligned}
$$

where B_d is the ball centered at x with radius d. Here C is independent of x and d. The Hölder seminorm $[f]_{k,\alpha,S}$ is given by

$$
[\beta]_{l,\alpha,S} = |\beta|_{W^{l,\infty}(S)} + \sum_{|\eta|=l} \sup_{x,y\in S} \frac{|D^\eta\beta(x) - D^\eta\beta(y)|}{|x-y|^\alpha}.
$$

Hence, using Proposition 3.15 we have for any $x \in \Omega\backslash\Omega_j^{(2)}$

$$
|D^{k+1}\vec{w}(x)| + |D^k q(x)| \leq C d_j^{-(k+1)} ||\vec{w}||_{L^\infty(S_{d_j/2})} + C d_j^{-k} ||q||_{L^\infty(S_{d_j/2})},
$$

where S_d is the intersection of Ω with the ball centered at x with radius d. Here we used that γ has support in $\Omega_j^{(1)}$ and that the seminorms of the constant $\text{avg}_\Omega(\gamma)$ are zero.

In fact, since $(\hat{\vec{w}}, q)$ where $\hat{\vec{w}} = \vec{w} - \text{avg}_{S_{d_j/2}}(\vec{w})$ satisfies (3.23) with the boundary condition $\hat{\vec{w}} = -\text{avg}_{S_{d_j/2}}(\vec{w})$ instead of zero, one has by Proposition 3.15

$$
\begin{aligned}
|D^{k+1}\vec{w}(x)| + |D^k q(x)| \quad &\leq \quad C d_j^{-(k+1)} ||\hat{\vec{w}}||_{L^\infty(S_{d_j/2})} + C d_j^{-k} ||q||_{L^\infty(S_{d_j/2})} \\
&\leq \quad C d_j^{-k} ||\nabla\vec{w}||_{L^\infty(S_{d_j/2})} + C d_j^{-k} ||q||_{L^\infty(S_{d_j/2})}.
\end{aligned}
$$

Here we used that and $||\hat{\vec{w}}||_{L^\infty(S_{d_j/2})} \leq Cd_j||\nabla\vec{w}||_{L^\infty(S_{d_j/2})}$. One can easily show using the Cauchy-Schwarz inequality and Poincare's inequality that

$$\text{avg}_\Omega(\gamma) \leq Cd_j^{N/2+1}$$

If we use this inequality and Proposition 3.14, we have

$$d_j^{-k}||\nabla\vec{w}||_{L^\infty(S_{d_j/2})} + d_j^{-k}||q||_{L^\infty(S_{d_j/2})} \leq Cd_j^{1-N/2-k}. \tag{3.25}$$

Therefore, we have have shown (3.24). This completes the proof of Lemma 3.13. □

3.4. **Proof of Theorem 2.3.**

Proof. Using an argument similar to the argument used in Theorem 2.3, we have

$$|(p-p_h)(x)| \leq C||p-\Pi(p)||_{L_h^\infty(\Omega),x,s} + Ch^{-N/2-1}||p-p_h||_{H^{-1}(T_x)}.$$

where $x \in \bar{T}_x$, $T_x \in \mathcal{T}_h$. We know that

$$h^{-N/2-1}||p-p_h||_{H^{-1}(T_x)} = \sup_{\substack{m\in C_c^\infty(T_x)\\ ||m||_{H^1(T_x)}=h^{-N/2-1}}} \int_{T_x} (p-p_h)m\,dx.$$

Let $m \in C_c^\infty$ with $||m||_{H^1(\Omega)} = h^{-N/2-1}$ and let $(\tilde{\vec{g}}, \tilde{\lambda})$ (with $\int_\Omega \tilde{\lambda} = 0$) solve

$$\begin{aligned} -\triangle\tilde{\vec{g}} + \nabla(\tilde{\lambda}) &= 0 \\ \nabla\cdot\tilde{\vec{g}} &= m - \text{avg}_\Omega(m) \\ \tilde{\vec{g}} &= 0 \end{aligned}$$

Let $(\tilde{\vec{g}}_h, \tilde{\lambda}_h) \in \vec{V}_h^k \times Q_h^k$ be the functions that satisfy

$$\begin{aligned} A_h(\tilde{\vec{g}}_h, \vec{v}) + B_h(\vec{v}, \tilde{\lambda}_h) &= 0 \\ -B_h(\tilde{\vec{g}}_h, q) + D_h(\tilde{\lambda}_h, q) &= \int_\Omega (m - \text{avg}_\Omega(m))q \qquad \forall(\vec{v}, q) \in \vec{V}_h^k \times Q_h^k. \end{aligned}$$

Then, by the consistency result of the LDG method and (2.3) we have

$$\begin{aligned}
\int_{T_x} (p-p_h)m\,dx &= \int_\Omega (p-p_h)m\,dx = \int_\Omega (p-p_h)(m-\mathrm{avg}_\Omega(m))\,dx \\
&= -B_h(\tilde{\vec g}, p-p_h) \\
&= -B_h(\tilde{\vec g}-\tilde{\vec g}_h, p-p_h) + A_h(\vec u-\vec u_h, \tilde{\vec g}_h) - R(\vec u, \tilde{\vec g}_h) \\
&= -B_h(\tilde{\vec g}-\tilde{\vec g}_h, p-p_h) - A_h(\vec u-\vec u_h, \tilde{\vec g}-\tilde{\vec g}_h) \\
&\quad -B_h(\vec u-\vec u_h, \tilde\lambda) - R(\vec u, \tilde{\vec g}_h) + R(\tilde{\vec g}, \vec u-\vec u_h) \\
&= -B_h(\tilde{\vec g}-\tilde{\vec g}_h, p-p_h) - A_h(\tilde{\vec g}-\tilde{\vec g}_h, \vec u-\vec u_h) \\
&\quad -B_h(\vec u-\vec u_h, \tilde\lambda-\tilde\lambda_h) - D_h(p-p_h, \tilde\lambda_h) \\
&\quad -R(\vec u, \tilde{\vec g}_h) + R(\tilde{\vec g}, \vec u-\vec u_h) \\
&= -B_h(\tilde{\vec g}-\tilde{\vec g}_h, p-p_h) - A_h(\tilde{\vec g}-\tilde{\vec g}_h, \vec u-\vec u_h) \\
&\quad -B_h(\vec u-\vec u_h, \tilde\lambda-\tilde\lambda_h) + D_h(p-p_h, \tilde\lambda-\tilde\lambda_h) \\
&\quad -R(\vec u, \tilde{\vec g}_h) + R(\tilde{\vec g}, \vec u-\vec u_h) \\
&= -B_h(\tilde{\vec g}-\tilde{\vec g}_h, p-\Pi(p)) - A_h(\tilde{\vec g}-\tilde{\vec g}_h, \vec u-\vec\Pi(\vec u)) \\
&\quad -B_h(\vec u-\vec\Pi(\vec u), \tilde\lambda-\tilde\lambda_h) + D_h(p-\Pi(p), \tilde\lambda-\tilde\lambda_h) \\
&\quad -R(\vec u, \tilde{\vec g}_h) + R(\tilde{\vec g}, \vec u-\vec\Pi(\vec u)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{T_x} (p-p_h)m\,dx \\
&\le C(||\vec u-\vec\Pi(\vec u)||_{W_h^{1,\infty}(\Omega),x,s} + ||p-\Pi(p)||_{L^\infty(\Omega),x,s} \\
&\qquad +||\sigma_x^s(\nabla\vec u-\underline{\Pi}(\nabla\vec u))||_{L^\infty(\Omega)}) \times \\
&(||\tilde{\vec g}-\tilde{\vec g}_h||_{W_h^{1,1}(\Omega),x,-s} + ||\tilde\lambda-\tilde\lambda_h||_{L^1(\Omega),x,-s} + h\sum_e ||\sigma_x^{-s}\{\!\{\nabla\tilde{\vec g}-\nabla\tilde{\vec g}_h\}\!\}||_{L^1(e)}).
\end{aligned}$$

We will be done once we show the following inequality

$$\begin{aligned}
&||\tilde{\vec g}-\tilde{\vec g}_h||_{W_h^{1,1}(\Omega),x,-s} + ||\tilde\lambda-\tilde\lambda_h||_{L^1(\Omega),x,-s} + h\sum_{e\in\mathcal{E}_h} ||\sigma_x^{-s}\{\!\{\nabla\tilde{\vec g}-\underline{\Pi}(\nabla\tilde{\vec g})\}\!\}||_{L^1(e)} \\
&\quad \le C\log(\frac{1}{h})^{\bar{\bar s}}.
\end{aligned}$$

Again, we omit the easier proof of the bound

$$h\sum_e ||\sigma_x^{-s}\{\!\{\nabla\tilde{\vec g}-\underline{\Pi}(\nabla\tilde{\vec g})\}\!\}||_{L^1(e)} \le C\log(\frac{1}{h})^{\bar{\bar s}}.$$

In order to prove the remaining inequality, we will need the two following lemmas. The proofs are very similar to the proofs of Lemmas 3.12 and 3.13. We leave the details to the reader.

Lemma 3.16. *If $d_j > 8h$, then*

$$\begin{aligned}
&||\tilde{\vec g}-\tilde{\vec g}_h||_{H_h^1(\Omega_j)} + ||\tilde\lambda-\tilde\lambda_h||_{L^2(\Omega_j)} \\
&\le Ch^k d_j^{-N/2-k} + d_j^{-1}(||\tilde{\vec g}-\tilde{\vec g}_h||_{L_2(\Omega_j^{(1)})} + ||\tilde\lambda-\tilde\lambda_h||_{H_<^{-1}(\Omega_j^{(1)})}).
\end{aligned}$$

Lemma 3.17. *If $d_j > 8h$, then*

$$\begin{aligned}&||\tilde{\vec{g}} - \tilde{\vec{g}}_h||_{L^2(\Omega_j^{(1)})} + ||\tilde{\lambda} - \tilde{\lambda}_h||_{H_<^{-1}(\Omega_j^{(1)})} \\ &\leq Ch^k d_j^{1-N/2-k}(||\tilde{\vec{g}} - \tilde{\vec{g}}_h||_{W_h^{1,1}(\Omega)} + ||\tilde{\lambda} - \tilde{\lambda}_h||_{L_h^1(\Omega)}) \\ &\quad + Ch(||\tilde{\vec{g}} - \tilde{\vec{g}}_h||_{H_h^1(\Omega_j^{(4)})} + ||\tilde{\lambda} - \tilde{\lambda}_h||_{L_h^2(\Omega_j^{(4)})}).\end{aligned}$$

Let M be a real number to be determined later and let J be an integer such that $d_J = Mh$. Set $\tilde{\vec{E}} = \tilde{\vec{g}} - \tilde{\vec{g}}_h$ and $\tilde{r} = \tilde{\lambda} - \tilde{\lambda}_h$. Notice that

$$\begin{aligned}||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega),x,-s} + ||\tilde{r}||_{L_h^1(\Omega),x,-s} \quad &\leq \quad ||\tilde{\vec{E}}||_{W_h^{1,1}(S_{Mh}),x,-s} + ||\tilde{r}||_{L_h^1(S_{Mh}),x,-s} \\ &\quad + \sum_{j=0}^{J}(||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega_j),x,-s} + ||\tilde{r}||_{L_h^1(\Omega_j),x,-s}).\end{aligned}$$

Using Hölder's inequality we can show

$$||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega_j),x,-s} + ||\tilde{r}||_{L_h^1(\Omega_j),x,-s} \leq Cd_j^{N/2+s}h^{-s}(||\tilde{\vec{E}}||_{H_h^1(\Omega_j)} + ||\tilde{r}||_{L_h^2(\Omega_j)}).$$

One also has

$$\begin{aligned}||\tilde{\vec{E}}||_{W_h^{1,1}(S_{Mh}),x,-s} + ||\tilde{r}||_{L_h^1(S_{Mh})} \quad &\leq \quad CM^{N/2+s}h^{N/2}(||\tilde{\vec{E}}||_{H_h^1(S_{Mh})} + ||\tilde{r}||_{L_h^2(S_{Mh})}) \\ &\leq \quad CM^{N/2+s}h^{N/2}h(||\tilde{\vec{g}}||_{H^2(\Omega)} + ||\tilde{\lambda}||_{H^1(\Omega)}) \\ &\leq \quad M^{N/2+s}h^{N/2}h||m - \text{avg}_\Omega(m)||_{H^1(\Omega)} \\ &\leq \quad CM^{N/2+s}.\end{aligned}$$

Here we used global error bounds and regularity results.
Therefore, we have

$$||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega),x,-s} + ||\tilde{r}||_{L_h^1(\Omega),x,-s} \leq CM^{N/2+s} + C\tilde{\eta}, \tag{3.26}$$

where

$$\tilde{\eta} = \sum_{j=0}^{J} d_j^{N/2+s}h^{-s}(||\tilde{\vec{E}}||_{H_h^1(\Omega_j)} + ||\tilde{r}||_{L_h^2(\Omega)}).$$

If we apply Lemma 3.16, we get

$$\tilde{\eta} \leq C\Theta(k-s) + C\sum_{j=0}^{J} d_j^{N/2-1+s}h^{-s}(||\tilde{\vec{E}}||_{L^2(\Omega_j^{(1)})} + ||\tilde{r}||_{H_<^{-1}(\Omega_j^{(1)})}),$$

Now applying Lemma 3.17, we have

$$\begin{aligned}
&\sum_{j=0}^{J} d_j^{N/2-1+s} h^{-s}(||\tilde{\vec{E}}||_{L_2(\Omega_j^{(1)})} + ||\tilde{r}||_{H^{-1}(\Omega_j^{(1)})}) \\
&\leq C\Theta(k-s)(||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega)} + ||\tilde{r}||_{L_h^1(\Omega)}) \\
&\quad + C\sum_{j=0}^{J} d_j^{N/2-1+s} h^{1-s}(||\tilde{\vec{E}}||_{H_h^1(\Omega_j^{(4)})} + ||\tilde{r}||_{L_h^2(\Omega_j^{(4)})}) \\
&\leq C\Theta(k-s)(||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega)} + ||\tilde{r}||_{L_h^1(\Omega)}) \\
&\quad + C d_J^{N/2-1+s} h^{1-s}(||\tilde{\vec{E}}||_{H_h^1(S_{Mh})} + ||\tilde{r}||_{L_h^2(S_{Mh})}) + \frac{C}{M}\tilde{\eta} \\
&\leq C\Theta(k-s)(||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega)} + ||\tilde{r}||_{L_h^1(\Omega)}) \\
&\quad + CM^{N/2-1+s} + \frac{C}{M}\tilde{\eta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{\eta} \quad \leq \quad & C\Theta(k-s) + C\Theta(k-s)(||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega)} + ||\tilde{r}||_{L_h^1(\Omega)}) \\
& + M^{N/2-1+s} + \frac{C}{M}\tilde{\eta}.
\end{aligned}$$

By Choosing M sufficiently large we have

$$\tilde{\eta} \leq C\Theta(k-s) + C\Theta(k-s)(||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega)} + ||\tilde{r}||_{L_h^1(\Omega)}) + M^{N/2-1+s}.$$

Substituting this bound into (3.26), we have

$$\begin{aligned}
||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega),x,-s} + ||\tilde{r}||_{L_h^1(\Omega),x,-s} \quad \leq \quad & CM^{N/2+s} + C\Theta(k-s) \\
& + C\Theta(k-s)(||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega)} + ||\tilde{r}||_{L_h^1(\Omega)}).
\end{aligned} \tag{3.27}$$

In particular, we see that

$$\begin{aligned}
||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega)} + ||\tilde{r}||_{L_h^1(\Omega)} \quad \leq \quad & CM^{N/2+s} + C\Theta(k) \\
& + C\Theta(k)(||\tilde{\vec{E}}|||_{W_h^{1,1}(\Omega)} + ||\tilde{r}||_{L_h^1(\Omega)}).
\end{aligned}$$

Hence, by choosing M large enough so that $C\Theta(k) \leq 1/2$, we get

$$||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega)} + ||\tilde{r}||_{L_h^1(\Omega)} \leq CM^{N/2+s} + C\Theta(k).$$

By this inequality and (3.27), we have

$$\begin{aligned}
&||\tilde{\vec{E}}||_{W_h^{1,1}(\Omega),x,-s} + ||\tilde{r}||_{L_h^1(\Omega),x,-s} \\
&\leq CM^{N/2+s} + C\Theta(k-s) + C\Theta(k-s)\Theta(k) \\
&\leq C\log(\frac{1}{h})^{\bar{\bar{s}}}.
\end{aligned}$$

This proves our result. □

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