ERROR ESTIMATES FOR THE RUNGE-KUTTA DISCONTINUOUS GALERKIN METHOD FOR THE TRANSPORT EQUATION WITH DISCONTINUOUS INITIAL DATA

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Abstract. We study the approximation of non-smooth solutions of the transport equation in one-space dimension by approximations given by a Runge-Kutta discontinuous Galerkin method of order two. We take an initial data which has compact support and is smooth except at a discontinuity, and show that, if the ratio of the time step size to the grid size is less than 1/3, the error at the time T in the $L^2(\mathbb{R}\setminus\mathcal{R}_T)-norm$ is the optimal order two when $\mathcal{R}_T$ is a region of size $O(T^{1/2}\,h^{1/2}\ \log 1/h)$ to the right of the discontinuity and of size $O(T^{1/3}\,h^{2/3}\ \log 1/h)$ to the left. Numerical experiments validating these results are presented.

Key words. discontinuous Galerkin methods, error estimates, hyperbolic problems

AMS subject classifications. 65N30

1. Introduction. In this paper, we present the first error estimates for the Runge-Kutta discontinuous Galerkin (RKDG) method for the transport equation with *discontinuous* initial data. These results are obtained for a formally second-order accurate RKDG method applied to the model problem

$$U_t + U_x = 0 \qquad \text{in } \mathbb{R}\times(0,T), \tag{1.1a}$$
$$U(\cdot,0) = U_0(\cdot) \quad \text{on } \mathbb{R}, \tag{1.1b}$$

where the initial condition U_0 has compact support; it has a discontinuity at $x = 0$ and is smooth everywhere else. Roughly speaking, we show that the quality of the approximation at time T is of second order in the size of the mesh, h, *outside* a region of size $O(T^{1/2}\,h^{1/2}\ \log 1/h)$ to the right of the discontinuity and of size $O(T^{1/3}\,h^{2/3}\ \log 1/h)$ to the left. An illustration of this result can be seen in Fig. 1.1.

The RKDG method was introduced by Cockburn and Shu *et al.* in a series of papers [10, 9, 8, 6, 11]; see also the monographs [4, 5] and the review [12]. Most a priori and a posteriori error analyzes of discontinuous Galerkin (DG) methods for hyperbolic problems have been carried out for either the semidiscrete version of the method or for DG methods using space-time elements; see [7], where the development of the DG methods up to the end of last century is described. To the knowledge of the authors, the only a priori error analysis for the RKDG method is due to Shu and Zhang [19] who proved, among other things, that the same method considered here (but applied to nonlinear scalar conservation laws in several space dimensions) converges with order two in the $L^\infty(0,T;L^2(\mathbb{R}^d))$-norm provided that the solution is *smooth*. In this paper, we continue this effort to understand the RKDG method and analyze it in case of solutions that have *discontinuities*. As a stepping stone towards the goal of solving the much more complicated case of nonlinear scalar conservation laws in several space dimensions, we consider here the simpler model problem (1.1) and find, for each time

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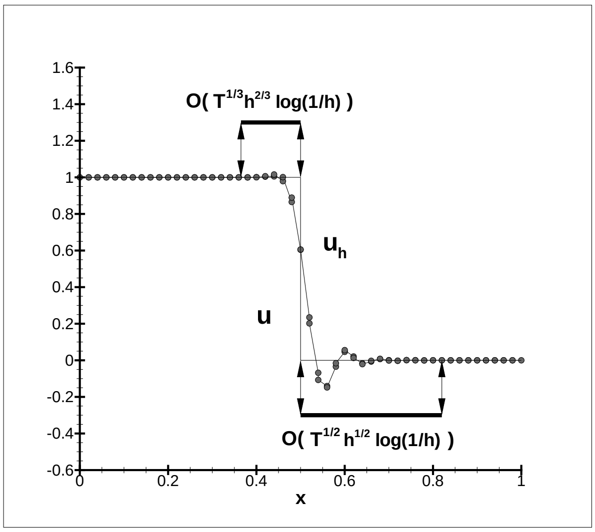

FIG. 1.1. *The two parts of the region $\mathcal{R}_T$ containing the numerical layer of the RKDG method: To the left (top) and to the right (bottom) of the discontinuity of the exact solution u. The approximate solution u_h was obtained by using $h = 1/100$ and $k/h = 0.3$. The time T is* 0.5.

T, the region $\mathcal{R}_T$ around the discontinuity of the exact solution $U(\cdot, T)$ such that the approximate solution $u_h(\cdot, T)$ given by the RKDG method converges with the optimal order of two in the $L^2(\mathbb{R} \setminus \mathcal{R}_T)$-norm.

To do that, we use an approach which is a modification of the classical L^2-argument to obtain error estimates in the $L^\infty(0, T; L^2(\mathbb{R}^d))$-norm, see, for example, subsection 2.7 of [5], where the semidiscrete case is treated. The modification has three main features. The first is the use of the decomposition of the error of the approximation given by the RKDG method proposed by Shu and Zhang [19]. The second, the use of special projections that allow us to obtain the full order of convergence of the approximation; see [3]. In our technique, without these projections, the order of convergence is reduced by 1/2. The third, the introduction of suitably chosen *weights* thanks to which we can localize the estimates and make the difference between the region to the left of the discontinuity and that to the right of it.

Similar weights were originally used by Johnson et al. [16] to prove local L^2 error estimates for a singularly perturbed reaction-convection-diffusion problem approximated by the streamline diffusion (SD) method; see [17] for L^∞ results. Recently, Guzmán [13] proved similar results for a DG method. Moreover, if one approximates (1.1) with either the standard SD or DG method and linear space-time elements, one can show using techniques in [16], [13] that the numerical layer resulting from discontinuous initial data is contained in a region whose size is at most $O(\log(1/h)h^{1/2})$ from either side of the discontinuity. In this article we show that in one side of the discontinuity, the size of the numerical layer can be reduced to $O(\log(1/h)h^{2/3})$ for the RKDG method. We accomplish this by taking advantage of the monotonicity of one of the two weight functions that we use; see Theorem 4.11 below.

Results of this type were obtained many years ago for finite difference methods for the model problem (1.1) with discontinuous initial data; see [1], [14], [2] and [15]. Indeed, by using of Fourier techniques, the size of the numerical layer to the left of the discontinuity was shown to be different to the size of the numerical layer to its right for some schemes; see (1.13) in [15]. In particular, the results concerning the size of the numerical layer for second-order accurate finite difference methods with fourth order dissipation in [15] are exactly the same result we prove in this paper for the RKDG method.

The organization of the paper is as follows. In section 2, we introduce the RKDG method under consideration and state and discuss the main result, Theorem 2.1. In section 4, we prove the theorem, and in section 5 we display numerical results showing that the result is sharp. We end in section 6 with some concluding remarks.

2. The main result. To state our main result, we first give a precise definition of the RKDG under consideration. The RKDG method is obtained by discretizing the equations in space by means of the DG method and then discretizing the resulting system of ODEs by a second-order TVD Runge-Kutta time stepping [18]; see [9].

We take uniform grids in both space and time. Let $\{I_j\}_j$ be a uniform partition of the real line where $I_j = (x_{j-1/2}, x_{j+1/2})$. Denote the mesh size by $h = x_{j+1/2} - x_{j-1/2}$ and the midpoint by $x_j = (x_{j+1/2}+x_{j-1/2})/2$. Accordingly, the RKDG approximation at each time $t^n = k\,n$, where k denotes the time step, is taken in the space

$$V_h = \{v \in L^2(\mathbb{R}), v|_{I_j} \in P^1(I_j) \;\; \forall j \in \mathbb{Z}\}.$$

For any function $v \in V_h$, we define the jump in v at the nodal point $x_{j+1/2}$ by

$$[v]_{j+1/2} = v(x^+_{j+1/2}) - v(x^-_{j+1/2}).$$

The RKDG approximation $\{\, u_h^n \,\}_{n=0}^{\infty}$ is defined as follows. For $n \geq 0$, u_h^{n+1} is the element of V_h such that

$$\sum_j (u_h^{n+1}, v_h)_j = \sum_j \{\frac{1}{2}(u_h^n, v_h)_j + \frac{1}{2}(w_h^n, v_h)_j\} + k\; H(w_h^n, v_h), \tag{2.1a}$$

where $w_h^n \in V_h$ is given by

$$\sum_j (w_h^n, v_h)_j = \sum_j (u_h^n, v_h)_j + kH(u_h^n, v_h), \tag{2.1b}$$

and

$$H(p,q) = \sum_j (p, q_x)_j + p(x^-_{j+1/2})[q]_{j+1/2}. \tag{2.1c}$$

Here $(w, v)_j = \int_{I_j} w\, v\, dx$. For $n = 0$, we set

$$u_h^0 = P_-(U_0). \tag{2.1d}$$

Here the projection $P_-(v)$ of a function v in the space

$$H^1_h(\mathbb{R}) = \{v : \mathbb{R} \mapsto \mathbb{R} \text{ such that } v|_{I_j} \in H^1(I_j)\; \forall j \in \mathbb{Z}\},$$

is defined as the element of V_h satisfying, on each interval I_j, $j \in \mathbb{Z}$,

$$(P_-(v), 1)_j = (v, 1)_j \quad \text{ and } \quad P_-(v)(x^-_{j+1/2}) = v(x^-_{j+1/2}).$$

This projection was introduced by [20] in the framework of DG methods for ODEs and was later used in [3] to obtain optimal error estimates for DG methods for one-dimensional convection-diffusion problems.

We can now state our main result.

Theorem 2.1. *Let U be the solution of the initial value problem* (1.1) *and let u_h^N be the solution of the* RKDG *method* (2.1) *at time $T = N\,k$. Suppose that $\lambda := \frac{k}{h} \le 1/3 - \epsilon$ for some fixed $\epsilon \in (0, 1/3)$, then there exist constants $\beta > 0$ and $C > 0$ such that for any $s \ge 4$*

$$\|\, U(T) - u_h^N \,\|_{L^2(\mathbb{R}\setminus\mathcal{R}_T)} \le C\,T\,h^2(1 + (\frac{k}{T\,\epsilon})^{1/2}) + Ch^s + ||U(T) - P_-(U(T))||_{L^2(\mathbb{R}\setminus\mathcal{R}_T)}.$$

where the interval $\mathcal{R}_T$ is given by

$$\mathcal{R}_T = T + \beta\,s\ \log(1/h)\epsilon^{-1}\ (-\lambda^{-7/3}T^{1/3}\,h^{2/3}, \lambda^{-1/2}T^{1/2}\,h^{1/2}),$$

and C is independent of h, k, T and ϵ.

A couple of remarks are in order. The first is that our hypothesis that the CFL condition $\lambda \le 1/3 - \epsilon$ is reasonable since it is well known that the RKDG under consideration is L^2-stable under the CFL condition

$$\lambda \le 1/3,$$

see, for example, [9, 12]. Taking $\epsilon > 0$, allows us to take advantage of the damping properties of the RKDG method. We would have to significantly modify our approach to deal the case $\epsilon = 0$ and the result might be very different.

The second remark is that this error estimate extends the result by Shu and Zhang [19] for smooth solutions to non-smooth solutions, in the special case considered here, of course. Indeed, in our case such result would read

$$\|\, U(T) - u_h^N \,\|_{L^2(\mathbb{R})} \le C\,h^2,$$

provided the initial condition U_0 is smooth. Our results says that the same rate of convergence in h holds if the region $\mathcal{R}_T$ around the discontinuity $x = t$ is removed from $\mathbb{R}$.

3. Idea of the Proof. Since the proof of Theorem 2.1 is very technical, we explain here the idea it is based upon. Suppose we are interested in establishing error estimates to the left of the line $t = x$, that is, suppose we want to prove

$$\begin{aligned}\|\, U(T) - u_h^N \,\|_{L^2(\mathbb{R}\setminus\mathcal{R}_T^-)} \le C||U(T) - P_-(U(T))||_{L^2(\mathbb{R}\setminus\mathcal{R}_T^-)} \\ + C\,T\,h^2(1 + (\frac{k}{T\,\epsilon})^{1/2}) + Ch^s \qquad (3.1)\end{aligned}$$

where $\mathcal{R}_T^- = (T - \beta\,s\ \log(1/h)\epsilon^{-1}\lambda^{-7/3}T^{1/3}\,h^{2/3}, -\infty)$.

To do this, we do not bound $U(T) - u_h^N$ directly. Instead, we compare u_h^N with a smooth approximation u of U so that

$$\|\, U(T) - u_h^N \,\|_{L^2(\mathbb{R}\setminus\mathcal{R}_T^-)} = \|u(T) - u_h^N\|_{L^2(\mathbb{R}\setminus\mathcal{R}_T^-)}.$$

We then obtain a bound of the form

$$\|u(T) - u_h^N\|_{L^2(\mathbb{R}\setminus\mathcal{R}_T^-)} \le C\|\phi_1(T)(u(T) - u_h^N)\|_{L^2(\mathbb{R})},$$

where ϕ_1 is a suitably chosen positive weight function.

The function ϕ_1 will be of order one in the region of interest, will satisfy

$$(\phi_1)_t + (\phi_1)_x = 0.$$

It will be a decreasing function in x which is exponentially small near the the line $x = t$; as a consequence, it will have an internal layer. Moreover, ϕ_1 will satisfy

$$\left|\frac{D_x^l(\phi_1(x,t))}{\phi_1(x,t)}\right| \le \frac{C}{M^l} \quad \text{for all } (x,t), l = 1,2,3,4. \tag{3.2}$$

It is clear that the parameter M is nothing but the size of the above mentioned layer.

In this way, we reduce the original problem to that of showing a weighted stability result for the error. In solving this problem, we will try to match the size of the layer of the weight function, M, with that of the internal layer produced by the numerical scheme to the left of the discontinuity. Indeed, roughly speaking, we will show that if we take $M = \epsilon^{-1}\lambda^{-7/3}T^{1/3}\,h^{2/3}$, we obtain that

$$\|\phi_1(T)(u(T) - u_h^N)\|_{L^2(\mathbb{R})} \le C\|\phi_1(0)(u(0) - u_h^0)\|_{L^2(\mathbb{R})} + C\,T\,h^2(1 + (\frac{k}{T\,\epsilon})^{1/2}). \tag{3.3}$$

Finally, if we take that $u_0 = U_0$ in the region of interest and since $\phi_1(0)$ is exponentially small when $u_0 \neq U_0$ we would arrive at (3.1).

A similar argument is used to estimate the error to the right of the discontinuity. In this case, however, it turns out that we cannot choose the corresponding weight function ϕ_2 to be a decreasing function in x. This leads to a larger numerical layer to the right.

Most of this paper is dedicated to proving a result similar to (3.3) for the two different weights ϕ_1 and ϕ_2. Next we illustrate one of the main ideas behind the proof of such an estimate by proving a similar result for a continuous problem; the techniques will be similar but more simple.

To this end, we have to find a continuous model for the equation satisfied by the error $\boldsymbol{e}$. Although the model equation for stable second-order accurate finite difference methods with constant coefficients is of the form

$$\boldsymbol{e}_t + \boldsymbol{e}_x = c_2\,h^2\boldsymbol{e}_{xxx} - c_3\,h^3\boldsymbol{e}_{xxxx},$$

where $c_3 > 0$, see [15], we are going to work with a simpler equation in order to simplify the computations; the final result will be the same. Thus, let $\boldsymbol{e}$ be the solution of

$$\boldsymbol{e}_t + \boldsymbol{e}_x = h^2\boldsymbol{e}_{xxx} \quad \text{in } \mathbb{R}\times(0,T), \tag{3.4a}$$
$$\boldsymbol{e}(x,0) = \boldsymbol{e}_0(x) \quad \text{on } \mathbb{R},$$

where $\boldsymbol{e}_0$ is smooth with compact support, and suppose we are interested in proving a weighted stability result of the form

$$\|\phi(T)\boldsymbol{e}(T)\|_{L^2(\mathbb{R})} \le C\|\phi(0)\boldsymbol{e}_0\|. \tag{3.5}$$

where ϕ solves $\phi_t + \phi_x = 0$, is a decreasing function in x and satisfies (3.2).

In obtaining the estimate (3.5), one important question is: how to pick M in an optimal way? To answer this question, we proceed as follows. First, we multiply both sides of (3.4a) by $\phi^2\boldsymbol{e}$ and integrate in space and time and to obtain

$$||\phi(T)\boldsymbol{e}(T)||^2_{L^2(\mathbb{R})} = ||\phi(0)\boldsymbol{e}(0)||^2_{L^2(\mathbb{R})} + h^2\int_0^T (\boldsymbol{e}_{xxx}(t), \phi^2(t)\boldsymbol{e})\,dt.$$

By using that $\boldsymbol{e}_{xxx}\boldsymbol{e} = \frac{1}{2}(\boldsymbol{e}^2)_{xxx} - \frac{3}{2}((\boldsymbol{e}_x)^2)_x$

$$
\begin{aligned}
h^2 \int_0^T (\boldsymbol{e}_{xxx}(t), \phi^2(t)\boldsymbol{e})\, dt &= \frac{h^2}{2}\int_0^T (\phi^2(t), (\boldsymbol{e}^2(t))_{xxx})\, dt - \frac{3h^2}{2}\int_0^T ((\phi(t))^2, ((\boldsymbol{e}_x)^2)_x)\, dt \\
&= \frac{h^2}{2}\int_0^T ((\phi^2(t))_{xxx}, \boldsymbol{e}^2(t))\, dt + \frac{3h^2}{2}\int_0^T ((\phi^2(t))_x, (\boldsymbol{e}_x)^2)\, dt
\end{aligned}
$$

In the last equation we used integration by parts and the fact that $\boldsymbol{e}$ has compact support. Now we use that $(\phi^2(t))_x < 0$ to get that

$$
\begin{aligned}
||\phi(T)\boldsymbol{e}(T)||^2_{L^2(\mathbb{R})} &\le ||\phi(0)\boldsymbol{e}(0)||^2_{L^2(\mathbb{R})} + \frac{h^2}{2}\int_0^T ((\phi^2(t))_{xxx}, \boldsymbol{e}^2(t))\, dt \\
&\le ||\phi(0)\boldsymbol{e}(0)||^2_{L^2(\mathbb{R})} + \frac{Ch^2}{M^3}\int_0^T \|\phi(t)\boldsymbol{e}^2(t)\|^2_{L^2(\mathbb{R})}\, dt
\end{aligned}
$$

In the last equation we used (3.4a). Finally, by Gronwall's inequality, we have

$$
||\phi(T)\boldsymbol{e}(T)||^2_{L^2(\mathbb{R})} \le ||\phi(0)\boldsymbol{e}_0||^2_{L^2(\mathbb{R})}\Big(1 + \frac{CTh^2}{M^3} e^{\frac{CTh^2}{M^3}}\Big),
$$

and we see that we must choose $M = CT^{1/3}h^{2/3}$ to obtain the wanted estimate.

Note that if ϕ is not a decreasing function in x then M will have to be larger in order to prove stability. This actually happens when dealing with the numerical layer to the right of the discontinuity. This is why the layer to the left of the discontinuity is smaller.

4. Proof. In this section we give a detailed proof of Theorem 2.1. We proceed in several steps.

4.1. Step 1: The Error Equations. First we define a suitable approximation u to U. The smooth approximation u will satisfy

$$
\begin{aligned}
u_t + u_x &= 0 \quad \text{in } \mathbb{R}\times(0,T), && (4.1a)\\
u(x,0) &= u_0(x) \quad \text{on } \mathbb{R}. && (4.1b)
\end{aligned}
$$

where u_0 is a smooth function which agrees with U_0 on $(-\infty,-h)\cup(h,\infty)$ such that

$$
|D^l u_0(x)| \le Ch^{-l} \quad \text{for } x\in[-h,h],\ l = 1,2,3,4. \tag{4.2}
$$

We then obtain the equations satisfied by the error $e_u^n = u(t^n) - u_h^n$. To capture the two-step nature of the RKDG method under consideration, we follow [19] and introduce the function

$$
w(x,t) = u(x,t) - k\, u_x(x,t), \tag{4.3}
$$

and the corresponding error, namely, $e_w^n = w(t^n) - w_h^n$. Finally, we write, for $p = u$ and $p = w$,

$$
\begin{aligned}
e_p^n &= \xi_p^n - \eta_p^n, && (4.4a)\\
\xi_p^n &= P_-(p(t^n)) - p_h^n \quad \text{(projection of the error } e_p^n), && (4.4b)\\
\eta_p^n &= P_-(p(t^n)) - p(t^n) \quad \text{(interpolation error)}. && (4.4c)
\end{aligned}
$$

In addition to the projection P_-, we also have to introduce the similar projection P_+ given on I_j by

$$(P_+(v), 1)_j = (v, 1)_j \quad \text{and} \quad P_+(v)(x^+_{j-1/2}) = v(x^+_{j-1/2}).$$

The projections $P_\pm$ are strongly related to the bilinear form $H(\cdot,\cdot)$ defined by (2.1c), as we can see in the next result.

LEMMA 4.1. *We have, for any $w \in H^1_h(\mathbb{R})$ and any $v_h \in V_h$,*

$$H(w - P_-(w), v_h) = 0, \tag{4.5a}$$
$$H(v_h, w - P_+(w)) = 0. \tag{4.5b}$$

Proof. The identity (4.5a) follows from the definitions of H and P_-. Using integration by parts we can rewrite $H(p,q)$ as

$$H(p,q) = -\sum_j \{(p_x, q)_j + q(x^+_{j+1/2})[p]_{j+1/2}\}.$$

The identity (4.5b) now follows from the above equation and the definition of P_+. □

The equations for the error are contained in the following result.

LEMMA 4.2. *We have, for any $v \in H^1_h(\mathbb{R})$,*

$$\sum_j (\xi^n_w, v)_j = \sum_j (\xi^n_u, v)_j + k\, H(\xi^n_u, v) + E_2(v) + E_3(v), \tag{4.6a}$$

$$\begin{aligned}\sum_j (\xi^{n+1}_u, v)_j = &\sum_j (\xi^n_u, v)_j + \frac{k}{2} H(\xi^n_w, v) + \frac{k}{2} H(\xi^n_u, v) \\ &+ E_1(P_+(v)) + \frac{1}{2} E_2(v) + \frac{1}{2} E_3(v) + E_4(v) + E_5(v),\end{aligned} \tag{4.6b}$$

where

$$\begin{aligned}
E_1(v) &= \frac{k^3}{6} \sum_j (u_{ttt}(\zeta^n), v)_j,\\
E_2(v) &= \sum_j (\xi^n_w - \xi^n_u, v - P_+(v))_j,\\
E_3(v) &= \sum_j (\eta^n_w - \eta^n_u, P_+(v))_j,\\
E_4(v) &= \sum_j (\xi^{n+1}_u - (\xi^n_u + \xi^n_w)/2, v - P_+(v))_j,\\
E_5(v) &= \sum_j (\eta^{n+1}_u - (\eta^n_u + \eta^n_w)/2, P_+(v))_j.
\end{aligned}$$

Here, ζ^n is a function of x lying in the interval (t^n, t^{n+1}).

Proof. Let us begin by proving the identity (4.6a). For any function v in $H^1_h(\mathbb{R})$, we have, by the definition of ξ_p and η_p, (4.4), that

$$\begin{aligned}\sum_j(\xi_w^n, v)_j &= \sum_j\{(e_w^n, v)_j + (\eta_w^n, v)_j\}\\ &= \sum_j\{(e_w^n, P_+(v))_j + (e_w^n, v - P_+(v))_j + (\eta_w^n, v)_j\}\\ &= \sum_j\{(e_w^n, P_+(v))_j + (\xi_w^n, v - P_+(v))_j + (\eta_w^n, P_+v)_j\}.\end{aligned}$$

To suitably rewrite the first term of the right-hand side, we notice that by definition of w, (4.3), and since u is smooth we have

$$\sum_j(w(t^n), v_h)_j = \sum_j(u(t^n), v_h)_j + k\, H(u(t^n), v_h),$$

for any $v_h \in V_h$. Hence, by subtracting the equation defining w_h^n, (2.1b), we obtain

$$\sum_j(e_w^n, v_h)_j = \sum_j(e_u^n, v_h)_j + k\, H(e_u^n, v_h). \tag{4.7}$$

Now, taking $v_h = P_+(v)$, we immediately get

$$\begin{aligned}\sum_j(e_w^n, P_+(v))_j &= \sum_j(e_u^n, P_+(v))_j + k\, H(e_u^n, P_+(v))\\ &= \sum_j(e_u^n, P_+(v))_j + k\, H(\xi_u^n, v),\end{aligned}$$

since, by the properties (4.5a) and (4.5b), we have

$$H(e_u^n, P_+(v)) \;=\; H(P_-(e_u^n), P_+(v)) \;=\; H(\xi_u^n, P_+(v)) \;=\; H(\xi_u^n, v).$$

As a consequence, we get

$$\begin{aligned}\sum_j(\xi_w^n, v)_j &= \sum_j(e_u^n, P_+(v))_j + k\, H(\xi_u^n, v)\\ &\quad + \sum_j\{(\xi_w^n, v - P_+(v))_j + (\eta_w^n, P_+(v))_j\}\\ &= \sum_j(\xi_u^n, v)_j + k\, H(\xi_u^n, v)\\ &\quad + \sum_j\{(\xi_w^n - \xi_u^n, v - P_+(v))_j + (\eta_w^n - \eta_u^n, P_+(v))\}\\ &= \sum_j(\xi_u^n, v)_j + k\, H(\xi_u^n, v) + E_2(v) + E_3(v),\end{aligned}$$

by definition of $E_2(v)$ and $E_3(v)$. This proves the identity (4.6a).

It remains to prove the identity (4.6b). For any function v in $H_h^1(\mathbb{R})$ we have that,

$$\begin{aligned}\sum_j(\xi_u^{n+1}, v)_j &= \sum_j\{(\xi_u^{n+1}, P_+(v))_j + (\xi_u^{n+1}, v - P_+(v))_j\}\\ &= \sum_j\{(e_u^{n+1}, P_+(v))_j + (\eta_u^{n+1}, P_+(v))_j + (\xi_u^{n+1}, v - P_+(v))_j\},\end{aligned}$$

by the definition of ξ_p and η_p, (4.4).

Next, we rewrite the first term of the right-hand side. By Taylor's theorem

$$u(x,t+k) = \frac{1}{2}u(x,t) + \frac{1}{2}w(x,t) - \frac{k}{2}w_x(x,t) + \frac{k^3}{6}u_{ttt}(x,\zeta),$$

where ζ depends on x and is $t \le \zeta \le t+k$. After a simple integration by parts, we have

$$\sum_j (u(t^{n+1}),v)_j = \sum_j \frac{1}{2}(u(t^n),v)_j + \frac{1}{2}(w(t^n),v)_j + \frac{k}{2}H(w(t^n),v) + E_1(v).$$

Subtracting the equation defining u_h^{n+1}, (2.1a), we get for any $v_h \in V_h$

$$\sum_j (e_u^{n+1},v_h)_j = \sum_j \frac{1}{2}(e_u^n,v_h)_j + \frac{1}{2}(e_w^n,v_h)_j + \frac{k}{2}H(e_w^n,v_h) + E_1(v_h),$$

and using (4.7), we obtain

$$\sum_j (e_u^{n+1},v_h)_j = \sum_j (e_u^n,v_h) + \frac{k}{2}H(e_w^n,v_h) + \frac{k}{2}H(e_u^n,v_h) + E_1(v_h).$$

Finally, taking $v_h = P_+(v)$ and using (4.5b) this equation becomes

$$\sum_j (e_u^{n+1},P_+(v))_j = \sum_j (e_u^n,P_+(v))_j + \frac{k}{2}H(\xi_w^n,v)_j + \frac{k}{2}H(\xi_u^n,v) + E_1(P_+(v)).$$

This implies

$$\begin{aligned}\sum_j (\xi_u^{n+1},v)_j &= \sum_j (e_u^n,P_+(v))_j + \frac{k}{2}H(\xi_w^n,v) + \frac{k}{2}H(\xi_u^n,v) + E_1(P_+(v)) \\ &\quad + \sum_j \{(\eta_u^{n+1},P_+(v))_j + (\xi_u^{n+1},v-P_+(v))_j\} \\ &= \sum_j (\xi_u^n,v)_j + \frac{k}{2}H(\xi_w^n,v) + \frac{k}{2}H(\xi_u^n,v) + E_1(P_+(v)) \\ &\quad + \sum_j \{(\eta_u^{n+1}-\eta_u^n,P_+(v))_j + (\xi_u^{n+1}-\xi_u^n,v-P_+(v))_j\} \\ &= \sum_j (\xi_u^n,v) + \frac{k}{2}H(\xi_w^n,v) + \frac{k}{2}H(\xi_u^n,v) + E_1(P_+(v)) \\ &\quad + \frac{1}{2}E_2(v) + \frac{1}{2}E_3(v) + E_4(v) + E_5(v).\end{aligned}$$

□

4.2. Step 2: The weights. Theorem 2.1 will follow from estimates of quantities of the form

$$\|\phi(\cdot,t^n)(u(t^n)-u_h^n)\|_{L^2(\mathbb{R})},$$

where $\phi(x,t)$ is a suitably chosen weight function. Here we describe the weights we are going to work with.

Since, as we stated in the introduction, we have different results on the left and on the right of the discontinuity, we consider two slightly different weight functions ϕ_1 and ϕ_2. The weight ϕ_1 will be of order one in the region $x < t + x_1$, for some number $x_1 < 0$, and "small" in the region $x > t + x_1 + d_1$ for some $d_1 > 0$, whereas the weight ϕ_2 is of order one on the region $x > t + x_2$, for some number $x_2 > 0$, and "small" in the region $x < t + x_2 - d_2$ for some $d_2 > 0$. A very important property that will allow better results to the left of the discontinuity is that $\phi_1(x,t)$ can be chosen as a decreasing function in x.

We take the functions ϕ_i, for $i = 1, 2$, as solutions of

$$\begin{aligned} \phi_{i,t} + \phi_{i,x} &= 0 \quad \text{in } R \times (0,T),\\ \phi_i(x,0) &= b_i(x) \quad \text{on } R, \end{aligned}$$

for some initial conditions b_1 and b_2. We choose the initial conditions b_1 and b_2 so that ϕ_1 and ϕ_2 satisfy:

PROPOSITION 4.3. *We have*

$$c \le \phi_i(x,t) \le C \qquad \text{for } (x,t) \in \Omega_i(0), \tag{4.8a}$$

$$\phi_i(x,t) \le h^s \qquad \text{for } (x,t) \in \mathbb{R}^2 \backslash \Omega_i(d_i), \tag{4.8b}$$

$$(-1)^{i+1} \phi_{i,x}(x,t) < 0 \qquad \text{for all } (x,t), \tag{4.8c}$$

$$\left| \frac{D_x^{l+1}(\phi_i(x,t))}{(\phi_i(x,t))_x} \right| + \left| \frac{D_x^{l}(\phi_i(x,t))}{\phi_i(x,t)} \right| \le \frac{C}{(\tilde{K}_i \tilde{T}^{1-\gamma_i} h^{\gamma_i})^l} \qquad \text{for all } (x,t), l = 1,2,3 \tag{4.8d}$$

$$RO(S,\phi_i) + RO(S,(\phi_i)_x) \le C \qquad \text{for all squares } S \text{ with sides of length } \tilde{T}^{1-\gamma_i} h^{\gamma_i}, \tag{4.8e}$$

where

$$\begin{aligned} \Omega_1(d) &= \{(x,t) : x \le t + x_1 + d\},\\ \Omega_2(d) &= \{(x,t) : x \ge t + x_2 - d\},\\ d_i &= s \log(1/h) \tilde{K}_i \tilde{T}^{1-\gamma_i} h^{\gamma_i}, \end{aligned}$$

$RO(D,\chi) = \max_{r\in D} |\chi(r)| \,/\, \min_{r\in D} |\chi(r)|$, *for any domain D, and c and C are positive constants independent of h, λ and T. Here $\tilde{T} = \frac{T}{\lambda}$ and $\tilde{K}_i = \frac{K_i}{\lambda^{m_i}}$. The parameters K_i, γ_i and m_i satisfy $K_i \ge 1$, $0 \le \gamma_i \le 1$ and $m_i \ge 0$ for $i = 1, 2$.*

In the rest of this paper we assume that the CFL number satisfies $\lambda \le 1/3$. Therefore, often we will use that $\frac{1}{\tilde{K}_i} \le \frac{1}{K_i} \le 1$. Also, since $\tilde{T}^{1-\gamma_i} h^{\gamma_i} = (\frac{T}{k})^{1-\gamma_i} h \ge h$, we see that $RO(S,\phi_i)$ and $RO(S,(\phi_i)_x)$ are bounded for squares S with sides of size h.

The construction of b_1 and b_2 is very similar to the construction of weight functions used in [16]. We include it in the appendix for completeness.

4.3. Step 3: The error in one time step. Next, we find how the weighted error changes in a single time step. This information is captured in a key identity contained in the following result.

LEMMA 4.4. *Let ϕ be any solution of the equation $\phi_t + \phi_x = 0$. Then we have*

$$\|\phi(t^{n+1})\xi_u^{n+1}\|^2_{L^2(\mathbb{R})} + \mathbb{J}_h = \|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} + \frac{k}{2}\Theta((\phi^2)_x) + \Psi_h(\phi) + E_h, \tag{4.9}$$

where

$$\mathbb{J}_h := \frac{k}{2}\sum_j \{[\phi(t^n)\xi_u^n]_{j+1/2}^2 + [\phi(t^{n+1})\xi_w^n]_{j+1/2}^2\}$$

$$\Theta(v) := \sum_j \Theta_j(v), \quad \Theta_j(v) := (\xi_w^n - \xi_u^n, v(t^n)(\xi_w^n - \xi_u^n))_j,$$

$$\Psi_h(\phi) := \|\phi(t^{n+1})(\xi_u^{n+1} - \xi_w^n)\|_{L^2(\mathbb{R})}^2.$$

Moreover,

$$\begin{aligned} E_h :=& 2\,E_1(P_+(\phi^2(t^{n+1})\xi_w^n)) + 2\,E_4(\phi^2(t^{n+1})\xi_w^n) + 2\,E_5(\phi^2(t^{n+1})\xi_w^n) \\ &+ E_2(\phi^2(t^n)\xi_u^n) + E_3(\phi^2(t^n)\xi_u^n) - \frac{k^2}{2}\sum_j (\xi_w^n - \xi_u^n, (\phi^2(t^n))_{xx}\xi_w^n)_j \\ &+ S_1 + S_2, \end{aligned}$$

where

$$S_1 = -\frac{k^3}{6}\sum_j (\xi_u^n, (\phi^2)_{xxx}(\theta_1^n)\xi_w^n)_j \quad \textit{and} \quad S_2 = \frac{k^3}{4}\sum_j (\xi_w^n, (\phi^2)_{xxx}(\theta_2^n)\xi_w^n)_j.$$

Here θ_i^n*,* $i = 1, 2$*, depend on* x *and* $t^n \le \theta_i^n \le t^{n+1}$*.*

Before proving this result, let us briefly discuss some of its salient features. Notice that:

• This result is completely independent of the fact that the approximate solution is piecewise linear in space. It only takes into account the nature of the Runge-Kutta time stepping method we are considering.

• The term $\mathbb{J}_h$ containing the jumps across inter-element boundaries captures the dissipative nature of the DG-space discretization. In the analysis, it will allow us to control the terms of the right-hand side.

• The term $\Theta((\phi^2)_x)$ will allow us to distinguish between the behavior of the error to the right and that to the left of the discontinuity. Since $\Theta((\phi_1^2)_x) \le 0$, this term enhances the damping properties of the method to the left of the discontinuity. This property does not hold for for $\Theta((\phi_2^2)_x)$ which will have to be controlled by the jumps in $\mathbb{J}_h$. This is the technicality that captures the fact that the approximation properties to the left and to the right of the discontinuity of the exact solution are very different. This results in a region $\mathcal{R}_T$ whose size is significantly *smaller* to the left of the discontinuity than to its right.

• If the initial condition U_0 were smooth we could then take $\phi \equiv 1$ and the above result would be the first step in the L^2-error analysis, see [19]. In such a case, the estimates of the terms of the right-hand side play a crucial role, with the exception of the term $\Theta((\phi^2)_x)$, which is identically equal to zero.

• Note that this lemma also contains the first step of the L^2-stability analysis of the RKDG method under consideration, which we obtain by setting $\phi \equiv 1$ and $u = 0$. In this case, the equation (4.9) becomes

$$||u_h^{n+1}||_{L^2(\mathbb{R})}^2 + \frac{k}{2}\sum_j \{[u_h^n]_j^2 + [w_h^n]_j^2\} = ||u_h^n||_{L^2(\mathbb{R})}^2 + ||u_h^{n+1} - w_h^n||_{L^2(\mathbb{R})}^2. \quad (4.10)$$

The second term of the left-hand side reflects the dissipative nature of the DG-space discretization of the method whereas the second term of the right-hand side captures the *anti*-dissipative nature typical of explicit schemes. The condition that the dissipative term dominates the *anti*-dissipative is nothing but the CFL condition. As we are going to see, a sharp estimate of the term $||u_h^{n+1} - w_h^n||_{L^2(\mathbb{R})}$ allows us to see that the method is L^2 stable provided $\lambda \le 1/3$.

Let us now prove Lemma 4.4.

Proof. We begin by considering the trivial identity

$$\begin{aligned}\|\phi(t^{n+1})\xi_u^{n+1}\|^2_{L^2(\mathbb{R})} =&\|\phi(t^{n+1})(\xi_u^{n+1}-\xi_w^n)\|^2_{L^2(\mathbb{R})}\\ &+2\sum_j(\xi_u^{n+1},\phi^2(t^{n+1})\xi_w^n)_j-\|\phi(t^{n+1})\xi_w^n\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

If we subtract (4.6a) from two times (4.6b) we get

$$\begin{aligned}2\sum_j(\xi_u^{n+1},v)_j=&\sum_j\{(\xi_w^n,v)_j+(\xi_u^n,v)_j\}+kH(\xi_w^n,v)\\ &+2E_1(P_+(v))+2E_4(v)+2E_5(v).\end{aligned}$$

Taking $v=\phi^2(t^{n+1})\,\xi_w^n$ in this equation and $v=\phi^2(t^n)\,\xi_u^n$ in (4.6a), we obtain

$$\begin{aligned}2\sum_j(\xi_u^{n+1},\phi^2(t^{n+1})\xi_w^n)_j=&\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})}+\|\phi(t^{n+1})\xi_w^n\|^2_{L^2(\mathbb{R})}\\ &+kH(\xi_u^n,\phi^2(t^n)\xi_u^n)+kH(\xi_w^n,\phi^2(t^{n+1})\xi_w^n)\\ &+2E_1(P_+(\phi^2(t^{n+1})\xi_w^n))+2E_4(\phi^2(t^{n+1})\xi_w^n)\\ &+2E_5(\phi^2(t^{n+1})\xi_w^n)+E_2(\phi^2(t^n)\xi_u^n)\\ &+E_3(\phi^2(t^n)\xi_u^n)+\sum_j(\xi_w^n,(\phi^2(t^{n+1})-\phi^2(t^n))\xi_u^n)_j,\end{aligned}$$

and hence,

$$\begin{aligned}\|\phi(t^{n+1})\xi_u^{n+1}\|^2_{L^2(\mathbb{R})}=&\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})}+\|\phi(t^{n+1})(\xi_u^{n+1}-\xi_w^n)\|^2_{L^2(\mathbb{R})}\\ &+kH(\xi_u^n,\phi^2(t^n)\xi_u^n)+kH(\xi_w^n,\phi^2(t^{n+1})\xi_w^n)\\ &+2E_1(P_+(\phi^2(t^{n+1})\xi_w^n))+2E_4(\phi^2(t^{n+1})\xi_w^n)\\ &+2E_5(\phi^2(t^{n+1})\xi_w^n)+E_2(\phi^2(t^n)\xi_u^n)\\ &+E_3(\phi^2(t^n)\xi_u^n)+\sum_j(\xi_u^n,(\phi^2(t^{n+1})-\phi^2(t^n))\xi_w^n)_j.\end{aligned}$$

Since, for any $v\in H_h^1(\mathbb{R})$

$$\begin{aligned}H(v,\phi^2(t)v)=&H(\phi(t)v,\phi(t)v)+\frac{1}{2}\sum_j(v,(\phi^2(t))_xv)_j\\ =&-\frac{1}{2}\sum_j[\phi(t)v]^2_{j+1/2}+\frac{1}{2}\sum_j(v,(\phi^2(t))_xv)_j,\end{aligned}$$

we get

$$
\begin{aligned}
\|\phi(t^{n+1})\xi_u^{n+1}\|_{L^2(\mathbb{R})}^2 + \mathbb{J}_h = \|\phi(t^n)\xi_u^n\|_{L^2(\mathbb{R})}^2 + \Psi_h(\phi) \\
+\frac{k}{2}\sum_j \{(\xi_u^n, (\phi^2(t^n))_x \xi_u^n)_j + (\xi_w^n, (\phi^2(t^{n+1}))_x \xi_w^n)_j\} \\
+2E_1(P_+(\phi^2(t^{n+1})\xi_w^n)) + 2E_4(\phi^2(t^{n+1})\xi_w^n) \\
+2E_5(\phi^2(t^{n+1})\xi_w^n) + E_2(\phi^2(t^n)\xi_u^n) \\
+E_3(\phi^2(t^n)\xi_u^n) + \sum_j (\xi_u^n, (\phi^2(t^{n+1}) - \phi^2(t^n))\xi_w^n)_j. \quad (4.11)
\end{aligned}
$$

Using Taylor's expansion, we see that

$$
\begin{aligned}
\sum_j (\xi_u^n, (\phi^2(t^{n+1}) - \phi^2(t^n))\xi_w^n)_j = -k\sum_j (\xi_u^n, (\phi^2(t^n))_x \xi_w^n)_j \\
+\frac{k^2}{2}\sum_j (\xi_u^n, (\phi^2(t^n))_{xx}\xi_w^n)_j + S_1.
\end{aligned}
$$

Here we used that $(\phi^2)_t = -(\phi^2)_x$. Similarly,

$$
\begin{aligned}
\frac{k}{2}\sum_j (\xi_w^n, (\phi^2(t^{n+1}))_x \xi_w^n)_j = \frac{k}{2}\sum_j \{(\xi_w^n, (\phi^2(t^n))_x \xi_w^n)_j \\
-\frac{k^2}{2}(\xi_w^n, (\phi^2(t^n))_{xx}\xi_w^n)_j\} + S_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{k}{2}\sum_j \{(\xi_u^n, (\phi^2(t^n))_x \xi_u^n)_j + (\xi_w^n, (\phi^2(t^{n+1}))_x \xi_w^n)_j\} \\
&\quad + \sum_j (\xi_u^n, (\phi^2(t^{n+1}) - \phi^2(t^n))\xi_w^n)_j \\
&= \frac{k}{2}\sum_j \{(\xi_u^n, (\phi^2(t^n))_x \xi_u^n)_j + (\xi_w^n, (\phi^2(t^n))_x \xi_w^n)_j\} \\
&\quad -k\sum_j (\xi_u^n, (\phi^2(t^n))_x \xi_w^n)_j - \frac{k^2}{2}\sum_j (\xi_w^n - \xi_u^n, (\phi^2(t^n))_{xx}\xi_w^n) + S_1 + S_2 \\
&= \frac{k}{2}\Theta((\phi^2)_x) - \frac{k^2}{2}\sum_j (\xi_w^n - \xi_u^n, (\phi^2(t^n))_{xx}\xi_w^n) + S_1 + S_2.
\end{aligned}
$$

The result follows after inserting the above identity into (4.11). This completes the proof. □

4.4. Step 4: Bound for Ψ_h. In order to bound Ψ_h we first need a bound for $||\xi_u^{n+1} - \xi_w^n||_{I_j}$ for each interval I_j. By exploiting the fact that $\xi_u^{n+1} - \xi_w^n$ is linear on each element, we express $\xi_u^{n+1} - \xi_w^n$ in terms of its mean and derivative on each element. More precisely, if $v \in V_h$ then for $x \in I_j$ we have

$$
v(x) = \bar{v}(x_j) + v_x(x_j)(x - x_j) \quad (4.12)
$$

where the average of v on I_j is $\bar{v}(x_j) = \frac{1}{h}(v,1)_j$. The representation we are seeking is contained in the following lemma which is proved in the appendix. For the rest of this paper θ^n will denote a number satisfying $t^n \le \theta^n \le t^{n+1}$ which depends on x; the values of θ^n may be different in separate occurrences. It will appear in different places when we use Taylor's Theorem.

LEMMA 4.5. *For $x \in I_j$, we have*

$$\begin{aligned}(\xi_u^{n+1} - \xi_w^n)(x) =& \frac{-3k^2}{h^2}[\xi_u^n]_{j-1/2} - \frac{k}{2h}[\xi_w^n - \xi_u^n]_{j-1/2} + \frac{k^3}{6h}R_{3,j} - \frac{k}{2h^3}R_{1,j} \\ &+ (\frac{3k}{h^2}[\xi_w^n - \xi_u^n]_{j-1/2} + \frac{k^3}{6h^3}R_{4,j} - \frac{6}{h^3}R_{1,j} + \frac{6}{h^3}R_{2,j})(x - x_j),\end{aligned} \tag{4.13a}$$

and

$$(\xi_w^n - \xi_u^n)(x) = - k(\xi_u^n)_x(x_j) - \frac{k}{h}[\xi_u^n]_{j-1/2} + (6\frac{k}{h^2}[\xi_u^n]_{j-1/2} + \frac{12}{h^3}R_{1,j})(x - x_j), \tag{4.13b}$$

where

$$\begin{aligned}R_{1,j} &= \int_{I_j}(\eta_w^n - \eta_u^n)(x - x_j)dx, \quad & R_{2,j} &= \int_{I_j}(\eta_u^{n+1} - (\eta_w^n + \eta_u^n)/2)(x - x_j)dx, \\ R_{3,j} &= \int_{I_j} u_{ttt}(\theta^n)dx, & R_{4,j} &= \int_{I_j} u_{ttt}(\theta^n)(x - x_j)dx.\end{aligned}$$

Now we calculate the term $\|(\xi_u^{n+1} - \xi_w^n)\|^2_{L^2(I_j)}$. We are going to express it in terms of the vector $[\boldsymbol{\xi}^n]_{j-1/2} := ([\xi_u^n]_{j-1/2}, [\xi_w^n]_{j-1/2})$.

LEMMA 4.6. *If $\lambda = \frac{k}{h}$, then*

$$\|\xi_u^{n+1} - \xi_w^n\|^2_{L^2(I_j)} = k\ [\boldsymbol{\xi}^n]_{j-1/2}\mathbb{A}[\boldsymbol{\xi}^n]^t_{j-1/2} + Y_j, \tag{4.14}$$

where

$$\mathbb{A} := \begin{pmatrix} Q(\lambda) & \frac{1}{2}Z(\lambda) \\ \frac{1}{2}Z(\lambda) & \lambda \end{pmatrix}, \quad Q(\lambda) = 9\lambda^3 - 3\lambda^2 + \lambda, \quad Z(\lambda) = 3\lambda^2 - 2\lambda,$$

and

$$\begin{aligned}Y_j =& h(\frac{k^3}{6h}R_{3,j} - \frac{k}{2h^3}R_{1,j})^2 + \frac{h^3}{12}(\frac{k^3}{6h^3}R_{4,j} - \frac{6}{h^3}R_{1,j} + \frac{6}{h^3}R_{2,j})^2 \\ &+ 2h(\frac{-3k^2}{h^2}[\xi_u^n]_{j-1/2} - \frac{k}{2h}[\xi_w^n - \xi_u^n]_{j-1/2})(\frac{k^3}{6h}R_{3,j} - \frac{k}{2h^3}R_{1,j}) \\ &+ 2\frac{h^3}{12}(\frac{3k}{h^2}[\xi_w^n - \xi_u^n]_{j-1/2})(\frac{k^3}{6h^3}R_{4,j} - \frac{6}{h^3}R_{1,j} + \frac{6}{h^3}R_{2,j}).\end{aligned}$$

Proof.

By using (4.13a) and the fact that $\int_{I_j}(x - x_j)^2 = \frac{h^3}{12}$ and $\int_{I_j}(x - x_j)dx = 0$, we get that

$$\begin{aligned}\|\xi_u^{n+1} - \xi_w^n\|^2_{L^2(I_j)} =& \quad h\,(\overline{(\xi_u^{n+1} - \xi_w^n)}(x_j))^2 + \frac{h^3}{12}((\xi_u^{n+1} - \xi_w^n)_x(x_j))^2 \\ =& h(\frac{-3k^2}{h^2}[\xi_u^n]_{j-1/2} - \frac{k}{2h}[\xi_w^n - \xi_u^n]_{j-1/2} + \frac{k^3}{6h}R_{3,j} - \frac{k}{2h^3}R_{1,j})^2 \\ &+ \frac{h^3}{12}(\frac{3k}{h^2}[\xi_w^n - \xi_u^n]_{j-1/2} + \frac{k^3}{6h^3}R_{4,j} - \frac{6}{h^3}R_{1,j} + \frac{6}{h^3}R_{2,j})^2,\end{aligned}$$

and the result follows after a few simple algebraic manipulations. □

Note that, by simply setting $\phi = 1$ and $u = 0$, we obtain an identity which is used for the L^2-stability analysis, namely,

$$\|u_h^{n+1} - w_h^n\|_{L^2(I_j)}^2 = k \sum_j [\boldsymbol{u}^n]_{j-1/2}\, \mathbb{A} [\boldsymbol{u}^n]_{j-1/2}^t,$$

where $[\boldsymbol{u}^n]_{j-1/2} := ([u_h^n]_{j-1/2}, [w_h^n]_{j-1/2})$. Indeed, inserting this expression in the identity (4.10), we get

$$\|u_h^{n+1}\|_{L^2(\mathbb{R})}^2 + k \sum_j [\boldsymbol{u}^n]_{j-1/2} (\frac{1}{2}\mathbb{I} - \mathbb{A}) [\boldsymbol{u}^n]_{j-1/2}^t = \| u_h^n \|_{L^2(\mathbb{R})}^2,$$

where $\mathbb{I}$ is the 2×2 identity matrix. Therefore, the method is L^2-stable if the matrix $\frac{1}{2}\mathbb{I} - \mathbb{A}$ is positive semi-definite. This occurs precisely for $\lambda \leq 1/3$.

Note also that, although we do not need to take $\lambda < 1/3$ to achieve L^2-stability, we need to take this choice in our error analysis. As we are going to see, taking $\lambda \leq 1/3 - \epsilon$ with $\epsilon > 0$ ensures that the matrix $\frac{1}{2}\mathbb{I} - \mathbb{A}$ is positive definite and, as a consequence, the terms involving the jumps of ξ_u^n and ξ_w^n in the right-hand side of the identity (4.9) of Lemma 4.4, can be controlled by the term $\mathbb{J}_h$.

We now state and prove the bound for $\Psi_h(\phi_i)$. We will need the following notation:

$$[\boldsymbol{\phi\xi}^n]_{j-1/2} := ([\phi(t^n)\, \xi_u^n]_{j-1/2},\, [\phi(t^{n+1})\, \xi_w^n]_{j-1/2}).$$

Lemma 4.7. *We have*

$$\begin{aligned}\Psi_h(\phi_i) \leq k \sum_j & [\boldsymbol{\phi}_i \boldsymbol{\xi}^n]_{j-1/2}\, \mathbb{B}_i [\boldsymbol{\phi}_i \boldsymbol{\xi}^n]_{j-1/2}^t \\ & + C\, K_i \|\phi_i(t^n)(\eta_w^n - \eta_u^n)\|_{L^2(I_j)}^2 \\ & + C\, K_i \|\phi_i(t^n)(\eta_u^{n+1} - (\eta_u^n + \eta_w^n)/2)\|_{L^2(I_j)}^2 \\ & + C\, K_i k^6 \|\phi_i(t^n) u_{ttt}(\theta^n)\|_{L^2(\mathbb{R})}^2,\end{aligned}$$

where

$$\mathbb{B}_i = \mathbb{A} + \frac{C}{K_i}\, \mathbb{I}.$$

Proof. To simplify the notation, we drop the subindex i. We have

$$\Psi_h(\phi) = T_1 + T_2,$$

where

$$\begin{aligned}T_1 &= \sum_j \phi^2(x_{j-1/2}, t^{n+1}) \|(\xi_u^{n+1} - \xi_w^n)\|_{L^2(I_j)}^2, \\ T_2 &= \sum_j \int_{I_j} (\phi^2(x, t^{n+1}) - \phi^2(x_{j-1/2}, t^{n+1}))(\xi_u^{n+1} - \xi_w^n)^2(x)\, dx.\end{aligned}$$

We first estimate the term T_1. We multiply (4.14) by $\phi^2(x_{j-1/2}, t^{n+1})$ and we bound the terms of the resulting right hand side separately. Since

$$\begin{aligned}\phi^2(x_{j-1/2}, t^{n+1}) =&\phi^2(x_{j-1/2}, t^n) + k(\phi^2)_t(x_{j-1/2}, \theta^n)\\ \phi(x_{j-1/2}, t^{n+1}) =&\phi(x_{j-1/2}, t^n) + k\phi_t(x_{j-1/2}, \theta^n),\end{aligned}$$

by using inequalities (4.8d) and (4.8e), we have

$$\begin{aligned}&k\phi^2(x_{j-1/2}, t^{n+1})[\boldsymbol{\xi}^n]_{j-1/2}\mathbb{A}[\boldsymbol{\xi}^n]^t_{j-1/2} \le k[\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{A}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}\\ &+ k(\frac{Ch^{1-\gamma}}{\tilde{T}^{1-\gamma}\tilde{K}})[\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{I}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}.\end{aligned}$$

Here we used that the entries of $\mathbb{A}$ are bounded for $\lambda \in [0, 1/3]$.

Using (4.8e) and the Cauchy Schwarz inequality, we get from the definition of $R_{l,j}$ for $l = 1, \cdots, 4$, that

$$\begin{aligned}&\phi(x_{j-1/2}, t^n)(\frac{k^6}{h}|R_{3,j}|^2 + \frac{1}{h^3}|R_{1,j}|^2 + \frac{k^6}{h^3}|R_{4,j}|^2 + \frac{1}{h^3}|R_{2,j}|^2)\\ &\le C(\|\phi(t^n)(\eta^n_w - \eta^n_u)\|^2_{L^2(I_j)} + \|\phi(t^n)(\eta^{n+1}_u - (\eta^n_u + \eta^n_w)/2)\|^2_{L^2(I_j)})\\ &\quad + Ck^6\|\phi(t^n)u_{ttt}(\theta^n)\|^2_{L^2(I_j)}.\end{aligned}$$

Therefore, if we again apply (4.8e), multiply and divide each term of Y_j by $K^{1/2}$ and apply Young's inequality we obtain

$$\begin{aligned}|\phi^2(x_{j-1/2}, t^{n+1})Y_j| \le\ & \frac{1}{K}k[\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{I}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}\\ &+C\,K\|\phi(t^n)(\eta^n_w - \eta^n_u)\|^2_{L^2(I_j)}\\ &+C\,K\|\phi(t^n)(\eta^{n+1}_u - (\eta^n_u + \eta^n_w)/2)\|^2_{L^2(I_j)}\\ &+C\,Kk^6\|\phi(t^n)u_{ttt}(\theta^n)\|^2_{L^2(I_j)}.\end{aligned}$$

Hence,

$$\begin{aligned}T_1 \le\ & k\sum_j [\boldsymbol{\phi\xi}^n]_{j-1/2}(\mathbb{A} + ((\frac{h}{\tilde{T}})^{1-\gamma}\frac{C}{\tilde{K}} + \frac{C}{K})\mathbb{I})[\boldsymbol{\phi\xi}^n]^t_{j-1/2}\\ &+C\,K\|\phi(t^n)(\eta^n_w - \eta^n_u)\|^2_{L^2(I_j)}\\ &+C\,K\|\phi(t^n)(\eta^{n+1}_u - (\eta^n_u + \eta^n_w)/2)\|^2_{L^2(I_j)} + C\,K\,k^6\|\phi(t^n)u_{ttt}(\theta^n)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

To bound T_2 we see from (4.14) that

$$T_2 \le \sum_j \|\phi^2(x_{j-1/2}, t^{n+1}) - \phi^2(t^{n+1})\|_{L^\infty(I_j)}(k[\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{A}[\boldsymbol{\phi\xi}^n]^t_{j-1/2} + Y_j).$$

By using the inequality

$$\|\phi^2(x_{j-1/2}, t^{n+1}) - \phi^2(t^{n+1})\|_{L^\infty(I_j)} \le \frac{Ch^{1-\gamma}}{\tilde{T}^{1-\gamma}\tilde{K}}\|\phi^2(t^{n+1})\|_{L^\infty(I_j)},$$

the inequality (4.8e) and the Young's inequality, we obtain

$$\begin{aligned}&\|\phi^2(x_{j-1/2}, t^{n+1}) - \phi^2(t^{n+1})\|_{L^\infty(I_j)}(k[\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{A}[\boldsymbol{\phi\xi}^n]^t_{j-1/2})\\ &\le \frac{Ch^{1-\gamma}}{\tilde{T}^{1-\gamma}\tilde{K}}k[\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{I}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}.\end{aligned}$$

By the triangle inequality and (4.8e) we have

$$\|\phi^2(x_{j-1/2},t^{n+1})-\phi^2(t^{n+1})\|_{L^\infty(I_j)}|Y_j|\le C|\phi(x_{j-1/2},t^{n+1})Y_j|.$$

Therefore,

$$\begin{aligned}T_2\le k\sum_j[\boldsymbol{\phi\xi}^n]_{j-1/2}((\frac{h}{\tilde T})^{1-\gamma}\frac{C}{\tilde K}+\frac{C}{K})\mathbb{I}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}\\ +C\,K\|\phi(t^n)(\eta^n_w-\eta^n_u)\|^2_{L^2(I_j)}\\ +C\,K\|\phi(t^n)(\eta^{n+1}_u-(\eta^n_u+\eta^n_w)/2)\|^2_{L^2(I_j)}+M(\epsilon)k^6\|\phi(t^n)u_{ttt}(\theta^n)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

By combining the estimates of T_1 and T_2 and using $\frac{h}{\tilde T}\le 1$, $\gamma\le 1$ and $\frac{1}{\tilde K}\le\frac{1}{K}$ concludes the proof. □

4.5. Step 6: Estimates of E_h. We state two slightly different estimates for $E_h(\phi_i)$. One would be applied with the weight ϕ_1 and the other for ϕ_2. To simplify notation, for the rest of this section we let $\phi=\phi_i$, $K=K_i$, $m=m_i$ and $\gamma=\gamma_i$ for $i=1$ or $i=2$.

LEMMA 4.8. *If K is sufficiently large, then*

$$\begin{aligned}E_h(\phi)\le(1+\lambda^{m-2})\frac{Ck}{K}\sum_j[\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{I}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}+\frac{k}{4}\Theta(|(\phi^2)_x|)\\ +\{\frac{k}{4T}+(1+(\frac{h}{\tilde T})^{(3-4\gamma)}+(\frac{h}{\tilde T})^{2-3\gamma})\frac{C\,k}{K\,T}\}||\phi(t^n)\xi^n_u||^2_{L^2(\mathbb{R})}\\ +(1+(\frac{h}{\tilde T})^{2(3/2-2\gamma)}+(\frac{h}{\tilde T})^{2-3\gamma})\frac{C\,T}{k}||\phi(t^n)(\eta^n_w-\eta^n_u)||^2_{L^2(\mathbb{R})}\\ +\frac{C\,T}{k}(||\phi(t^n)(\eta^{n+1}_u-\eta^n_u)||^2_{L^2(\mathbb{R})}+k^6||\phi(t^n)u_{ttt}(\theta^n)||^2_{L^2(\mathbb{R})}).\end{aligned}$$

and

$$\begin{aligned}E_h(\phi)\le\frac{Ck}{K}\sum_j[\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{I}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}+\frac{k}{4}\Theta(|(\phi^2)_x|)\\ +\{\frac{k}{4T}+(1+(\frac{h}{\tilde T})^{2(3/2-2\gamma)}+(\frac{h}{\tilde T})^{2-3\gamma}+(\frac{h}{\tilde T})^{1-2\gamma})\frac{C\,k}{K\,T}\}||\phi(t^n)\xi^n_u||^2_{L^2(\mathbb{R})}\\ +(1+(\frac{h}{\tilde T})^{2(3/2-2\gamma)}+(\frac{h}{\tilde T})^{2-3\gamma}+(\frac{h}{\tilde T})^{1-2\gamma})\frac{C\,T}{k}||\phi(t^n)(\eta^n_w-\eta^n_u)||^2_{L^2(\mathbb{R})}\\ +\frac{C\,T}{k}(||\phi(t^n)(\eta^{n+1}_u-\eta^n_u)||^2_{L^2(\mathbb{R})}+k^6||\phi(t^n)u_{ttt}(\theta^n)||^2_{L^2(\mathbb{R})}).\end{aligned}$$

The proof of this lemma is contained in the appendix. Here we would like to point out that the main tool in proving this result is super-approximation. The super-approximation result involving are weights are similar to the ones used in [16] and [13]. For the sake of completeness we include proof of the following super-approximation lemma in the appendix.

LEMMA 4.9. *If $P = P_-$ or $P = P_+$ and $l = 0, 1, 2$, then there exists $C > 0$ independent of h such that for all $v \in V_h$ the following estimates hold:*

$$\begin{aligned}
&\|\phi^{-1}(t)(\phi^2(t)v - P(\phi^2(t)v)\|_{L^2(I_j)} \\
&+ h\|\phi^{-1}(t)(\phi^2(t)v - P(\phi^2(t)v))_x\|_{L^2(I_j)} \\
&\qquad \le \frac{Ch^{2-2\gamma}}{\tilde{T}^{2(1-\gamma)}\tilde{K}^2}\|\phi(t)v\|_{L^2(I_j)} \\
&\qquad + \frac{Ch^{2-\gamma/2}}{\tilde{T}^{(1-\gamma)/2}\tilde{K}^{1/2}}\|(|\phi(t)\phi_x(t)|)^{1/2}v_x\|_{L^2(I_j)}, \qquad (4.15a) \\
&\|\phi^{-1}(t)(D_x^l(\phi^2(t))v - P(D_x^l(\phi^2(t))v)\|_{L^2(I_j)} \\
&+ h\|\phi^{-1}(t)(D_x^l(\phi^2(t))v - P(D_x^l(\phi^2(t))v))_x\|_{L^2(I_j)} \\
&\qquad \le \left(\frac{Ch^{2-(l+2)\gamma}}{\tilde{T}^{(1-\gamma)(l+2)}\tilde{K}^{2+l}} + \frac{Ch^{1-(l+1)\gamma}}{\tilde{T}^{(1-\gamma)(l+1)}\tilde{K}^{l+1}}\right)\|\phi(t)v\|_{L^2(I_j)}, \qquad (4.15b)
\end{aligned}$$

Let us illustrate how we can use Lemma 4.9 to prove Lemma 4.8 by showing how we can bound one of the terms appearing in the definition of E_h.

LEMMA 4.10. *Let $t^n \le t \le t^{n+1}$ and $v_h \in V_h$, then for $l = 0, 1, 2$,*

$$\begin{aligned}
&h^l|E_2(D_x^l(\phi^2(t))v_h)| + h^l|E_4(D_x^l(\phi^2(t))v_h)| \\
&\le \frac{C\,k}{K^{l+1}}\sum_j [\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{I}[\boldsymbol{\phi\xi}^n]^t_{j-1/2} \\
&+\left(1 + \left(\frac{h}{\tilde{T}}\right)^{2(l+1/2-(l+1)\gamma)} + \left(\frac{h}{\tilde{T}}\right)^{2(l+3/2-(l+2)\gamma)}\right)\frac{C\,k}{K^{l+1}\,T}\|\phi(t^n)v_h\|^2_{L^2(\mathbb{R})} \qquad (4.16) \\
&+\frac{CT}{k}(\|\phi(t^n)(\eta_w^n - \eta_u^n)\|^2_{L^2(\mathbb{R})} + \|\phi(t^n)(\eta_u^{n+1} - \eta_u^n)\|^2_{L^2(\mathbb{R})} + k^6\|u_{ttt}(\theta^n)\|^2_{L^2((R)}).
\end{aligned}$$

If we take $l = 0$, $v_h = \xi_u^n$ and $t = t^n$, then we get the E_2 term appearing in the definition of E_h.

Proof. Using (4.13b) and the properties of P_+ we get that

$$\begin{aligned}
h^l|E_2(D_x^l(\phi^2(t))v_h)| &= h^l|\sum_j(\xi_w^n - \xi_u^n, D_x^l(\phi^2(t))v_h - P_+(D_x^l(\phi^2(t))v_h))_j| \\
&= h^l|\sum_j\left(\frac{6k}{h^2}[\xi_u^n]_{j-1/2} + \frac{12}{h^3}R_{1,j}\right) \times \\
&\qquad (x - x_j, D_x^l(\phi^2(t))v_h - P_+(D_x^l(\phi^2(t))v_h))_j|.
\end{aligned}$$

By multiplying and dividing the last equation by $\phi(t^n, x_{j-1/2})$ and using (4.8e) we get

$$\begin{aligned}
&h^l|E_2(D_x^l(\phi^2(t))v_h)| \\
&\le h^{l+3/2}\sum_j\left(\frac{6k}{h^2}|[\phi(t^n)\xi_u^n]_{j-1/2}| + \frac{12}{h^3}|R_{1,j}\phi(t^n, x_{j-/2})|\right) \times \\
&\qquad ||\phi^{-1}(t)(D_x^l(\phi^2(t))v_h - P_+(D_x^l(\phi^2(t))v_h))||_{L^2(I_j)}, \qquad (4.17)
\end{aligned}$$

where we used that $||x-x_j||_{L^2(I_j)} \leq Ch^{3/2}$. If we use (4.15b) we see that

$$kh^{l-1/2} \sum_j |[\phi(t^n)\xi_u^n]_{j-1/2}| \, ||\phi^{-1}(t)(D_x^l(\phi^2(t))v_h - P_+(D_x^l(\phi^2(t))v_h))||_{L^2(I_j)}$$
$$\leq (\frac{C}{K^{l+1}} + \frac{C}{K^{l+2}})k \sum_j [\phi(t^n)\xi_u^n]_{j-1/2}^2$$
$$((\frac{h}{\tilde{T}})^{2(l+1/2-(l+1)\gamma)} \frac{C}{K^{l+1}} + (\frac{h}{\tilde{T}})^{2(l+3/2-(l+2)\gamma)} \frac{C}{K^{l+2}}) \frac{k}{T} \|\phi(t^n)v_h\|_{L^2(\mathbb{R})}^2. \quad (4.18)$$

By the Cauchy-Schwarz inequality we get that $|R_{1,j}| \leq Ch^{3/2}\|\eta_w^n - \eta_u^n\|_{L^2(I_j)}$. Hence,

$$h^{l-3/2} \sum_j |R_{1,j}\phi(t^n, x_{j-1/2})| \, ||\phi^{-1}(t)(D_x^l(\phi^2(t))v_h - P_+(D_x^l(\phi^2(t))v_h))||_{L^2(I_j)}$$
$$\leq (\frac{C}{K^{2(l+1)}} + \frac{C}{K^{2(l+2)}}) \frac{k}{T} \|\phi(t^n)v_h\|_{L^2(\mathbb{R})}^2$$
$$+((\frac{h}{\tilde{T}})^{2(1-\gamma)(l+1)} + (\frac{h}{\tilde{T}})^{2(1-\gamma)(l+2)}) \frac{CT}{k} \|\eta_w^n - \eta_u^n\|_{L^2(I_j)}^2. \quad (4.19)$$

Therefore, if we plug (4.18) and (4.19) into (4.17) and use $\frac{h}{\tilde{T}} \leq 1$, $\gamma \leq 1$ and $K \geq 1$ we can bound $h^l|E_2(D_x^l(\phi^2(t))v_h)|$ by the right hand side of (4.16). In a similar fashion we can bound $h^l|E_4(D_x^l(\phi^2(t))v_h)|$ to conclude the proof. ☐

4.6. Step 7: Estimates for $\Theta((\phi^2)_x)$ and the final estimates for the error in one time step.

4.6.1. Case 1: The weight ϕ_1. In this case, the only estimate we need for $\Theta((\phi_1^2)_x)$ is simply

$$\Theta((\phi_1^2)_x) < 0, \quad (4.20)$$

which follows from (4.8c). Collecting the estimates for Ψ_h and E_h and $\Theta((\phi_1^2)_x)$ we can give the right bound for the weighted ϕ_1 error produced after one time step.

THEOREM 4.11. *Let $0 < \epsilon < \frac{1}{3}$ and suppose that $\lambda \leq 1/3 - \epsilon$. If $\gamma_1 = 2/3$, $m_1 = 2$ and*

$$\frac{1}{K_1} = c\,\epsilon, \quad (4.21)$$

for a sufficiently small fixed constant $c > 0$, then

$$\|\phi_1(t^{n+1})\xi_u^{n+1}\|_{L^2(\mathbb{R})}^2 \leq (1 + \frac{k}{T})\|\phi_1(t^n)\xi_u^n\|_{L^2(\mathbb{R})}^2$$
$$+C(\frac{T}{k} + \epsilon^{-1})\|\phi_1(t^n)(\eta_u^{n+1} - \eta_u^n)\|_{L^2(\mathbb{R})}^2$$
$$+C(\frac{T}{k} + \epsilon^{-1}) \, \|\phi_1(t^n)(\eta_w^n - \eta_u^n)\|_{L^2(\mathbb{R})}^2$$
$$+C(\frac{T}{k} + \epsilon^{-1}) \, k^6 \, \|\phi_1(t^n)u_{ttt}(\theta^n)\|_{L^2(\mathbb{R})}^2.$$

Proof. To simplify notation we let $\phi = \phi_1$, $\gamma = \gamma_1$, $m_1 = m$ and $K = K_1$. If we insert (4.20), the result of Lemma 4.7 and (4.15) into Lemma 4.4 we get

$$\begin{aligned}
&\|\phi(t^{n+1})\xi_u^{n+1}\|_{L^2(\mathbb{R})} + \frac{k}{4}\Theta(|(\phi^2)_x|)\\
&+k\sum_j [\boldsymbol{\phi\xi}^n]_{j-1/2}((\frac{1}{2} - \frac{C}{K}(1+\lambda^{m-2}))\mathbb{I} - \mathbb{A})[\boldsymbol{\phi\xi}^n]^t_{j-1/2}\\
&+\{1 + \frac{k}{4T} + (1 + (\frac{h}{\tilde{T}})^{2(3/2-2\gamma)} + (\frac{h}{\tilde{T}})^{2-3\gamma})\frac{C\,k}{K\,T}\}||\phi(t^n)\xi_u^n||^2_{L^2(\mathbb{R})}\\
&+\{CK + (1 + (\frac{h}{\tilde{T}})^{2(3/2-2\gamma)} + (\frac{h}{\tilde{T}})^{2-3\gamma})\frac{C\,T}{k}\}||\phi(t^n)(\eta_w^n - \eta_u^n)||^2_{L^2(\mathbb{R})}\\
&+\{CK + \frac{C\,T}{k}\}(||\phi(t^n)(\eta_u^{n+1} - \eta_u^n)||^2_{L^2(\mathbb{R})} + k^6||\phi(t^n)u_{ttt}(\theta^n)||^2_{L^2(\mathbb{R})}),
\end{aligned}$$

where we used that $|(\phi^2)_x| = -(\phi^2)_x$ in this case where $\phi = \phi_1$. We see from the term

$$(\frac{h}{\tilde{T}})^{2-3\gamma}\frac{C\,k}{K\,T}\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})}$$

that we are forced to take $\gamma = 2/3$. Substituting $\gamma = 2/3$ and $m = 2$ and using our hypothesis (4.21) we see that

$$\begin{aligned}
&\|\phi(t^{n+1})\xi_u^{n+1}\|_{L^2(\mathbb{R})} + k\sum_j [\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{D}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}\\
&\le (1 + \frac{k}{T})\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})}\\
&\quad+(\frac{CT}{k}1 + C\epsilon^{-1})(\|\phi(t^n)(\eta_w^n - \eta_u^n)\|^2_{L^2(\mathbb{R})} + \|\phi(t^n)(\eta_u^{n+1} - (\eta_u^n + \eta_w^n)/2)\|^2_{L^2(\mathbb{R})})\\
&\quad+(\frac{CT}{k}1 + C\epsilon^{-1})k^6\,\|\phi(t^n)u_{ttt}(\theta^n)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

where

$$\mathbb{D} = ((\frac{1}{2} - \epsilon)\mathbb{I} - \mathbb{A}). \tag{4.22}$$

Here we used that $\frac{h}{\tilde{T}} \le 1$.

The proof will be complete if we can show that

$$k\sum_j [\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{D}[\boldsymbol{\phi\xi}^n]^t_{j-1/2} \ge 0.$$

This occurs when $\mathbb{D}$ is positive semi-definite. This is guaranteed by our hypothesis $\lambda \le 1/3 - \epsilon$. □

4.6.2. Case 2: The weight ϕ_2. In this case we do not have that $\Theta((\phi_2^2)_x) < 0$, therefore we must find an appropriate bound for this term. The bound actually is true for both ϕ_1 and ϕ_2.

LEMMA 4.12. *Let $\phi = \phi_i$, $K = K_i$, $\gamma = \gamma_i$ for either $i = 1$ or $i = 2$. Then,*

$$k\Theta((\phi^2)_x) \le \frac{Ck}{K}\sum_j [\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{I}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}$$
$$+\{\frac{k}{4T} + (1 + (\frac{h}{\tilde{T}})^{(1-2\gamma)} + (\frac{h}{\tilde{T}})^{(3-4\gamma)} + (\frac{h}{\tilde{T}})^{(5-6\gamma)})\frac{C\,k}{K\,T}\}\|\phi(t^n)\xi^n_u\|^2_{L^2(\mathbb{R})}$$
$$+(1 + (\frac{h}{\tilde{T}})^{2(1/2-\gamma)} + (\frac{h}{\tilde{T}})^{(3-4\gamma)} + (\frac{h}{\tilde{T}})^{(5-6\gamma)})\frac{C\,T}{k}\|\phi(t^n)(\eta^n_w - \eta^n_u)\|^2_{L^2(\mathbb{R})}$$
$$+\frac{C\,T}{k}(\|\phi(t^n)(\eta^{n+1}_u - \eta^n_u)\|^2_{L^2(\mathbb{R})} + \|\phi(t^n)u_{ttt}(\theta^n)\|^2_{L^2(\mathbb{R})}).$$

Proof. By using (4.6a), we have

$$\frac{k}{2}\sum_j(\xi^n_w - \xi^n_u, (\phi^2(t^n))_x(\xi^n_w - \xi^n_u))_j = \frac{k^2}{2}H(\xi^n_u, (\phi^2(t^n))_x(\xi^n_w - \xi^n_u))$$
$$+\frac{k}{2}E_2((\phi^2(t^n))_x(\xi^n_w - \xi^n_u)) + \frac{k}{2}E_3((\phi^2(t^n))_x(\xi^n_w - \xi^n_u)).$$

The result now follows if we apply Lemmas 7.9, 4.10, 7.7, and 7.3. □

Now we can bound the weighted ϕ_2 error in one time step.

THEOREM 4.13. *Let $0 < \epsilon < \frac{1}{3}$ and suppose that $\lambda \le 1/3 - \epsilon$. If $\gamma_2 = 1/2$, $m_2 = 0$ and*

$$\frac{1}{K_2} = c\,\epsilon, \tag{4.23}$$

for a sufficiently small fixed constant $c > 0$, then

$$\|\phi_2(t^{n+1})\xi^{n+1}_u\|^2_{L^2(\mathbb{R})} \le (1 + \frac{k}{T})\|\phi_2(t^n)\xi^n_u\|^2_{L^2(\mathbb{R})}$$
$$+C(\frac{T}{k} + \epsilon^{-1})\|\phi_2(t^n)(\eta^{n+1}_u - \eta^n_u)\|^2_{L^2(\mathbb{R})}$$
$$+C(\frac{T}{k} + \epsilon^{-1})\ \|\phi_2(t^n)(\eta^n_w - \eta^n_u)\|^2_{L^2(\mathbb{R})}$$
$$+C(\frac{T}{k} + \epsilon^{-1})\ k^6\ \|\phi_2(t^n)u_{ttt}(\theta^n)\|^2_{L^2(\mathbb{R})}.$$

Proof. Again, to simplify notation we let $\phi = \phi_2$, $\gamma = \gamma_2$, $m = m_2$ and $K = K_2$. If we insert (4.15) and the results of Lemmas 4.12, 4.7 into Lemma 4.4, we get

$$\|\phi(t^{n+1})\xi^{n+1}_u\|_{L^2(\mathbb{R})} + k\sum_j[\boldsymbol{\phi\xi}^n]_{j-1/2}((\frac{1}{2} - \frac{C}{K})\mathbb{I} - \mathbb{A})[\boldsymbol{\phi\xi}^n]^t_{j-1/2}$$
$$\le (1 + \frac{k}{2T} + \frac{C\,k}{K\,T})\|\phi(t^n)\xi^n_u\|^2_{L^2(\mathbb{R})}$$
$$+(C\,K + \frac{C\,T}{k})(\|\phi(t^n)(\eta^n_w - \eta^n_u)\|^2_{L^2(\mathbb{R})} + \|\phi(t^n)(\eta^{n+1}_u - \eta^n_u)\|^2_{L^2(\mathbb{R})})$$
$$+(C\,K + \frac{C\,T}{k})k^6\|\phi(t^n)u_{ttt}(\theta^n)\|^2_{L^2(\mathbb{R})},$$

where we used $\frac{h}{T} \leq 1$ and our hypothesis $\gamma = 1/2$ and $m = 0$. In fact, we only needed our original assumption $m \geq 0$ for this inequality. We choose $m = 0$ which minimizes the numerical layer. If we now use our hypothesis (4.23), we see that

$$
\begin{aligned}
&\|\phi_2(t^{n+1})\xi_u^{n+1}\|^2_{L^2(\mathbb{R})} + k\sum_j [\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{D}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}\\
&\leq (1+\frac{k}{T})\|\phi_2(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})}\\
&\quad +(\frac{CT}{k} + C\epsilon^{-1})\ \|\phi(t^n)(\eta_u^{n+1} - \eta_u^n)\|^2_{L^2(\mathbb{R})}\\
&\quad +(\frac{CT}{k} + C\epsilon^{-1})\ \|\phi(t^n)(\eta_w^{n} - \eta_u^n)\|^2_{L^2(\mathbb{R})}\\
&\quad +(\frac{CT}{k} + C\epsilon^{-1})k^6\ \|\phi(t^n)u_{ttt}(\theta^n)\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where $\mathbb{D}$ is defined in (4.22). By our hypothesis $\lambda \leq 1/3 - \epsilon$, $\mathbb{D}$ is positive definite. This proves Theorem 4.13. □

4.7. Step 9: Conclusion of the proof of Theorem 2.1. Let K_1 and ϕ_1 be as in Theorem 4.11. Choose $x_1 = -2sK_1\log(1/h)\lambda^{-7/3}T^{1/3}h^{2/3}$ for some $s > 0$, where x_1 appears in the definition of ϕ_1. We choose s sufficiently large below. Since $x_1 < -h$

$$
\begin{aligned}
&\|U(T) - u_h^N\|_{L^2(-\infty,T+x_1)} = \|u(T) - u_h^N\|_{L^2(-\infty,T+x_1)}\\
&\leq \|\eta_u^N\|_{L^2(-\infty,T+x_1)} + \|\xi_u^N\|^2_{L^2(-\infty,T+x_1)}\\
&= ||P_-(U(T)) - U(T)||_{L^2(-\infty,T+x_1)} + \|\xi_u^N\|_{L^2(-\infty,T+x_1)}.
\end{aligned}
$$

By (4.8a), we have $\|\xi_u^N\|_{L^2(-\infty,T+x_1)} \leq C\|\phi_1(t^N)(\xi_u^N)\|_{L^2(\mathbb{R})}$. Hence, we only need to bound $\|\phi_1(t^N)\xi_u^N\|_{L^2(\mathbb{R})}$. Now we apply Theorem 4.11 for $n \leq N$

$$
\begin{aligned}
\|\phi_1(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} \leq{}& (1+\frac{k}{T})\|\phi_1(t^{n-1})\xi_u^{n-1}\|^2_{L^2(\mathbb{R})}\\
&+(\frac{CT}{k} + C\,\epsilon^{-1})(\|\phi_1(t^{n-1})(\eta_u^n - \eta_u^{n-1})\|^2_{L^2(\mathbb{R})}\\
&+\|\phi_1(t^{n-1})(\eta_w^{n-1} - \eta_u^{n-1})\|^2_{L^2(\mathbb{R})})\\
&+(\frac{CT}{k} + C\,\epsilon^{-1})\,k^6\,\|\phi_1(t^{n-1})u_{ttt}(\theta^{n-1})\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

We decompose the real line as $\mathbb{R} = S_n \cup S_n^c$ where $S_n = (-\infty,\ t^n + x_1 + sK_1\log(1/h)\lambda^{-7/3}T^{1/3}h^{2/3}) = (-\infty,\ t^n - sK_1\log(1/h)\lambda^{-7/3}T^{1/3}h^{2/3})$. Notice that $u(\cdot,t^n) = U(\cdot,t^n)$ in S_n. Hence, by approximation properties of the projection operator P_-, we have

$$
\begin{aligned}
&(\frac{CT}{k} + C\,\epsilon^{-1})\|\phi_1(t^{n-1})(\eta_u^n - \eta_u^{n-1})\|^2_{L^2(S_{n-1})}\\
&+(\frac{CT}{k} + C\,\epsilon^{-1})\|\phi_1(t^{n-1})(\eta_w^{n-1} - \eta_u^{n-1})\|^2_{L^2(S_{n-1})}\\
&+(\frac{CT}{k} + C\,\epsilon^{-1})k^6\ \|\phi_1(t^{n-1})u_{ttt}(\theta^n)\|^2_{L^2(S_{n-1})} \leq (\frac{CT}{k} + C\,\epsilon^{-1})\,k^2\,h^4.
\end{aligned}
$$

It follows from (4.8b) that $\phi(x,t^{n-1}) \le h^s$ for $x \in \mathbb{R}\backslash S_{n-1}$. By using (4.2) and choosing $s \ge 4$, we can easily show that

$$\begin{aligned}
&(\frac{C\,T}{k} + C\,\epsilon^{-1})\|\phi(t^{n-1})(\eta_u^n - \eta_u^{n-1})\|^2_{L^2(\mathbb{R}\backslash S_{n-1})} \\
&+(\frac{C\,T}{k} + C\,\epsilon^{-1})\|\phi(t^{n-1})(\eta_w^{n-1} - \eta_u^{n-1})\|^2_{L^2(\mathbb{R}\backslash S_{n-1})} \\
&+(\frac{C\,T}{k} + C\,\epsilon^{-1})k^6\ \|\phi(t^{n-1})u_{ttt}(\theta^n)\|^2_{L^2(\mathbb{R}\backslash S_{n-1})} \le (\frac{C\,T}{k} + C\,\epsilon^{-1})k^2\,h^4.
\end{aligned}$$

Hence,

$$\|\phi_1(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} \le (1+\frac{k}{T})\|\phi_1(t^{n-1})\xi_u^{n-1}\|^2_{L^2(\mathbb{R})} + (\frac{C\,T}{k} + C\,\epsilon^{-1})k^2\,h^4,$$

and it follows by Gronwall's inequality that

$$\begin{aligned}
&\|\phi_1(T)\xi_u^N\|^2_{L^2(\mathbb{R})} \le (1+e)\|\phi_1(0)\xi_u^0\|^2_{L^2(\mathbb{R})} + (\frac{C\,T}{k} + C\,\epsilon^{-1})T\,k\,h^4 \\
&= (1+e)\|\phi_1(0)(P_-(U_0) - P_-(u_0))\|^2_{L^2(\mathbb{R})} + (\frac{C\,T}{k} + C\,\epsilon^{-1})T\,k\,h^4.
\end{aligned}$$

Again, using that $U_0 = u_0$ in $(-\infty, -h)$ and using the decay properties of $\phi_1(0)$, we have

$$\|\phi_1(t^N)\xi_u^N\|^2_{L^2(\mathbb{R})} \le (\frac{C\,T}{k} + C\,\epsilon^{-1})T\,k\,h^4 + Ch^s.$$

Therefore, using this result and (4.8a) we get that

$$\begin{aligned}
&\|U(T) - u_h^N\|_{L^2(-\infty,\,T-2\,s\,K_1\,\log(1/h)\lambda^{-7/3}T^{1/3}h^{2/3})} \le CTh^2 + C(\frac{Tk}{\epsilon})^{1/2}h^2 + Ch^s \\
&+\|P_-(U(T)) - U(T)\|_{L^2(-\infty,\,T-2\,s\,K_1\,\log(1/h)\lambda^{-7/3}T^{1/3}h^{2/3})}
\end{aligned}$$

Since $K_1 = \frac{1}{c\,\epsilon}$, we can choose $\beta = \frac{2}{c}$ to obtain

$$\begin{aligned}
&\|U(T) - u_h^N\|_{L^2(-\infty,\,T-\beta\,s\log(1/h)\lambda^{-7/3}\epsilon^{-1}T^{1/3}h^{2/3})} \le CTh^2 + C(\frac{Tk}{\epsilon})^{1/2}h^2 + Ch^s \\
&+\|P_-(U(T)) - U(T)\|_{L^2(-\infty,\,T-\beta\,s\log(1/h)\lambda^{-7/3}\epsilon^{-1}T^{1/3}h^{2/3})}
\end{aligned}$$

The estimate in the region $x > T + \beta\,s\log(1/h)\lambda^{-1/2}\epsilon^{-1}T^{1/2}h^{1/2}$ can be established in a similar fashion.

5. Numerical Experiments. The purpose of this section is to verify our main result, Theorem 2.1, and to present numerical evidence suggesting that it is sharp.

5.1. L^2 **errors.** To verify the estimate of Theorem 2.1, we consider the problem (1.1) with periodic initial condition $U_0(x) = \sin(2\pi x) + \chi(x)$ for $x \in [0,1]$ where χ is the characteristic function of the interval $(\frac{1}{4}, \frac{3}{4})$. We use uniform spatial meshes, uniform time stepping and take $CFL = \lambda = k/h = 0.33$.

We will investigate the error at time $T = 1$. The mesh ℓ will denote a mesh with size $h_l = \frac{2^{-\ell}}{1000}$. We compute the L^2 error to the left and right of the singularity $x = \frac{1}{4}$ and $T = 1$. Specifically, we compute the following errors

$$\begin{aligned}
eL_\ell :=& \|U(T) - u_h(T)\|_{L^2(0,\frac{1}{4}-5h_l^{2/3})} \\
eR_\ell :=& \|U(T) - u_h(T)\|_{L^2(\frac{1}{4}+5h_l^{1/2},\frac{7}{10})}.
\end{aligned}$$

We then compute the orders of convergence defined by

$$oL_\ell := \frac{\log(\frac{eL_\ell}{eL_{\ell+1}})}{\log(2)}$$

$$oR_\ell := \frac{\log(\frac{eR_\ell}{eR_{\ell+1}})}{\log(2).}$$

We list the results in Table 5.1. As we can see, the optimal orders of convergence are realized; this confirms the prediction of Theorem (2.1). Note that we did not use a logarithmic factor in this computational experiment. We simply took the region around the discontinuity as $\mathcal{R}_T = \frac{1}{4} + (-5\,h^{2/3}, 5\,h^{1/2})$.

Table 5.1
Error to the left and to the right of the singularity

ℓ	eL_ℓ	oL_ℓ	eR_ℓ	oR_ℓ
1	.44e-8	-	.58e-8	-
2	.11e-8	1.98	.15e-8	1.96
3	.28e-9	1.99	.38e-9	1.98
4	.70e-10	1.99	.95e-10	1.99
5	.18e-10	2.00	.24e-10	1.99
6	.44e-11	2.00	.60e-11	1.99
7	.20e-11	2.00	.15e-11	2.00

5.2. Behavior of the error near discontinuities. To show that our results are sharp, we explore a very precise form of the error near the discontinuity for the special periodic initial condition $U_0(x) = \chi(x)$ for $x \in [0, 1]$.

We would study the error *near* the discontinuity $x = 1/4$ and $T = 1$. We do this by scaling the error near the discontinuity and plotting the results for different h. To the left of the discontinuity, for each fix mesh size h we plot the scaled error

$$\mathrm{errorL}(y, h) := \left| U(1/4 + h^{2/3}\,y, T) - u_h(1/4 + h^{2/3}\,y, T) \right|.$$

Note that here $y \leq 0$. The results are given in the top of Figure 5.1. Note that as we decrease h the graphs seem to be converging. This reflects the fact that the scaling $h^{2/3}$ is the correct one and that our results are sharp.

Similarly, to the right of the discontinuity we define, for $y \geq 0$,

$$\mathrm{errorR}(y, h) := \left| U(1/4 + h^{1/2}\,y, T) - u_h(1/4 + h^{1/2}\,y, T) \right|.$$

Since the above function is very oscillatory, we are going to consider its "envelope" which we denote by "envelopeR(y, h)"; see the middle of Figure 5.1. In the bottom of Figure 5.1, we plot those envelopes for various h in Figure 5.1. Notice that the graphs seem to be converging as we decrease h. Again, this reflects the fact that the scaling $h^{1/2}$ is the correct one and that our results are sharp.

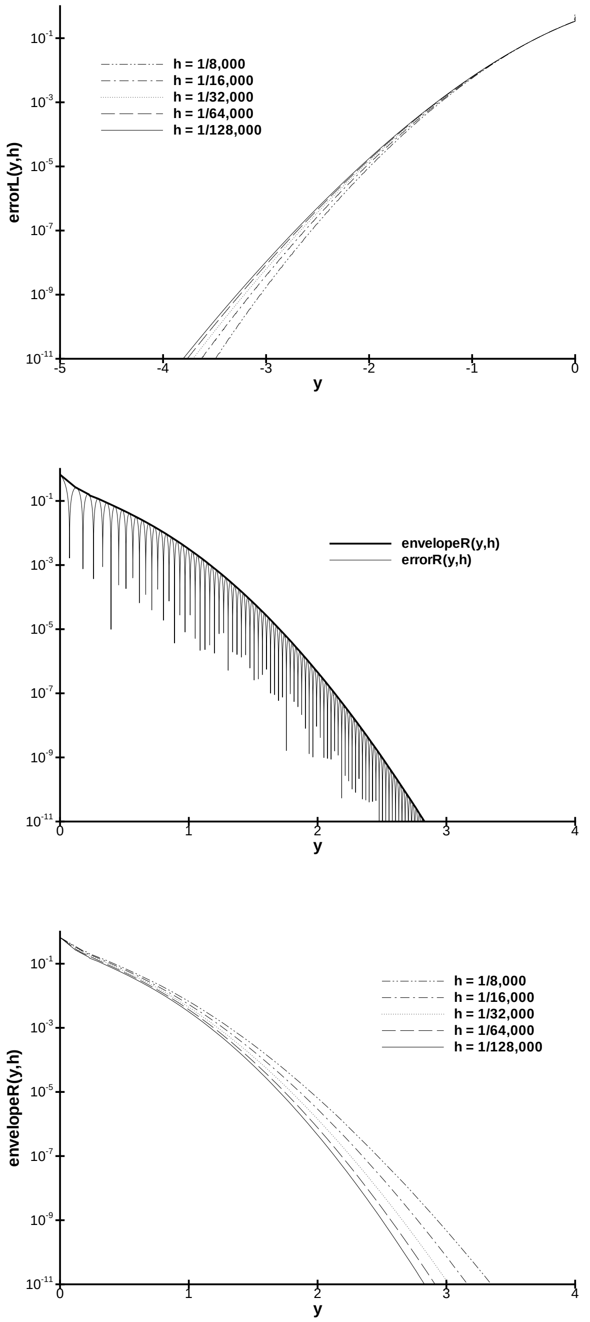

Fig. 5.1. *History of convergence of the scaled error to the left of the discontinuity (top) and of the "envelope" of the scaled error to the right of the discontinuity (bottom) for different values of the meshsize h. In the middle, we see an example of the scaled error to the right of the discontinuity and its "envelope".*

6. Concluding remarks. The approach developed in this paper could be applied to the study of general RKDG methods for the model problem under consideration. The only difficulty in carrying out such a task is that the use of high-order accurate Runge-Kutta methods quickly complicates the calculations. If we solve exactly in time, however, a characterization of the region containing the numerical layer should be fairly easy to obtain for any given polynomial degree p for the DG discretization in space. This constitutes the subject of ongoing work.

7. Appendix.

7.1. Construction of weights. To finish the construction of the weights ϕ_1 and ϕ_2 we only need to construct the initial conditions b_1 and b_2.

$$b_1(x) = \psi\left((x-x_1)/\tilde{K}_1\tilde{T}^{1-\gamma_1}\, h^{\gamma_1}\right), \quad \text{and} \quad b_2(x) = \psi\left((x_2-x)/\tilde{K}_2\tilde{T}^{1-\gamma_2}\, h^{\gamma_2}\right).$$

The function ψ is given by

$$\psi(r) = \int_r^\infty e^{-g(s)} ds,$$

where $g \in C^5(\mathbb{R})$ and is such that $g(s) = |s|$ for $|s| \geq 1$.

Now we can prove Proposition 4.3

Proof. Since we have the following representation of ϕ for $i = 1, 2$:

$$\phi_1(x,t) = b_1(x-t) = \psi\left(((x-t)-x_1)/\tilde{K}_1\tilde{T}^{1-\gamma_1}\, h^{\gamma_1}\right),$$
$$\phi_2(x,t) = b_2(x-t) = \psi\left((x_2-(x-t))/\tilde{K}_2\tilde{T}^{1-\gamma_2}\, h^{\gamma_2}\right),$$

it is not difficult to show that these results can be easily obtained by using the following properties of ψ:

$$\begin{aligned} c \leq \psi(r) \leq C \quad &\text{for} \quad r \leq 1, \\ \psi'(x) < 0 \quad &\text{for} \quad x \in \mathbb{R} \\ \psi(r) = e^{-r} \quad &\text{for} \quad r > 1, \\ |\psi'(r)| + |\psi''(r)| + |\psi'''(r)| \leq C|\psi(r)| \quad &\text{for} \quad r \in \mathbb{R}, \\ |\psi''(r)| + |\psi'''(r)| + |\psi''''(r)| \leq C|\psi'(r)| \quad &\text{for} \quad r \in \mathbb{R}, \end{aligned}$$

and that, for any interval I of unit length, $RO(I,\psi) + RO(I,\psi') \leq C$. This completes the sketch of the proof. □

7.2. Proof of Lemma 4.9. The proof is very similar to the super-approximation proofs contained in [16] and [13].

Proof. To simplify notation we drop t. By approximation properties of P, we have that

$$\begin{aligned} &\|D_x^l(\phi^2)v - P(D_x^l(\phi^2)v)\|_{L^2(I_j)} + h\|(D_x^l(\phi^2)v)_x - P(D_x^l(\phi^2)v)_x\|_{L^2(I_j)} \\ &\leq Ch^2\|(D_x^l(\phi^2)v)_{xx}\|_{L^2(I_j)}. \end{aligned}$$

Since $v_{xx} = 0$, we have that

$$h^2\|(D_x^l(\phi^2)v)_{xx}\|_{L^2(I_j)} \leq h^2\|D_x^{l+2}(\phi^2)v\|_{L^2(I_j)} + 2h^2\|D_x^{l+1}(\phi^2)v_x\|_{L^2(I_j)}.$$

Using (4.8d), we obtain

$$\|D_x^l(\phi^2)v - P(D_x^l(\phi^2)v)\|_{L_h^2(I_j)} + h\|(D_x^l(\phi^2)v)_x - P(D_x^l(\phi^2)v)_x\|_{L^2(I_j)}$$
$$\leq \frac{Ch^{2-(l+1)\gamma}}{\tilde{T}^{(1-\gamma)(l+2)}\tilde{K}^{l+2}}\|\phi^2 v\|_{L^2(I_j)} + \frac{h^{2-\frac{1}{2}(l+1)\gamma}}{\tilde{T}^{(1-\gamma)(l+1/2)}K^{(l+1/2)}}\|\phi|\phi\phi_x|^{1/2}v_x\|_{L^2(I_j)} \quad (7.1)$$

If we multiply both sides by ϕ^{-1} (here we use (4.8e)) and set $l = 0$ we arrive at (4.15a). The inequality (4.15b) follows from (7.1), (4.8d), (4.8e) and (7.4). □

7.3. Proof of Lemma 4.5. We start by proving (4.13b). If χ_j is the characteristic function of I_j, then

$$\overline{\xi_w^n - \xi_u^n}(x_j) = \frac{1}{h}(\xi_w^n - \xi_u^n, 1)_j = \frac{k}{h}H(\xi_u^n, \chi_j) + \frac{1}{h}(E_2(\chi_j) + E_3(\chi_j)).$$

Since $\chi_j \in V_h$, $E_2(\chi_j) = 0$. By the properties of P_- we have that $E_3(\chi_j) = 0$. Also, from the definition of H we have $H(\xi_u^n, \chi_j) = -h(\xi_u^n)_x(x_j) - [\xi_u^n]_{j-1/2}$. This shows that

$$\overline{\xi_w^n - \xi_u^n}(x_j) = -k(\xi_u^n)_x(x_j) - \frac{k}{h}[\xi_u^n]_{j-1/2}. \quad (7.2)$$

Using (4.12), we have

$$(\xi_w^n - \xi_u^n)_x(x_j) = (\xi_w^n - \xi_u^n, x - x_j)_j / \int_{I_j}(x - x_j)^2 dx$$
$$= \frac{12}{h^3}\{kH(\xi_u^n, \chi_j(x - x_j)) + E_2(\chi_j(x - x_j)) + E_3(\chi_j(x - x_j))\}.$$

We used that $\int_{I_j}(x - x_j)^2 = \frac{h^3}{12}$.

Again, since $\chi_j(x - x_j) \in V_h$, we see that $E_2(\chi_j(x - x_j)) = 0$. By the definition of E_3, we have $E_3(\chi_j(x - x_j)) = R_{1,j}$. Also, $H(\xi_u^n, \chi_j(x - x_j)) = -\int_{I_j}(x - x_j)(\xi_u^n)_x dx - (x - x_j)(x_{j-1/2}^+)[\xi_u^n]_{j-1/2} = \frac{h}{2}[\xi_u^n]_{j-1/2}$ since $(\xi_u^n)_x$ is constant on I_j and $\int_{I_j}(x - x_j)dx = 0$.

Therefore,

$$(\xi_w^n - \xi_u^n)_x(x_j) = \frac{6k}{h^2}[\xi_u^n]_{j-1/2} + \frac{12}{h^3}R_{1,j}.$$

Equation (4.13b) now follows from (4.12), (7.2) and the above identity.

Subtracting (4.6a) from (4.6b) we have

$$(\xi_u^{n+1} - \xi_w^n, v) = \frac{k}{2}H(\xi_w^n - \xi_u^n, v) + E_1(P_+(v)) - E_2(v)/2 - E_3(v)/2$$
$$+ E_4(v) + E_5(v).$$

Using this representation and following the techniques used to prove (4.13b) we can easily show (4.13a).

7.4. Proof of Lemma 4.8. In order to prove Lemma 4.8 we will state and prove a series of lemmas. Most of these lemmas are bounds for the terms of E_h. We first state standard inverse estimates.

Lemma 7.1. *There exists a $C > 0$ such that for every $v \in P^1(I_j)$*

$$|v(x_{j+1/2})| + |v(x_{j-1/2})| \le Ch^{-1/2}\|v\|_{L^2(I_j)}. \tag{7.3}$$

and

$$\|v'\|_{L^2(I_j)} \le Ch^{-1}\|v\|_{L^2(I_j)}. \tag{7.4}$$

We also need the following preliminary estimates.

Lemma 7.2. *Let $t^n \le t \le t^{n+1}$, then for $v_h \in V_h$*

$$\|\phi^{-1}(t)(P_+(\phi^2(t)v_h) - \phi^2(t)v_h)\|_{L^2(I_j)} \le \frac{C}{K}\|\phi(t^n)v_h\|_{L^2(I_j)}, \tag{7.5}$$

and for K sufficiently large

$$\|\phi^{-1}(t)P_+(\phi^2(t)v_h)\|_{L^2(I_j)} \le 2\|\phi(t^n)v_h\|_{L^2(I_j)}. \tag{7.6}$$

Proof. By (4.15b) we have

$$\|\phi^{-1}(t)(P_+(\phi^2(t)v_h) - \phi^2(t)v_h)\|_{L^2(I_j)} \le ((\frac{h}{\tilde{T}})^{1-\gamma}\frac{C}{K} + (\frac{h}{\tilde{T}})^{2(1-\gamma)}\frac{C}{K^2})\|\phi(t^n)v_h\|_{L^2(I_j)}.$$

The inequality (7.5) now follows from $K \ge 1$ and $\frac{h}{\tilde{T}} \le 1$. The inequality (7.6) easily follows from (7.5). ∎

The following result compares ξ_w^n to ξ_u^n.

Lemma 7.3. *If $t^n \le t \le t^{n+1}$ and K is sufficiently large, then*

$$\|\phi(t)\xi_w^n\|^2_{L^2(\mathbb{R})} \le C\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} + C\|\phi(t^n)(\eta_w^n - \eta_u^n)\|^2_{L^2(\mathbb{R})}.$$

Proof. Using (4.6a), we have

$$\|\phi(t)\xi_w^n\|^2_{L^2(\mathbb{R})} = \sum_j(\xi_u^n, \phi^2(t)\xi_w^n)_j + kH(\xi_u^n,\, \phi^2(t)\xi_w^n) + E_2(\phi^2(t)\xi_w^n) + E_3(\phi^2(t)\xi_w^n).$$

By applying (4.8e) and the Cauchy-Schwarz inequality we get

$$\sum_j(\xi_u^n, \phi^2(t)\xi_w^n)_j \le C\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} + \frac{1}{8}\|\phi(t)\xi_w^n\|^2_{L^2(\mathbb{R})}.$$

If we use (4.8e), (7.3) and (7.4) we get

$$kH(\xi_u^n,\, \phi^2(t)\xi_w^n) \le C\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} + \frac{1}{8}\|\phi(t)\xi_w^n\|^2_{L^2(\mathbb{R})}.$$

By using (7.5) and taking K sufficiently large we get

$$E_2(\phi^2(t^n)\xi_w^n) \le C\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} + \frac{1}{8}\|\phi(t)\xi_w^n\|^2_{L^2(\mathbb{R})}.$$

If we use (7.6) we get

$$E_3(\phi^2(t)\xi_w^n) \le C\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} + \frac{1}{8}\|\phi(t)\xi_w^n\|^2_{L^2(\mathbb{R})}.$$

Combining the last four inequalities proves Lemma 7.3. □

We will need a different bound than what is given in Lemma 4.10 for the case $l = 0$ and $v_h = \xi_u^n$ or $v_h = \xi_w^n$.

LEMMA 7.4. *Let* $t^n \le t \le t^{n+1}$, *then*

$$
\begin{aligned}
&|E_2(\phi^2(t)\xi_u^n)| + |E_4(\phi^2(t)\xi_w^n)| \\
&\le (1+\lambda^{m-2})\frac{Ck}{K}\sum_j [\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{I}[\boldsymbol{\phi\xi}^n]_{j-1/2}^t + \frac{k}{6}\Theta(|(\phi^2)_x|) \\
&\quad + (1+(\frac{h}{\tilde T})^{2(3/2-2\gamma)})\frac{C\,k}{K\,T}||\phi(t^n)\xi_u^n||^2_{L^2(\mathbb{R})} \\
&\quad + (1+(\frac{h}{\tilde T})^{2(3/2-2\gamma)})\frac{T}{k}||\phi(t^n)(\eta_w^n - \eta_u^n)||^2_{L^2(\mathbb{R})} \\
&\quad + \frac{C\,T}{k}(||\phi(t^n)(\eta_u^{n+1} - \eta_u^n)||^2_{L^2(\mathbb{R})} + k^6||\phi(t^n)u_{ttt}(\theta^n)||^2_{L^2(\mathbb{R})}).
\end{aligned}
$$

Proof. As was done in the proof of Lemma 4.10, we see that

$$
\begin{aligned}
|E_2(\phi^2(t)\xi_u^n)| &= |\sum_j(\frac{6k}{h^2}[\xi_u^n]_{j-1/2}(\phi(t)(x-x_j), \phi^{-1}(t)(\phi^2(t)\xi_u^n - P_+(\phi^2(t)\xi_u^n)))_j \\
&\qquad + \sum_j \frac{12}{h^3}R_{1,j}(\phi(t)(x-x_j), \phi^{-1}(t)(\phi^2(t)\xi_u^n - P_+(\phi^2(t)\xi_u^n)))_j| \\
&\le M_1 + M_2,
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &:= \frac{C\,k}{h^{1/2}}\sum_j |[\phi(t^n)\xi_u^n]_{j-1/2}|\,\|\phi^{-1}(t)(\phi^2(t)\xi_u^n - P_+(\phi^2(t)\xi_u^n))\|_{L^2(I_j)} \\
M_2 &:= \frac{C}{h^{3/2}}\sum_j |R_{1,j}\phi(t^n, x_{j-1/2})|\,\|\phi^{-1}(t)(\phi^2(t)\xi_u^n - P_+(\phi^2(t)\xi_u^n))\|_{L^2(I_j)}.
\end{aligned}
$$

We first bound M_2. Using (4.19) with $v_h = \xi_u^n$ and $l = 0$ combined with the inequalities $K \ge 1$, $\frac{h}{\tilde T} \le 1$ and $\gamma \le 1$ we have

$$
M_2 \le \frac{C\,k}{K\,T}\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} + \frac{C\,T}{k}\|\phi(t^n)(\eta_w^n - \eta_u^n)\|^2_{L^2(I_j)}. \tag{7.7}
$$

For the next term we use (4.8e) and (4.15a), to obtain

$$
\begin{aligned}
M_1 &\le \frac{Ckh^{3/2-2\gamma}}{\tilde T^{2(1-\gamma)}\tilde K^2}\sum_j |[\phi(t^n)(\xi_u^n)]_{j-1/2}|\ \|\phi(t^n)\xi_u^n\|_{L^2(I_j)} \\
&\quad + \frac{Ckh^{3/2-\gamma/2}}{\tilde T^{(1-\gamma)/2}\tilde K^{1/2}}\sum_j |[\phi(t^n)(\xi_u^n)]_{j-1/2}|\ \|(|\phi(t^n)(\phi)_x(t^n)|)^{1/2}(\xi_u^n)_x\|_{L^2(I_j)}.
\end{aligned}
$$

Using Young's inequality, we have

$$
\begin{aligned}
&\frac{Ckh^{3/2-2\gamma}}{\tilde T^{2(1-\gamma)}\tilde K^2}\sum_j |[\phi(t^n)(\xi_u^n)]_{j-1/2}|\ \|\phi(t^n)\xi_u^n\|_{L^2(I_j)} \\
&\le \frac{C}{K^2}k\sum_j[\phi(t^n)\xi_u^n]^2_{j+1/2} + (\frac{h}{\tilde T})^{2(3/2-2\gamma)}\frac{C\,k}{K^2\,T}\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{7.8}
$$

If we use (4.13b), the triangle inequality and the fact that $|x - x_j| \leq h$ for $x \in I_j$, then

$$\|(|\phi(t^n)(\phi)_x(t^n)|)^{1/2}(\xi_u^n)_x\|_{L^2(I_j)} \leq \frac{1}{k}\Theta_j(|(\phi^2)_x|)^{1/2}$$
$$+\frac{Ch^{-(1+\gamma)/2}}{\tilde{K}^{1/2}\tilde{T}^{(1-\gamma)/2}}|[\phi(t^n)\xi_u^n]_{j-1/2}| + \frac{Ch^{-\gamma/2}}{k\tilde{K}^{1/2}\tilde{T}^{(1-\gamma)/2}}||\phi(t^n)(\eta_w^n - \eta_u^n)||_{L^2(I_j)},$$

where we also used $|R_{1,j}| \leq h^{3/2}||\eta_w^n - \eta_u^n||_{L^2(I_j)}$, (4.8d) and (4.8e).

Therefore,

$$\frac{Ckh^{3/2-\gamma/2}}{\tilde{T}^{(1-\gamma)/2}\tilde{K}^{1/2}}\sum_j |[\phi(t^n)(\xi_u^n)]_{j-1/2}|\ \|(|\phi(t^n)(\phi)_x(t^n)|)^{1/2}(\xi_u^n)_x\|_{L^2(I_j)}$$
$$\leq (\frac{C\lambda^{-2}}{\tilde{K}}(\frac{h}{\tilde{T}})^{1-\gamma} + \frac{C}{\tilde{K}}(\frac{h}{\tilde{T}})^{2(1-\gamma)} + \frac{C\lambda^{-1}}{\tilde{K}^2}(\frac{h}{\tilde{T}})^{2(3/2-\gamma)})k\sum_j[\phi(t^n)\xi_u^n]_{j-1/2}^2$$
$$+\frac{k}{12}\Theta(|(\phi^2)_x|) + \frac{T}{k}||\eta_w^n - \eta_u^n||_{L^2(\mathbb{R})}^2$$
$$\leq (\lambda^{m-2} + \lambda^m + \lambda^{2m-1})\frac{Ck}{K}\sum_j[\phi(t^n)\xi_u^n]_{j-1/2}^2 + \frac{k}{12}\Theta(|(\phi^2)_x|)$$
$$+\frac{CT}{k}||\phi(t^n)(\eta_w^n - \eta_u^n)||_{L^2(\mathbb{R})}^2.$$

Here we used that $\tilde{K} = \lambda^{-m}K$, $\frac{h}{\tilde{T}} \leq 1$, $\gamma \leq 1$ and $K \geq 1$. Combining this inequality with (7.8) we get

$$M_1 \leq (1 + \lambda^{m-2})\frac{Ck}{K}\sum_j[\phi(t^n)\xi_u^n]_{j-1/2}^2 + \frac{k}{12}\Theta(|(\phi^2)_x|)$$
$$+(\frac{h}{\tilde{T}})^{2(3/2-2\gamma)}\frac{C\,k}{K\,T}\|\phi(t^n)\xi_u^n\|_{L^2(\mathbb{R})}^2$$
$$+\frac{CT}{k}||\phi(t^n)(\eta_w^n - \eta_u^n)||_{L^2(\mathbb{R})}^2,$$

where we used that $K \geq 1$, $\lambda \leq 1$ and $m \geq 0$. Therefore, combining the estimate of M_1 and M_2 gives the right bound for $|E_2(\phi^2(t)\xi_u^n)|$. We are left to bound $|E_4(\phi^2(t)\xi_w^n)|$. To this end, we write

$$|E_4(\phi^2(t)\xi_w^n)| = |E_4(\phi^2(t)\xi_u^n) + E_4(\phi^2(t)(\xi_w^n - \xi_u^n))|.$$

We can follow the proof of the bound for $E_2(\phi^2(t)\xi_u^n)$ to obtain

$$|E_4(\phi^2(t)\xi_u^n)| \leq (1 + \lambda^{m-2})\frac{Ck}{K}\sum_j[\phi(t^n)\xi_w^n]_{j-1/2}^2$$
$$+\frac{k}{24}\Theta(|(\phi^2)_x|) + (1 + (\frac{h}{\tilde{T}})^{2(3/2-2\gamma)})\frac{Ck}{TK}||\phi(t^n)\xi_u^n||_{L^2(\mathbb{R})}^2$$
$$+(1 + (\frac{h}{\tilde{T}})^{2(3/2-2\gamma)})\frac{CT}{k}||\phi(t^n)(\eta_w^n - \eta_u^n)||_{L^2(\mathbb{R})}^2$$
$$+\frac{CT}{K\,k}(||\phi(t^n)(\eta_u^{n+1} - \eta_u^n)||_{L^2(\mathbb{R})}^2) + k^6||\phi(t^n)u_{ttt}(\theta^n)||_{L^2(\mathbb{R})}^2).$$

If we use (4.13a) and (4.13b) and the properties of P_+, we have

$$|E_4(\phi^2(t)(\xi_w^n - \xi_u^n))| = \sum_j (\frac{3k}{h^2}[\xi_w^n]_{j-1/2} + \frac{k^3}{6h^3}R_{4,j} + \frac{12}{h^3}R_{2,j}) \times$$
$$(x - x_j,\ \phi^2(t)(\xi_w^n - \xi_u^n) - P_+(\phi^2(t)(\xi_w^n - \xi_u^n)))_j$$
$$\leq Q_1 + Q_2 + Q_3,$$

where

$$Q_1 := \frac{C}{h^{3/2}} \sum_j |R_{2,j}\phi(t^n, x_{j-1/2})|\, D_j,$$
$$Q_2 := \frac{C\,k^3}{h^{3/2}} \sum_j |R_{4,j}\phi(t^n, x_{j-1/2})|\, D_j,$$
$$Q_3 := \frac{C\,k}{h^{1/2}} \sum_j |[\phi(t^n)\xi_w^n]_{j-1/2}|\, D_j,$$

and

$$D_j := \|\phi^{-1}(t)(\phi^2(t)(\xi_w^n - \xi_u^n) - P_+(\phi^2(t)(\xi_w^n - \xi_u^n)))\|_{L^2(I_j)}.$$

Following the proof of the bound for M_2, we can show

$$Q_1 \leq \frac{C\,k}{K\,T}\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} + \frac{C\,T}{k}(\|\phi(t^n)(\eta_w^n - \eta_u^n)\|^2_{L^2(\mathbb{R})} + \|\phi(t^n)(\eta_u^{n+1} - \eta_u^n)\|^2_{L^2(\mathbb{R})}),$$

where in addition we used Lemma 7.3. Similarly, we can show

$$Q_2 \leq \frac{C\,k}{K\,T}\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} + \frac{C\,T}{k}(\|\phi(t^n)(\eta_w^n - \eta_u^n)\|^2_{L^2(\mathbb{R})} + k^6\|\phi(t^n)u_{ttt}(\theta^n)\|^2_{L^2(\mathbb{R})}).$$

In order to bound Q_3 we first bound D_j using (4.15a)

$$|D_j| \leq \frac{Ch^{2-2\gamma}}{\tilde{T}^{2(1-\gamma)}\tilde{K}^2}(||\phi(t^n)\xi_u^n||_{L^2(I_j} + ||\phi(t^n)\xi_w^n||_{L^2(I_j)})$$
$$\frac{Ch^{2-\gamma/2}}{\tilde{T}^{(1-\gamma)/2}\tilde{K}^{1/2}}||(|\phi(t^n)(\phi(t^n))_x|)^{1/2}(\xi_w^n - \xi_u^n)_x||_{L^2(I_j)}$$
$$\leq \frac{Ch^{2-2\gamma}}{\tilde{T}^{2(1-\gamma)}\tilde{K}}(||\phi(t^n)\xi_u^n||_{L^2(I_j} + \|\phi(t^n)(\eta_w^n - \eta_u^n)\|_{L^2(I_j)})$$
$$+ \frac{Ch^{1-\gamma/2}}{\tilde{T}^{(1-\gamma)/2}\tilde{K}^{1/2}}\Theta_j(|(\phi^2)_x|)^{1/2}.$$

In the last inequality we used Lemma 7.3 and the inverse estimate (7.4). Therefore,

$$Q_3 \leq \frac{Ck}{K}\sum_j[\phi(t^{n+1})\xi_w^n]^2_{j-1/2} + (\frac{h}{\tilde{T}})^{2(3/2-2\gamma)}\frac{C\,k}{KT}||\phi\xi_u^n||^2_{L^2((R)}$$
$$+ (\frac{h}{\tilde{T}})^{2(3/2-2\gamma)}\frac{C\,T}{k}\|\phi(t)(\eta_w^n - \eta_u^n)\|^2_{L^2(\mathbb{R})}$$
$$+ \frac{k}{24}\Theta(|(\phi^2)_x|),$$

where we used that $\gamma \leq 1$, $K \geq 1$, and $\lambda \leq 1$. Hence,

$$|E_4(\phi^2(t)(\xi_w^n - \xi_u^n))| \leq \frac{Ck}{K}\sum_j [\phi(t^{n+1})\xi_w^n]_{j-1/2}^2$$
$$+\frac{k}{24}\Theta(|(\phi^2)_x|) + (1+(\frac{h}{\tilde{T}})^{2(3/2-2\gamma)})\frac{C\,k}{KT}||\phi(t^n)\xi_u^n||_{L^2((R)}^2$$
$$+(1+(\frac{h}{\tilde{T}})^{2(3/2-2\gamma)})\frac{C\,T}{k}\|\phi(t^n)(\eta_w^n - \eta_u^n)\|_{L^2(\mathbb{R})}^2$$
$$+\frac{CT}{k}(\|\phi(t^n)(\eta_u^{n+1} - \eta_w^n)\|_{L^2(\mathbb{R})}^2 + k^6||\phi(t^n)u_{ttt}(\theta^n)||_{L^2((R)}^2).$$

This completes the proof of Lemma 7.4. □

Lemma 7.5.

$$|\frac{k^2}{2}\sum_j(\xi_w^n - \xi_u^n, (\phi^2(t^n))_{xx}\xi_w^n)_j| \leq (\frac{h}{\tilde{T}})^{2-3\gamma}\frac{C\,k}{K^3\,T}\|\phi(t^n)\xi_u^n\|_{L^2(\mathbb{R})}^2 + \frac{k}{12}\Theta(|(\phi^2)_x|)$$
$$+(\frac{h}{\tilde{T}})^{2-3\gamma}\frac{C\,T}{K^3\,k}\|\phi(t^n)(\eta_w^n - \eta_u^n)\|_{L^2(\mathbb{R})}^2. \qquad (7.9)$$

and

$$|\frac{k^2}{2}\sum_j(\xi_w^n - \xi_u^n, (\phi^2(t^n))_{xx}\xi_w^n)_j| \leq (\frac{h}{\tilde{T}})^{1-2\gamma}\frac{C\,k}{K^2\,T}\|\phi(t^n)\xi_u^n\|_{L^2(\mathbb{R})}^2$$
$$+(\frac{h}{\tilde{T}})^{1-2\gamma}\frac{C\,T}{K^2\,k}\|\phi(t^n)(\eta_w^n - \eta_u^n)\|_{L^2(\mathbb{R})}^2. \qquad (7.10)$$

Proof. The results are simple consequences of (4.8d), (4.8e), the Cauchy-Scwharz inequality and Lemma 7.3. □

Lemma 7.6. *For K sufficiently large*

$$|E_1(P_+(\phi^2(t^{n+1})\xi_w^n)| \leq \frac{k}{8\,T}\|\phi(t^n)\xi_u^n\|_{L^2(\mathbb{R})}^2 + CTk^5\|\phi(t^n)u_{ttt}(\theta^n)\|_{L^2(\mathbb{R})}^2.$$

Proof. By applying Cauchy-Schwarz inequality, (7.6) and Young's inequality we obtain Lemma 7.6. □

Lemma 7.7. *Let $t^n \leq t \leq t^{n+1}$ and $v_h = \xi_u^n$ or $v_h = \xi_w^n$. For $l = 0, 1, 2$, we have*

$$h^l|E_3(D_x^l(\phi^2(t))v_h)| + h^l|E_5(D_x^l(\phi^2(t))v_h)|$$
$$\leq \frac{k}{16\,T}\|\phi(t^n)\xi_u^n\|_{L^2(\mathbb{R})}^2 + \frac{C\,T}{k}(\|\phi(t^n)(\eta_u^{n+1} - \eta_u^n)\|_{L^2(\mathbb{R})}^2 + \|\phi(t^n)(\eta_w^n - \eta_u^n)\|_{L^2(\mathbb{R})}^2).$$

Proof. This result easily follows from the Cauchy-Schwarz inequality, (7.6), Lemma 7.3, and Young's inequality. □

Lemma 7.8. *We have*

$$|S_1| + |S_2| \leq (\frac{h}{\tilde{T}})^{2-3\gamma}\frac{Ck}{K^3T}\|\phi(t^n)\xi_u^n\|_{L^2(\mathbb{R})}^2 + (\frac{h}{\tilde{T}})^{2-3\gamma}\frac{CT}{k}||\phi(t^n)(\eta_w^n - \eta_u^n)||_{L^2(\mathbb{R})}^2.$$

Proof. This follows from (4.8d), (4.8e), the Cauchy-Schwarz inequality and Lemma 7.3. □

Now we can prove Lemma 4.8 which is a simple consequence of the lemmas of this section.

Proof. We start with the proof of (4.15). If we apply Lemma 7.6 , Lemma 7.4, Lemma 7.7, (7.9) and Lemma 7.8 we get (4.15).

To prove (4.15) we proceed as the proof for (4.15), but instead of using Lemma 7.4 we apply Lemma 4.10 only and we use (7.10) rather than (7.9). We also use Lemma 7.3. ▯

7.5. Auxiliary Lemma. The following lemma is used to prove (4.12).

LEMMA 7.9. *Let $\phi = \phi_i$, $\gamma = \gamma_i$ and $K = K_i$ for $i = 1$ or $i = 2$. If $t^n < t < t^{n+1}$, then*

$$k^2|H(\xi_u^n, (\phi^2(t))_x(\xi_w^n - \xi_u^n))| \le \frac{Ck}{K}\sum_j [\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{I}[\boldsymbol{\phi\xi}^n]^t_{j-1/2}$$
$$+((\frac{h}{\tilde{T}})^{2(1/2-\gamma)} + 1)\frac{C\,k}{K\,T}\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})}$$
$$+(1 + (\frac{h}{\tilde{T}})^{2(1/2-\gamma)})\frac{C\,T}{k}\|\phi(t^n)(\eta_w^n - \eta_u^n)\|^2_{L^2(\mathbb{R})}.$$

Proof.

$$k^2|H(\xi_u^n, (\phi^2(t))_x(\xi_w^n - \xi_u^n))|$$
$$= k^2|\sum_j (\xi_u^n, ((\phi^2)_x(t)(\xi_w^n - \xi_u^n))_x)_j + \xi_u^n(x^-_{j+1/2})[(\phi^2)_x(t)(\xi_w^n - \xi_u^n)]_{j+1/2}|.$$

By applying the Cauchy-Schwarz inequality , (4.8d), (4.8e),(7.3) and Young's inequality we have

$$k^2|\sum_j \xi_u^n(x^-_{j+1/2})[(\phi^2)_x(t)(\xi_w^n - \xi_u^n)]_{j+1/2}|$$
$$\le \frac{C\,k}{K}\sum_j\{[\boldsymbol{\phi\xi}^n]_{j-1/2}\mathbb{I}[\boldsymbol{\phi\xi}^n]^t_{j-1/2} + (\frac{h}{\tilde{T}})^{2(1/2-\gamma)}\frac{C\,k}{T}\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})}.$$

The product rule gives

$$|\sum_j (\xi_u^n, ((\phi^2)_x(t)(\xi_w^n - \xi_u^n))_x)_j| = |\sum_j\{(\xi_u^n, (\phi^2)_{xx}(t)(\xi_w^n - \xi_u^n))_j$$
$$+(\xi_u^n, (\phi^2)_x(t)(\xi_w^n - \xi_u^n)_x)_j|\}|.$$

Applying the Cauchy-Schwarz inequality (4.8d), Lemma 7.3 and Young's inequality we have

$$k^2|\sum_j\{(\xi_u^n, (\phi^2)_{xx}(t^n)(\xi_w^n - \xi_u^n))_j \le (\frac{h}{\tilde{T}})^{1-2\gamma}\frac{C\,k}{K^2T}(\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})}$$
$$+\|\phi(t^n)(\eta_w^n - \eta_u^n)\|^2_{L^2(\mathbb{R})}).$$

If we use (4.13b), (4.8e), (4.8d) and Young's inequality, we see that

$$
\begin{aligned}
k^2|\sum_j(\xi_u^n,(\phi^2)_x(t)(\xi_w^n-\xi_u^n)_x)_j| \leq \frac{C\,k}{K}\sum_j[\phi(t^n)\xi_u^n]^2_{j-1/2} \\
+((\frac{h}{\tilde{T}})^{2(1/2-\gamma)}+1)\frac{C\,k}{K\,T}\|\phi(t^n)\xi_u^n\|^2_{L^2(\mathbb{R})} \\
+\frac{C\,T}{k}\|\phi(t^n)(\eta_w^n-\eta_u^n)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

This concludes the proof. □

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