POINTWISE ERROR ESTIMATES FOR DISCONTINUOUS GALERKIN METHODS WITH LIFTING OPERATORS FOR ELLIPTIC PROBLEMS

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ABSTRACT. In this article, we prove some weighted pointwise estimates for three discontinuous Galerkin methods with lifting operators appearing in their corresponding bilinear forms. We consider a Dirichlet problem with a general second order elliptic operator.

1. INTRODUCTION

Discontinuous Galerkin (DG) methods for elliptic problems have received considerable attention in the last few years. A unified analysis of L_2-based global estimates was given by Arnold et. al. [2] for nine DG Methods. In that article, in order to do the unified analysis, they cast all the methods in their primal forms (although some methods are more natural in their mixed forms). Four of the methods were shown to be consistent, to be adjoint consistent and to have coercive bilinear forms for the Laplacian. With these properties, they were able to show optimal convergence rates for the gradient and function values. For these four methods, a natural question arises: How do these methods behave pointwise? Kanschat and Rannacher [8] gave a quasi-optimal convergence result in L_∞ for the interior penalty (IP) method, and Chen and Chen [6] gave weighted pointwise estimates for the same method, which implies the result in [8]. In this paper, we show weighted pointwise error estimates for the three remaining methods.

One main difference between the IP method and the three methods considered here is that the latter have terms with lifting operators appearing in their bilinear forms. As pointed out in [2], the IP method can be problematic since the penalty parameters must be chosen sufficiently large to make the method stable. The three remaining methods do not have this problem.

Once one has local H^1 estimates, weighted pointwise estimates are easily obtained following the pointwise estimates proof of Schatz [12] for the standard continuous Galerkin method or a similar proof in [6] for the IP method. Therefore, our main contribution is to prove local H^1 estimates for these methods. The local H^1 analysis becomes more difficult because of the presence of lifting operators. Here we define one of the lifting operators. Let e be the edge shared by triangles T_1 and

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T_2, and let $q \in [L_2(e)]^2$. We define $r_{M,e} : [L^2(e)]^2 \to [V_h]^2$ by the following identity:

$$\int_{T_1 \bigcup T_2} r_{M,e}(q)^T M\tau dx = -\int_e q^T \langle M\tau \rangle \, ds, \quad \forall \tau \in [V_h]^2.$$

Here M is a symmetric, uniformly positive definite matrix, $\langle \cdot \rangle$ is the average operator across e and V_h denotes our subspace of discontinuous functions. One of the difficulties that we overcame is determining proper bounds for terms of the form: $||\omega^2 r_{M,e}([u_h]) - r(\omega^2[u_h])||_{L_2(T_1 \bigcup T_2)}$, where ω is a cut-off function and $[u_h]$ denotes the jump of our approximation across the edge e. In order to do this, we use L_2-type projection operators and a modification of super-approximation (see Lemmas 2.2 and 2.3).

Chen [5] proved some local H^1 error estimates for the local discontinuous Galerkin (LDG) method in its mixed formulation. In this paper, we do the analysis for the LDG method in its primal form, and we repeat the analysis for two other methods. Lifting operators do not appear in the mixed formulation for the LDG method. Therefore, using the mixed formulation avoids the difficulties of analyzing lifting operators. However, one cannot avoid these difficulties in the remaining two methods, because lifting operators also appear in their mixed formulations.

The pointwise estimates obtained here and in [6] are modeled on the pointwise estimates obtained for the standard continuous Galerkin method in [12] . Let V_h be the space of discontinuous functions such that the restriction of a function to an element is a polynomial of degree $r-1$. The pointwise estimates take on the following forms (compare to Theorems 2.1, 3.1 in [12] and Theorems 5.1, 5.2 in [6]):

$$|(u - u_h)(x)| \le Ch \inf_{\chi \in V_h} |||u - \chi|||_{W_h^{1,\infty}(\Omega),x,s}, \quad 0 \le s < r-2 \tag{1.1}$$

and

$$|\nabla_h(u - u_h)(x)| \le C \inf_{\chi \in V_h} |||u - \chi|||_{W_h^{1,\infty}(\Omega),x,s}, \quad 0 \le s < r-1. \tag{1.2}$$

Here, $\nabla_h \phi$ denotes the piecewise defined function such that $\nabla_h \phi = \nabla \phi$ on each element of the triangulation. The weighted norm appearing on the right-hand sides of (1.1),(1.2) are precisely defined in Section 2.3, and it will be clear that we can bound that norm by the weighted norm defined in [12] if χ is continuous. More precisely, $|||u-\chi|||_{W_h^{1,\infty}(\Omega),x,s} \le C(||\sigma_x^s(u-\chi)||_{L_\infty(\Omega)} + ||\sigma_x^s \nabla(u-\chi)||_{L_\infty(\Omega)})$, where $\sigma_x(y) = h/(|x-y|+h)$. Therefore, if $s = 0$ (no weight) we get estimates in the L_∞-norm. However, if $s > 0$ our error will be localized around x. Consequently, we can also show expansion inequalities ([12]) for these DG methods. The inequalities (1.1), (1.2) will hold if $s = r-2$ and $s = r-1$, respectively, as long as we add a logarithmic factor to the right hand side of the inequalities (see Theorems 4.1, 4.2).

The rest of this paper is organized as follows: In the next section, we present some preliminaries. We define the problem in a precise way, and we introduce our bilinear forms. Then, in Section 2.4 we develop some important approximation results. We end the preliminaries by proving some estimates for lifting operators and by bounding the bilinear forms. In Section 3, we prove local H^1 estimates. Finally, in Section 4, we state our pointwise estimates.

2. Preliminaries

2.1. **Dirichlet Problem.** Let $\Omega \subset R^2$ be bounded with smooth boundary. We consider the following Dirichlet problem:

$$Lu \equiv -\nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u + c(x)u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

The components of $A = (a_{ij})_{1\leq i,j\leq 2}$, $b = (b_i)_{1\leq i\leq 2}$ and c are assumed to be smooth and bounded. Furthermore, we assume that A is symmetric and uniformly positive definite in Ω. That is, there exists a constant $C_{ell} > 0$ such that

$$\psi^T A(x)\psi \geq C_{ell}|\psi|^2, \quad \forall\ x \in \Omega \text{ and } \psi \in R^2.$$

We assume that (2.1) has a unique solution in $H_0^1(\Omega)$ for all $f \in L_2(\Omega)$.

In this paper we are not going to trace the dependence of constants on the ellipticity factor C_{ell} and upper bounds for $A(x)$, $b(x)$ or $c(x)$.

2.2. **Discontinuous Approximating Spaces and Bilinear Forms.** Suppose we have a family of triangulations $\mathcal{T}_h$ (possibly with hanging nodes) that fit Ω exactly, where $\Omega = \cup_{T\in\mathcal{T}_h} T$. Let $h = \sup_{T\in\mathcal{T}_h} h_T$, where $h_T = diam(T)$. Let $V_{h,p}$ denote the finite-dimensional space of functions that are polynomials of degree at most p on each element and define $\Sigma_{h,p} = V_{h,p} \times V_{h,p}$. From now on, we set $V_h = V_{h,r-1}$ and $\Sigma_h = \Sigma_{h,r-1}$. We naturally define, the collection of interior edges as $\mathcal{E}_h^0 = \{\partial T \cap \partial T' : T, T' \in \mathcal{T}_h, T \neq T', \text{meas}_1(\partial T \cap \partial T') \neq 0\}$ and the collection of boundary edges, in general curved, as $\mathcal{E}_h^\partial = \{\partial T \cap \partial\Omega : T \in \mathcal{T}_h, \text{meas}_1(\partial T \cap \partial\Omega) \neq 0\}$. The collection of all edges will be denoted by $\mathcal{E}_h = \mathcal{E}_h^\partial \cup \mathcal{E}_h^0$. S_e will denote the union of elements that have e as an edge. We assume that our elements are non-degenerate; that is, there exists a constant $C_{nd} > 0$, independent of T, such that $h_T \leq C_{nd}\, diam(B_T)$, where B_T is the largest ball contained in T. Furthermore, we assume the existence of a constant $C_E > 0$, independent of h and e, with $h \leq C_E h_e$, where $h_e = length(e)$. This is the quasi-uniform condition that was used in [6].

We say that T and T' are neighbors if $\partial T \cap \partial T' \in \mathcal{E}_h^0$. From the quasi-uniform condition, it follows that there exists a positive integer K independent of h such that each $T \in \mathcal{T}_h$ has at most K neighbors (if our meshes do not have hanging nodes then K can be 3).

On each edge, as in [2], we define the average and jump operators as follows for $e \in \mathcal{E}_h^0$:

$$\langle q\rangle = \tfrac{1}{2}(q_1 + q_2), \quad [q] = q_1 \cdot n_1 + q_2 \cdot n_2,$$
$$\langle \phi\rangle = \tfrac{1}{2}(\phi_1 + \phi_2), \quad [\phi] = \phi_1 n_1 + \phi_2 n_2,$$

for q vector valued and ϕ scalar valued. Here $S_e = T_1 \cup T_2, q_i = q|_{T_i},\ \phi_i = \phi|_{T_i}$, and n_i is the exterior normal to $T_i,\ i = 1, 2$. For $e \in \mathcal{E}_h^\partial$,

$$\langle q\rangle = q, \quad [\phi] = \phi n$$

where n is the outward unit normal. Note that $[q]$ is a scalar and $[\phi]$ a vector. The quantities $[q]$ and $\langle\phi\rangle$ on boundary edges are not required, so they are left undefined.

Now we present some local lifting operators as in [2]. Let M be a symmetric, smooth, bounded and uniformly positive definite matrix in Ω. Let $r_{M,e} : [L^2(e)]^2 \rightarrow$

Σ_h and $l_{M,e} : L^2(e) \rightarrow \Sigma_h$ be given by:

$$\begin{aligned}\int_{S_e} r_{M,e}(q)^T M\tau dx &= -\int_e q^T \langle M\tau\rangle\, ds \\ \int_{S_e} l_{M,e}(\phi)^T M\tau dx &= -\int_e \phi[M\tau] ds, \qquad \forall \tau \in \Sigma_h.\end{aligned}$$

We set the global lifting operators to be $r_M(q) = \sum_{e\in\mathcal{E}_h} r_{M,e}(q)$ and $l_M(\phi) = \sum_{e\in\mathcal{E}_h^0} l_{M,e}(\phi)$.

Now, we are ready to define the bilinear forms. They are the modified BRMPS [3], modified BMMPR [4] and local discontinuous Galerkin bilinear forms. (See [2] for the bilinear forms for the Laplacian.) The following term is common to all three:

$$\begin{aligned}\theta(u,v) &= \textstyle\sum_{T\in\mathcal{T}_h}\int_T (\nabla u^T A\nabla v + (b^T\nabla u)v + cuv)dx \\ &\quad -\textstyle\sum_{e\in\mathcal{E}_h}\int_e (\langle A\nabla_h u\rangle [v] + \langle A\nabla_h v\rangle [u] + b^T[u]\langle v\rangle)ds.\end{aligned}$$

Modified BRMPS

$$B(u,v) = \theta(u,v) + \textstyle\sum_{e\in\mathcal{E}_h}\eta_e\int_{S_e} r_{A,e}([u])^T A r_{A,e}([v])dx + \sum_{e\in\mathcal{E}_h^\partial}\frac{1}{h_e}\int_e [u][v]ds.$$

Modified BMMPR

$$\begin{aligned}B(u,v) &= \theta(u,v) + \textstyle\int_\Omega r_A([u])^T A r_A([v])dx \\ &\quad + \textstyle\sum_{e\in\mathcal{E}_h}\eta_e\int_{S_e} r_{I,e}([u])^T r_{I,e}([v])dx + \sum_{e\in\mathcal{E}_h^\partial}\frac{1}{h_e}\int_e [u][v]ds.\end{aligned}$$

Local Discontinuous Galerkin

$$\begin{aligned}B(u,v) &= \theta(u,v) + \textstyle\int_\Omega (r_A([u]) + l_A(\beta^T[u]))^T A(r_A([v]) + l_A(\beta^T[v]))dx \\ &\quad -\textstyle\sum_{e\in\mathcal{E}_h^0}\int_e ([A\nabla_h u]\beta^T[v] + \beta^T[u][A\nabla_h v])ds + \sum_{e\in\mathcal{E}_h}\frac{\eta_e}{h_e}\int_e [u][v]ds.\end{aligned}$$

Here η_e is constant for each e, and is bounded for all e. If we let $\eta = \inf_{e\in\mathcal{E}_h}\eta_e$, then we require that $K < \eta$ (see section 2.2 for the definition of K) for the Modified Bassi form and $0 < \eta$ for the two remaining forms. Also, β is a constant vector on each interior edge and is bounded component-wise for all e. Our modifications of the first two methods is solely motivated by our analysis. It consists of adding the last terms over the boundary edges. Without this modification we were not able to prove Theorem 3.1 since we were unable to show (2.19) for curved edges. Whether this modification is necessary in practice we do not know. We intend to investigate this question in the future.

The discontinuous approximation u_h solves

$$B(u_h, v) = (f, v), \quad \forall v \in V_h. \tag{2.2}$$

For each method we use the corresponding bilinear form.

2.3. **Discontinuous Sobolev Norms.** If $D \subset \Omega$, we define our discontinuous Sobolev space as in [6]:

$$W_h^{l,p}(D) = \{v : v \in W^{l,p}(T\cap D),\ \forall\ T \in \mathcal{T}_h\}.$$

This space is equipped with the norm ($1 \le p < \infty$)

$$||v||_{W_h^{l,p}(D)} = (\sum_{j=0}^{l} |v|^p_{W_h^{j,p}(D)})^{\frac{1}{p}},\ \ |v|_{W_h^{j,p}(D)} = (\sum_{T\in\mathcal{T}_h} |v|^p_{W^{j,p}(T\bigcap D)})^{\frac{1}{p}},$$

with the appropriate modification for $p = \infty$. When $p = 2$, we set $H_h^l(D) = W_h^{l,p}(D)$.

Let $D \subset \Omega$ and $1 \leq p < \infty$. The norms that occur in most of our analysis take on the following form:

$$\begin{aligned} |||v|||^p_{W_h^{l,p}(D)} &= ||v||^p_{W_h^{l,p}(D)} + \sum_{e\in\mathcal{E}_h} h_e^{1-p}||[v]||^p_{L_p(e\bigcap D)} \\ &+ \sum_{e\in\mathcal{E}_h} h_e|| \langle \nabla_h v\rangle ||^p_{L_p(e\bigcap D)} + \sum_{e\in\mathcal{E}_h^0} h_e||[\nabla_h v]||^p_{L_p(e\bigcap D)}. \end{aligned}$$

For $p = \infty$,

$$\begin{aligned} |||v|||_{W_h^{l,\infty}(D)} &= ||v||_{W_h^{l,\infty}(D)} + \sup_{e\in\mathcal{E}_h} \tfrac{1}{h_e}||[v]||_{L_\infty(e\bigcap D)} \\ &+ \sup_{e\in\mathcal{E}_h} || \langle \nabla_h v\rangle ||_{L_\infty(e\bigcap D)} + \sup_{e\in\mathcal{E}_h^0} ||[\nabla_h v]||_{L_\infty(e\bigcap D)}. \end{aligned}$$

We consider the weight $\sigma_x^s(y) = (\frac{h}{|x-y|+h})^s$ as in [12]. If we let $||v||^p_{W_h^{l,p}(\Omega),x,s} = ||\sigma_x^s v||^p_{L_p(\Omega)} + ||\sigma_x^s \nabla_h v||^p_{L_p(\Omega)}$ for $1 \leq p < \infty$, then for fixed x we define the norms

$$\begin{aligned} |||v|||^p_{W_h^{l,p}(\Omega),x,s} &= ||v||^p_{W_h^{l,p}(\Omega),x,s} + \textstyle\sum_{e\in\mathcal{E}_h} h_e^{1-p}||\sigma_x^s[v]||^p_{L_p(e)} \\ &+ \textstyle\sum_{e\in\mathcal{E}_h} h_e||\sigma_x^s \langle \nabla_h v\rangle ||^p_{L_p(e)} + \sum_{e\in\mathcal{E}_h^0} h_e||\sigma_x^s[\nabla_h v]||^p_{L_p(e)}, \end{aligned}$$

again with the appropriate modification for $p = \infty$.

2.4. **Approximation.** We start by stating well-known trace inequalities. Let e be an edge of $T \in \mathcal{T}_h$ and ϕ be either a scalar- or vector-valued function. Then, for $1 \leq p \leq \infty$, we have

$$||\phi||_{L_p(e)} \leq C(h^{-\frac{1}{p}}||\phi||_{L_p(T)} + h^{1-\frac{1}{p}}|\phi|_{W^{1,p}(T)}). \tag{2.3}$$

If we restrict ϕ to $V_{h,i}$ or $\Sigma_{h,i}$, for some fixed $i > 0$, we can state some inverse inequalities that are also well-known:

$$||\phi||_{W^{l,t}(T)} \leq Ch^{[\frac{2}{l}-\frac{2}{q}]+s-t}||\phi||_{W^{q,s}(T)}, \tag{2.4}$$

$$||\phi||_{L_p(e)} \leq Ch^{-\frac{1}{p}}||\phi||_{L_p(T)}, \tag{2.5}$$

where C does not depend on ϕ, h, e, or T.

The following is a standard elementwise approximation result. Let $v \in W_h^{j,p}(\Omega)([W_h^{j,p}(\Omega)]^2)$ with $0 \leq i \leq j \leq r$. Then, there exists a $\phi \in V_h(\Sigma_h)$ with

$$||v - \phi||_{W^{i,p}(T)} \leq Ch^{j-i}|v|_{W^{j,p}(T)}, \quad \forall T \in \mathcal{T}_h, \tag{2.6}$$

where C does not depend on v, h, or T.

We present some function spaces, as in [13], that will help us in stating further approximation results. If $S \subset R \subset \Omega$, let $\partial_<(S, R) = dist(\partial S \setminus \partial\Omega, \partial R \setminus \partial\Omega)$. The spaces are defined as follows:

$$V_h^<(A) = \{v \in V_h : \partial_<(supp(v), A) > 0\},$$

and

$$C_<^\infty(A) = \{v \in C^\infty : \partial_<(supp(v), A) > 0\}.$$

The following lemma follows from trace inequalities and elementwise approximation. (See section 4.3 in [2] for a similar result.)

Lemma 2.1. *Let $D_0 \subset D_d$ with $\partial_<(D_0, D_d) = d \geq \kappa h$ (for a fixed $\kappa > 1$ sufficiently large). Let $v \in H_h^r(D_d)$. Then, there exists $\psi \in V_h$ such that*

$$|||v - \psi|||_{H_h^1(D_0)} \leq Ch^{r-1}|v|_{H_h^r(D_d)}, \tag{2.7}$$

where C is independent of v, h, D_0 and D_d. Furthermore, if $supp(v) \subset D_0$, then $\psi \in V_h^<(D_d)$.

Throughout this paper, we are going to be estimating functions of the form $v = \omega\chi$ or $v = \omega^2\chi$, where $\chi \in V_h$ or Σ_h and ω is a cut-off function. Hence, we develop some approximation results for these functions.

Lemma 2.2. *Let $\chi \in V_h$, and let ω be a smooth function. Suppose there exist constants $C > 0$ and $d \geq \kappa h$ for some constant $\kappa > 1$ such that $|\omega|_{W^{l,\infty}(\Omega)} \leq Cd^{-l}$ for $l = 0, 1, \cdots, r+1$. Then,*

$$|\omega^2\chi|_{H^r(T)} \leq C\frac{1}{h^{r-2}}(d^{-1}|\omega\chi|_{H^1(T)} + d^{-2}||\chi||_{L_2(T)}), \tag{2.8}$$

$$|\omega^2\chi|_{H^r(T)} \leq C\frac{1}{h^{r-2}}(d^{-1}h^{-1}||\omega\chi||_{L_2(T)} + d^{-2}||\chi||_{L_2(T)}), \tag{2.9}$$

$$|\omega\nabla(\omega\chi)|_{H^r(T)} \leq C\frac{1}{h^{r-1}}(d^{-1}||\nabla(\omega\chi)||_{L_2(T)} + d^{-2}||\chi||_{L_2(T)}), \tag{2.10}$$

$$|\omega\chi|_{H^k(T)} \leq C\frac{1}{h^{k-1}}(|\omega\chi|_{H^1(T)} + d^{-1}||\chi||_{L_2(T)}) \quad \text{for } k = 1, \cdots, r-1, \tag{2.11}$$

and

$$|\omega\chi|_{H^r(T)} \leq C\frac{1}{h^{r-1}}d^{-1}||\chi||_{L_2(T)}. \tag{2.12}$$

Here C is independent of ω, χ, T, and h.

Proof. By Leibniz's rule, the fact that the rth derivatives of χ vanish in T and inverse estimates, we get that

(2.13)
$$|\omega^2\chi|_{H^r(T)} \leq C(\sum_{l=2}^{r} h^{l-r}|\omega^2|_{W^{l,\infty}(T)})||\chi||_{L_2(T)} + \sum_{|\alpha|=1,|\beta|=r-1} ||D^\alpha\omega^2 D^\beta\chi||_{L_2(T)}).$$

By the decay properties of ω and the fact that $hd^{-1} < 1$, we see that

$$(\sum_{l=2}^{r} h^{l-r}|\omega^2|_{W^{l,\infty}(T)})||\chi||_{L_2(T)} \leq Cd^{-2}h^{2-r}||\chi||_{L_2(T)}.$$

Now we handle the second sum in (2.13). Note that $D^\alpha\omega^2 = 2\omega D^\alpha\omega$ since $|\alpha| = 1$. Therefore,

$$\sum_{|\alpha|=1,|\beta|=r-1} ||D^\alpha\omega^2 D^\beta\chi||_{L_2(T)} \leq Cd^{-1} \sum_{|\beta|=r-1} ||\omega D^\beta\chi||_{L_2(T)}.$$

Now we let $\hat{\omega} = \frac{1}{|T|}\int_T \omega dx$. By the triangle inequality, we have that

$$||\omega D^\beta\chi||_{L_2(T)} \leq ||(\omega - \hat{\omega})D^\beta\chi||_{L_2(T)} + ||\hat{\omega}D^\beta\chi||_{L_2(T)}.$$

Using approximation properties, we see that

$$||(\omega-\hat{\omega})D^\beta\chi||_{L_2(T)} \leq Ch|\omega|_{W^{1,\infty}(T)}||D^\beta\chi||_{L_2(T)}.$$

Using inverse estimates and decay properties of ω, we have

$$||(\omega-\hat{\omega})D^\beta\chi||_{L_2(T)} \leq Cd^{-1}h^{2-r}||\chi||_{L_2(T)}.$$

To handle the next term we again use an inverse estimate, to get

$$||\hat{\omega}D^\beta\chi||_{L_2(T)} \leq Ch^{2-r}|\hat{\omega}\chi|_{H^1(T)}.$$

Using the triangle inequality, we have

$$|\hat{\omega}\chi|_{H^1(T)} \leq |(\hat{\omega}-\omega)\chi|_{H^1(T)} + |\omega\chi|_{H^1(T)}.$$

By using the product rule, approximation properties, and inverse estimates, we obtain

$$|(\hat{\omega}-\omega)\chi|_{H^1(T)} \leq Cd^{-1}||\chi||_{L_2(T)}.$$

Therefore,

$$|\hat{\omega}\chi|_{H^1(T)} \leq C(d^{-1}||\chi||_{L_2(T)} + |\omega\chi|_{H^1(T)}).$$

Combining these estimates, we have that

$$\sum_{|\alpha|=1,|\beta|=r-1} ||D^\alpha\omega^2 D^\beta\chi||_{L_2(T)} \leq d^{-1}h^{2-r}|\omega\chi|_{H^1(T)} + d^{-2}h^{2-r}||\chi||_{L_2(T)}.$$

This proves (2.8). By introducing again $\hat{\omega}$ we can bound $|\omega\chi|_{H^1(T)}$ by the right hand side of (2.9). This will prove (2.9). By using the triangle inequality we see that $|\omega\nabla(\omega\chi)|_{H^r(T)} \leq |\omega^2\nabla(\chi)|_{H^r(T)} + |\omega\chi\nabla\omega|_{H^r(T)}$. We can then use (2.9) to bound $|\omega^2\nabla(\chi)|_{H^r(T)}$ by the right hand side of (2.10), and we can use Leibniz's rule and inverse estimates to bound $|\omega\chi\nabla\omega|_{H^r(T)}$. This would prove (2.10). By Leibniz's rule, inverse estimates and using the decay properties of ω we can prove (2.11) and (2.12). We omit the proofs. ☐

Now we can state a super-approximation result similar to that in [12]. The differences between this approximation result and the one contained in [12] is that ω appears in the right-hand side of our result. This will allow us to perform "kick back" arguments. (See the proof of Theorem 3.1.)

Lemma 2.3. *Let $\partial_<(D_0, D_d) = d > \kappa h$, where $\omega \in C^\infty_<(D_0)$. Suppose $|\omega|_{W^{l,\infty}(D_0)} \leq Cd^{-l}$ for $l = 0, 1, \cdots, r+1$. Then, for all $\chi \in V_h$ there exists $\psi \in V_h^<(D_d)$ with*

$$|||\omega^2\chi - \psi|||_{H^1_h(\Omega)} \leq Ch(d^{-1}|\omega\chi|_{H^1_h(D_d)} + d^{-2}||\chi||_{L_2(D_d)})$$

where C is independent of χ and ω.

Proof. This easily follows from Lemma 2.1 and (2.8). ☐

2.5. **Boundedness and Consistency of the Forms.** We start by defining and stating some properties of certain projection operators that we use throughout this paper. In this direction, let M be a symmetric, smooth, bounded matrix that is uniformly positive definite in Ω. If $g \in L_2(\Omega)$, we define $P_M(g) \in \Sigma_h$ by the following equation:

$$\int_\Omega \chi^T M g = \int_\Omega \chi^T M P_M(g), \quad \forall \chi \in \Sigma_h. \tag{2.14}$$

Note that P_M can be defined elementwise since Σ_h is a space of discontinuous functions. If $M = I$, then this will simply be the vector-valued L_2-projection operator. L_2-projection operators have been analyzed extensively. The proof of the following lemma, which we omit, follows the proof for the L_2-projection operator ([7]); however, it is much simpler since V_h consists of discontinuous functions.

Lemma 2.4. *Let M be a symmetric, smooth and bounded matrix that is uniformly positive definite. Let P_M be defined by (2.14) and $1 \le p \le \infty$. Then, for all $T \in \mathcal{T}_h$,*

$$||P_M(g)||_{L_p(T)} \le C_1 ||g||_{L_p(T)}$$

and

$$||g - P_M(g)||_{H^k(T)} \le C_2 h^{r-k} |g|_{H^r(T)} \quad \text{for } k = 1, \cdots, r$$

where C_1 and C_2 are independent of g and T.

In order to show boundedness of our forms, we need some estimates for our lifting operators. The following lemma is an extension of Lemma 2(ii) in [4].

Lemma 2.5. *Let M be smooth, bounded, and uniformly positive definite in Ω. Let $1 \le p, q \le \infty$,$\frac{1}{p} + \frac{1}{q} = 1$. Then, there exists a constant C such that for all $\phi \in [L_p(e_1)]^2$ and $\chi \in [L_q(e_2)]^2$,*

$$\begin{aligned} |\int_{S_{e_1} \cap S_{e_2}} r_{M,e_1}(\phi)^T M r_{M,e_2}(\chi) dx| \quad &+ \quad |\int_{S_{e_1} \cap S_{e_2}} l_{M,e_1}(\phi)^T M r_{M,e_2}(\chi) dx| \\ &+ \quad |\int_{S_{e_1} \cap S_{e_2}} l_{M,e_1}(\phi)^T M l_{M,e_2}(\chi) |dx| \\ &\le \quad C \frac{1}{h} ||\phi||_{L_p(e_1)} ||\chi||_{L_q(e_2)}. \end{aligned} \tag{2.15}$$

In the case that e_1 and e_2 do not belong to the same triangle the, left hand side of (2.15) will be zero.

Proof. The last statement of the lemma follows by the definition of the lifting operators. In the other case, by the boundedness of M and (2.5), we have

$$\begin{aligned} |\int_{S_{e_1} \cap S_{e_2}} r_{M,e_1}(\phi)^T M r_{M,e_2}(\chi) dx| \quad &= \quad |\int_{e_1} \phi^T M \langle r_{M,e_2}(\chi) \rangle ds| \\ &\le \quad C ||\phi||_{L_p(e_1)} || \langle r_{M,e_2}(\chi) \rangle ||_{L_q(e_1)} \\ &\le \quad C h^{-\frac{1}{q}} ||\phi||_{L_p(e_1)} ||r_{M,e_2}(\chi)||_{L_q(S_{e_2})}. \end{aligned}$$

Now we use a duality argument to bound $||r_{M,e_2}(\chi)||_{L_q(S_{e_2})}$. If $g = M^{-1} z$, then

$$\int_{S_{e_2}} r_{M,e_2}(\chi)^T z dx = - \int_{e_2} \chi^T M \langle P_M(g) \rangle ds.$$

By Hölder's inequality, (2.5), and Lemma 2.4, we easily get

$$-\int_{e_2} \chi^T M \langle P_M(g)\rangle \, ds \le Ch^{-\frac{1}{p}} ||\chi||_{L_q(e_2)} ||g||_{L_p(S_{e_2})}.$$

Therefore, by duality $||r_{M,e_2}(\chi)||_{L_q(S_{e_2})} \le Ch^{-\frac{1}{p}}||\chi||_{L_q(e_2)}$ since $||g||_{L_p(S_{e_2})} \le C||z||_{L_p(S_{e_2})}$. Hence, we have established the following:

$$|\int_{S_{e_1}\cap S_{e_2}} r_{M,e_1}(\phi)^T M r_{M,e_2}(\chi) dx| \le C\frac{1}{h}||\phi||_{L_p(e_1)}||\chi||_{L_q(e_2)}. \tag{2.16}$$

The last two terms can be bounded following similar steps. □

In the case that e_1 and e_2 belong to a common triangle, a simple exercise shows $max_{y\in e_1}(\sigma_x^s(y)) \le 2^s min_{y\in e_2}(\sigma_x^s(y))$. Therefore,

$$\frac{1}{h}||\phi||_{L_p(e_1)}||\chi||_{L_q(e_2)} \le C\frac{1}{h}||\sigma^{-s}\phi||_{L_p(e_1)}||\sigma^s\chi||_{L_q(e_2)}. \tag{2.17}$$

If we take into account that $r_M = \sum_{e\in\mathcal{E}_h} r_{M,e}$, $l_M = \sum_{e\in\mathcal{E}_h^0} l_{M,e}$, make use of Hölder's inequality (both for the integrals and summation), apply Lemma 2.5, and (2.17), we can show the following boundedness of our forms. (See [6],[2] for analogous results.)

Lemma 2.6. *Let* $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *For all three forms, there exists a positive constant* C *such that for all* $u \in W_h^{1,p}(\Omega)$ *and* $v \in W_h^{1,q}(\Omega)$

$$|B(v,u)| + |B(u,v)| \le C|||u|||_{W_h^{1,p}(\Omega),x,s}|||v|||_{W_h^{1,q}(\Omega),x,-s} \tag{2.18}$$

where C *is independent of* x, u *and* v.

The next important inequality is an extension of Lemma 2(i) in [4].

Lemma 2.7. *Let* M *be given as in Lemma* 2.5. *Let* $e \in \mathcal{E}_h^0$, $\phi \in [P_{r-1}(e)]^2$ *and suppose* $\omega \in C^\infty(\Omega)$. *Then,*

$$h_e^{-1}\int_e (\omega\phi)^T M(\omega\phi) ds \le C(\int_{S_e} \omega^2 r_{M,e}(\phi)^T M r_{M,e}(\phi) dx + h_e|\omega|^2_{W^{1,\infty}(S_e)}||\phi||^2_{L_2(e)}). \tag{2.19}$$

Proof. Let $v \in [P_p(e)]^2$ for some fixed p. Then, v is defined naturally on the line containing e, call it l. As in [4], we define $\overline{v} \in [P_p(R^2)]^2$ as the vector-valued polynomial satisfying $v = \overline{v}$ on l and which is constant on the lines perpendicular to l. As pointed out in [4],

$$||\overline{v}||^2_{L_2(S_e)} \le Ch_e||v||^2_{L_2(e)}. \tag{2.20}$$

Also, let $\tilde{\omega} \equiv \min_{\bar{S}_e} \omega$. It follows by the mean value theorem, possibly applying it twice, that

$$||\tilde{\omega} - \omega||_{L_\infty(S_e)} \le Ch|\omega|_{W^{1,\infty}(S_e)}. \tag{2.21}$$

We easily see by our definitions that

$$\int_e (\omega\phi)^T M(\omega\phi) ds = E_1 + E_2 \tag{2.22}$$

where

$$E_1 = h_e^{-1}\int_e ((\omega\phi)^T M(\omega\phi) - (\tilde{\omega}\phi)^T M(\tilde{\omega}\phi))ds$$
$$E_2 = -h_e^{-1}\tilde{\omega}^2 \int_{S_e} (\overline{\phi})^T M r_{M,e}(\phi)ds.$$

We rewrite E_1,

$$E_1 = Ch_e^{-1}\int_e (\omega - \tilde{\omega})(\omega + \tilde{\omega})\phi^T M\phi ds.$$

By the Cauchy Schwarz inequality and the arithmetic-geometric mean inequality we have

$$E_1 \leq Ch_e^{-1}\int_e (\omega - \tilde{\omega})^2\phi^T M\phi ds + \epsilon h_e^{-1}\int_e (\omega + \tilde{\omega})^2\phi^T M\phi ds.$$

Later we will choose $\epsilon > 0$ sufficiently small. Finally, using (2.21) and $(\omega + \tilde{\omega})^2 \leq 2(\omega^2 + \tilde{\omega}^2) \leq 2\omega^2$, we see that

$$E_1 \leq Ch_e|\omega|^2_{W^{1,\infty}(S_e)}||\phi||^2_{L_2(e)} + 2\epsilon h^{-1}\int_e \omega^2\phi^T M\phi ds.$$

Again, using the Cauchy Schwarz inequality and the arithmetic-geometric mean inequality, we have

$$E_2 \leq \tilde{\omega}^2(\epsilon h^{-2}\int_{S_e} \overline{\phi}^T M\overline{\phi} dx + C\int_{S_e} r_{M,e}(\phi)^T M r_{M,e}(\phi)dx).$$

By using (2.20) and the positive definiteness of M, we see that

$$E_2 \leq C_1\epsilon \int_e \omega^2\phi^T M\phi ds + C\int_{S_e} \omega^2 r_{M,e}(\phi)^T M r_{M,e}(\phi)dx$$

where C_1 is a constant independent of ϵ.

Finally, taking ϵ small enough, we arrive at our conclusion. □

The next lemma can easily be shown by applying integration by parts (see Section 3.3 in [2] for similar results).

Lemma 2.8. *For all the forms we have consistency and adjoint consistency. That is, if $Lu = f$ or $L^*w = f$, with $u, w \in H_0^1(\Omega)$, then*

$$B(u, v) = (f, v) \text{ or } B(v, w) = (f, v), \quad \forall v \in V_h. \tag{2.23}$$

Until now, we have not addressed if (2.2) is well defined. Coerciveness, of course, will not hold for a general second order elliptic operator. However, using techniques similar to those in Section 4.2 of [2], and in addition taking care of the lower order terms, we can show the following lemma.

Lemma 2.9. *For all three forms there exists a constant $C > 0$ such that $\forall \chi \in V_h$*

$$|||\chi|||^2_{H_h^1(\Omega)} \leq C(B(\chi, \chi) + ||\chi||^2_{L_2(\Omega)}). \tag{2.24}$$

If we use this fact, Lemma 2.8 and use the techniques in [10], we could show that our problem is well defined for sufficiently small h.

3. Local H^1 Estimates

We start with our main theorem.

Theorem 3.1. *Let $D_0 \subset D_d \subset \Omega$ with $\partial_<(D_0, D_d) = d > \kappa h$ (where $\kappa > 1$ is a sufficiently large fixed constant). Suppose u and $u_h \in V_h$ satisfy*

$$B(u - u_h, v) = 0, \quad \forall v \in V_h \tag{3.1}$$

for any of the forms above. Then,

$$|||u-u_h|||_{H^1_h(D_0)} \leq C \inf_{\chi \in V_h} (|||u-\chi|||_{H^1_h(D_d)} + d^{-1}||u-\chi||_{L_2(D_d)}) + Cd^{-1}||u-u_h||_{L_2(D_d)}$$

where C is independent of d.

Proof. Since $u - u_h = (u - \chi) - (u_h - \chi)$, it suffices to show

$$|||u - u_h|||_{H^1_h(D_0)} \leq C(|||u|||_{H^1_h(D_d)} + d^{-1}||u||_{L_2(D_d)}) + Cd^{-1}||u - u_h||_{L_2(D_d)}.$$

By the triangle inequality, it will be enough to establish

$$|||u_h|||_{H^1_h(D_0)} \leq C|||u|||_{H^1_h(D_d)} + Cd^{-1}||u_h||_{L_2(D_d)}. \tag{3.2}$$

To this end, let $\omega \in C^\infty_<(D_{d/4})$ with $\omega \equiv 1$ on $D_{d/8}$ and $|\omega|_{W^{l,\infty}(\Omega)} \leq Cd^{-l}$ for $l = 0, 1, \ldots, r+1$. For a moment, let us assume that we can show the following inequality for all the forms:

$$|||\omega u_h|||^2_{H^1_h(\Omega)} \leq CB(u_h, \omega^2 u_h) + Cd^{-2}||u_h||^2_{L_2(D_d)}. \tag{3.3}$$

By (3.1), we can write

$$B(u_h, \omega^2 u_h) = B(u_h, \omega^2 u_h - \chi) - B(u, \omega^2 u_h - \chi) + B(u, \omega^2 u_h) \tag{3.4}$$

for any $\chi \in V_h$. Since our forms are bounded, we have

$$\begin{aligned} B(u, \omega^2 u_h) &\leq C|||u|||_{H^1_h(D_{d/2})}|||\omega^2 u_h|||_{H^1_h(D_{d/2})} \\ &\leq C|||u|||_{H^1_h(D_d)}(|||\omega u_h|||_{H^1_h(D_d)} + d^{-1}||u_h||_{L_2(D_d)}). \end{aligned}$$

Now, applying the arithmetic-geometric mean inequality, we see that

$$B(u, \omega^2 u_h) \leq C|||u|||^2_{H^1_h(D_d)} + \epsilon|||\omega u_h|||^2_{H^1_h(D_d)} + Cd^{-2}||u_h||^2_{L_2(D_d)}. \tag{3.5}$$

Using the boundedness of our forms and Lemma 2.3, we obtain

$$\begin{aligned} B(u, \omega^2 u_h - \chi) &\leq C|||u|||_{H^1_h(D_{d/2})}|||\omega^2 u_h - \chi|||_{H^1_h(D_{d/2})} \\ &\leq C|||u|||_{H^1_h(D_d)}(hd^{-1}|\omega u_h|_{H^1_h(D_d)} + hd^{-2}||u_h||_{L_2(D_d)}). \end{aligned}$$

Taking into account that $hd^{-1} < 1$, and applying the arithmetic-geometric mean inequality, we have

$$B(u, \omega^2 u_h - \chi) \leq C|||u|||^2_{H^1_h(D_d)} + \epsilon|\omega u_h|^2_{H^1_h(D_d)} + Cd^{-2}||u_h||^2_{L_2(D_d)}. \tag{3.6}$$

Similarly,

$$B(u_h, \omega^2 u_h - \chi) \leq Ch|||u_h|||_{H^1_h(D_{d/2})}(d^{-1}|\omega u_h|_{H^1_h(D_d)} + d^{-2}||u_h||_{L_2(D_d)}).$$

Using (2.4) and (2.5), we can show the inverse inequality $h|||u_h|||_{H^1_h(D_{d/2})} \leq C||u_h||_{L_2(D_d)}$. Therefore, applying the arithmetic-geometric mean inequality one more time gives

$$B(u_h, \omega^2 u_h - \chi) \leq \epsilon|\omega u_h|^2_{H^1_h(D_d)} + Cd^{-2}||u_h||^2_{L_2(D_d)}. \tag{3.7}$$

Finally, using (3.3), (3.4) and making ϵ small enough in (3.5),(3.6), and (3.7), we arrive at

$$|||\omega u_h|||^2_{H^1_h(\Omega)} \le C(|||u|||^2_{H^1_h(D_d)} + d^{-2}||u_h||^2_{L_2(D_d)}). \tag{3.8}$$

This will imply (3.2), which in turn implies our theorem.

We are left to show (3.3). We first prove this for the Modified BRMPS form. First, by applying (2.3) and (2.11), we see that

$$\begin{aligned}\sum_{e\in\mathcal{E}_h} h_e|| \langle\nabla(\omega u_h)\rangle ||^2_{L_2(e)} + \sum_{e\in\mathcal{E}_h^0} h_e||[\nabla(\omega u_h)]||^2_{L_2(e)} \\ \le C\sum_{T\in\mathcal{T}_h}\int_T \nabla(\omega u_h)^2 dx + Cd^{-2}||u_h||^2_{L_2(\Omega)}.\end{aligned} \tag{3.9}$$

By using (3.9), the positive definiteness of A, (2.19) and (2.5), we have

$$|||\omega u_h|||^2_{H^1_h(\Omega)} \le CI + Cd^{-2}||u_h||^2_{L_2(D_d)} \tag{3.10}$$

where

$$\begin{aligned} I &\equiv \sum_{T\in\mathcal{T}_h}\int_T \nabla(\omega u_h)^T A\nabla(\omega u_h)dx + \sum_{e\in\mathcal{E}_h}\int_{S_e}\omega^2 r_{A,e}([u_h])^T A r_{A,e}([u_h])dx \\ &\quad + \sum_{e\in\mathcal{E}_h^\partial}\frac{1}{h_e}\int_e \omega^2[u_h]^2 ds.\end{aligned}$$

In the last two inequalities we used the fact that each element has at most K neighbors.

Let

$$\begin{aligned} G &\equiv \sum_{T\in\mathcal{T}_h}\int_T \nabla(\omega u_h)^T A\nabla(\omega u_h)dx + 2\sum_{e\in\mathcal{E}_h}\int_{S_e}\nabla_h(\omega u_h)^T A\omega r_{A,e}([u_h])dx \\ &\quad + \sum_{e\in\mathcal{E}_h}\eta_e\int_{S_e}\omega^2 r_{A,e}([u_h])^T A r_{A,e}([u_h])dx.\end{aligned}$$

Using the arithmetic-geometric mean inequality on the middle term and the fact that each triangle has at most K neighbors, we get

$$\begin{aligned} G &\ge (1-\frac{K}{\epsilon_1})\sum_{T\in\mathcal{T}_h}\int_T \nabla(\omega u_h)^T A\nabla(\omega u_h)dx \\ &\quad +(\eta-\epsilon_1)\sum_{e\in\mathcal{E}_h}\int_{S_e}\omega^2 r_{A,e}([u_h])^T A r_{A,e}([u_h])dx.\end{aligned}$$

Choosing ϵ_1 to satisfy $K < \epsilon_1 < \eta$, we see that

$$I \le C(G + \sum_{e\in\mathcal{E}_h^\partial}\frac{1}{h_e}\int_e \omega^2[u_h]^2 ds). \tag{3.11}$$

Using the definition of G and adding and subtracting the terms of $B(u_h, \omega^2 u_h)$ (for the Modified BRMPS form) to the right hand side of (3.11), we arrive at

$$\begin{aligned}
I &\leq C(B(u_h, \omega^2 u_h) \\
&+ \sum_{T \in \mathcal{T}_h} \int_T \nabla(\omega u_h)^T A \nabla(\omega u_h) - \nabla(\omega^2 u_h)^T A \nabla u_h dx \\
&+ \{\sum_{e \in \mathcal{E}_h} (2 \int_{S_e} \nabla_h(\omega u_h)^T A \omega r_{A,e}([u_h]) dx \\
&+ \int_e (\left\langle \nabla_h(\omega^2 u_h)\right\rangle^T A[u_h] + \left\langle \nabla_h u_h \right\rangle^T A[\omega^2 u_h]) ds)\} \\
&+ \sum_{e \in \mathcal{E}_h} \eta_e \int_{S_e} (\omega^2 r_{A,e}([u_h])^T A r_{A,e}([u_h]) - r_{A,e}(\omega^2 [u_h])^T A r_{A,e}([u_h])) dx \\
&- \sum_{T \in \mathcal{T}_h} \int_T (c(\omega u_h)^2 + u_h b \cdot \nabla(\omega^2 u_h) dx + \sum_{e \in \mathcal{E}_h} \int_e \langle u_h \rangle \, b \cdot [\omega^2 u_h] ds)) \\
&= C(B(u_h, \omega^2 u_h) + I_1 + I_2 + I_3 + I_4).
\end{aligned}$$

Note that I_4 consists of the lower order terms of our bilinear form. By applying the product rule to each term of I_1, we see that

$$I_1 = |\sum_{T \in \mathcal{T}_h} \int_T u_h^2 \nabla \omega^T A \nabla \omega dx| \leq C d^{-2} ||u_h||^2_{L_2(D_d)} \tag{3.12}$$

where we also used that $|\nabla \omega|_{L_\infty(\Omega)} \leq C d^{-1}$ in the last inequality.

For I_2, we use the definitions of P_A, (2.14) and of our lifting operator $r_{A,e}$, to rewrite it as

$$\begin{aligned}
I_2 &= -(\sum_{e \in \mathcal{E}_h} \int_e 2 \left\langle P_A(\omega \nabla_h(\omega u_h))\right\rangle^T A[u_h] ds \\
&\quad - \int_e (\left\langle \nabla_h(\omega^2 u_h)\right\rangle^T A[u_h] + \left\langle \nabla_h u_h \right\rangle^T A[\omega^2 u_h]) ds).
\end{aligned}$$

By the product rule, $\left\langle \nabla_h(\omega^2 u_h))\right\rangle = \left\langle \omega^2 \nabla_h(u_h) + 2\omega \nabla(\omega) u_h \right\rangle$. Adding a term to this and applying the product one more time, we see that

$$\left\langle \nabla_h(\omega^2 u_h)\right\rangle + \left\langle \omega^2 \nabla_h(u_h)\right\rangle = 2\omega \left\langle \nabla_h(\omega u_h)\right\rangle.$$

Therefore, we can express I_2 in the following form:

$$I_2 = -(2 \sum_{e \in \mathcal{E}_h} \int_e \left\langle P_A(\omega \nabla_h(\omega u_h)) - \omega \nabla_h(\omega u_h)\right\rangle^T A[u_h] ds.$$

By the Cauchy-Schwarz inequality, (2.3) and (2.5),

$$\begin{aligned}
I_2 \leq C \sum_{e \in \mathcal{E}_h} h^{-1/2} ||u_h||_{L_2(S_e)} (h^{-1/2} ||P_A(\omega \nabla_h(\omega u_h)) - \omega \nabla_h(\omega u_h)||_{L_2(S_e)} \\
+ h^{1/2} ||P_A(\omega \nabla_h(\omega u_h)) - \omega \nabla_h(\omega u_h)||_{H_h^1(S_e)}).
\end{aligned}$$

Moreover, by Lemma 2.4 and (2.10),

$$I_2 \leq C \sum_{e \in \mathcal{E}_h} ||u_h||_{L_2(S_e)} (d^{-1} ||\nabla_h(\omega u_h)||_{L_2(S_e)} + d^{-2} ||u_h||_{L_2(S_e)}).$$

Finally, using the positive definiteness of A and the arithmetic-geometric mean inequality, we see that

$$I_2 \leq \epsilon \sum_{T \in \mathcal{T}_h} \int_T \nabla(\omega u_h)^T A \nabla(\omega u_h) dx + C d^{-2} ||u_h||_{L_2(D_d)}. \tag{3.13}$$

Now we bound I_3. Again, using the definition of P_A and $r_{A,e}$, we get that

$$I_3 = -\sum_{e \in \mathcal{E}_h} \eta_e \int_e \left\langle P_A(\omega^2 r_{A,e}([u_h])) - \omega^2 r_{A,e}([u_h]) \right\rangle^T A[u_h] ds.$$

After using (2.5), (2.3), Lemma 2.4 and (2.9), we have

$$I_3 \leq C \sum_{e \in \mathcal{E}_h} ||u_h||_{L_2(S_e \cap D_d)} (d^{-1} ||\omega r_{A,e}([u_h])||_{L_2(S_e)} + d^{-2} h ||r_{A,e}([u_h])||_{L_2(S_e)}).$$

By Lemma 2.5 and (2.5),

$$h ||r_{A,e}([u_h])||_{L_2(S_e)} \leq C ||u_h||_{L_2(S_e)}.$$

Therefore, after applying the arithmetic-geometric mean inequality, we get

$$I_3 \leq C d^{-2} ||u_h||^2_{L_2(D_d)} + \epsilon \sum_{e \in \mathcal{E}_h} \int_{S_e} \omega^2 r_{A,e}([u_h])^T A r_{A,e}([u_h]) dx.$$

Now we handle I_4. By applying the product rule and the Cauchy-Schwarz inequality, using the boundedness of c and b, the positive definiteness of A and (2.5), we have that

$$\begin{aligned} I_4 \quad &\leq \quad C d^{-1} ||u_h||^2_{L_2(D_d)} + \epsilon \sum_{T \in \mathcal{T}_h} \int_T \nabla(\omega u_h)^T A \nabla(\omega u_h) dx \\ &\quad + \epsilon \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \int_e (\omega [u_h])^T A (\omega [u_h]) ds. \end{aligned}$$

If we apply Lemma 2.7 followed by (2.5), we see that

$$\begin{aligned} I_4 \quad &\leq \quad C d^{-2} ||u_h||^2_{L_2(D_d)} + \epsilon \sum_{T \in \mathcal{T}_h} \int_T \nabla(\omega u_h)^T A \nabla(\omega u_h) dx \\ &+ \quad C_1 \epsilon \sum_{e \in \mathcal{E}_h} \int_{S_e} \omega^2 r_{A,e}([u_h])^T A r_{A,e}([u_h]) dx \end{aligned}$$

where C_1 does not depend on ϵ. Finally, by taking ϵ small enough to "kick back" we arrive at

$$I \leq C B(u_h, \omega^2 u_h) + C d^{-2} ||u_h||_{L_2(D_d)}. \tag{3.14}$$

Therefore, (3.3) holds for the Modified BRMPS form.

In order to work with the LDG form, we define $R(v) = r_A([v]) + l_A(\beta^T [v])$. Let us assume, for a moment, the following inequality:

$$\begin{aligned} &\sum_{e \in \mathcal{E}_h} \int_{S_e} \omega^2 r_{A,e}([u_h])^T A r_{A,e}([u_h]) dx + \sum_{e \in \mathcal{E}_h^0} \int_e \omega^2 l_{A,e}(\beta^T [u_h])^T A l_{A,e}(\beta^T [u_h]) dx \\ &\qquad \leq C \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \int_e \omega^2 [u_h]^2 ds + C d^{-2} ||u_h||_{L_2(D_d)}. \end{aligned} \tag{3.15}$$

Using this inequality together with the fact that $R(u_h) = \sum_e r_{A,e}([u_h]) + \sum_{e\in\mathcal{E}_h^0} l_{A,e}(\beta^T[u_h])$ and that $r_{A,e}$ and $l_{A,e}$ are both supported in S_e, we have

$$\int_\Omega \omega^2 R(u_h)^T AR(u_h)dx \le C \sum_{e\in\mathcal{E}_h} \frac{1}{h_e}\int_e \omega^2[u_h]^2 ds + Cd^{-2}||u_h||_{L_2(D_d)}. \tag{3.16}$$

Using (3.9), we get

$$|||\omega u_h|||^2_{H^1_h(\Omega)} \le CJ + Cd^{-2}||u_h||^2_{L_2(D_d)}$$

$$\text{where } J \equiv \sum_{T\in\mathcal{T}_h}\int_T \nabla(\omega u_h)^T A\nabla(\omega u_h)dx + \sum_{e\in\mathcal{E}_h}\frac{1}{h_e}\int_e [\omega u_h]^2 ds.$$

Recall that for the LDG form $\eta > 0$. Using this fact, the arithmetic-geometric mean inequality and (3.16) we have (see (4.16) in [2] for a similar inequality),

$$\begin{aligned} J \quad &\le \quad C(\sum_{T\in\mathcal{T}_h}\int_T \nabla(\omega u_h)^T A\nabla(\omega u_h)dx + 2\sum_{T\in\mathcal{T}_h}\int_T \omega R(u_h)^T A\nabla(\omega u_h)dx \\ &+ \quad \int_\Omega \omega^2 R(u_h)^T AR(u_h)dx + \sum_{e\in\mathcal{E}_h}\frac{\eta_e}{h_e}\int_e [\omega u_h]^2 ds) + Cd^{-2}||u_h||_{L_2(D_d)}. \end{aligned} \tag{3.17}$$

Adding and subtracting the terms of $B(u_h, \omega^2 u_h)$ from the right hand side of (3.17) we see that

$$J \le CB(u_h, \omega^2 u_h) + C(J_1 + J_2 + J_3 + J_4) + Cd^{-2}||u_h||_{L_2(D_d)}$$

where

$$J_1 = |\sum_{T\in\mathcal{T}_h}\int_T \nabla(\omega u_h)^T A\nabla(\omega u_h) - \nabla(\omega^2 u_h)^T A\nabla u_h dx|,$$

$$\begin{aligned} J_2 \quad &= |2\sum_{T\in\mathcal{T}_h}\int_T \omega R(u_h)^T A\nabla(\omega u_h)dx \\ &+ \sum_{e\in\mathcal{E}_h}\int_e (\langle A(\nabla_h(\omega^2 u_h)\rangle [u_h]) + \langle A\nabla_h u_h\rangle [\omega^2 u_h])ds \\ &+ \sum_{e\in\mathcal{E}_h^0}\int_e ([A(\nabla_h(\omega^2 u_h)]\beta^T[u_h]) + [A\nabla_h u_h]\beta^T[\omega^2 u_h])ds|, \end{aligned}$$

$$J_3 = |\int_\Omega \omega^2 R(u_h)^T AR(u_h)dx - \int_\Omega R(\omega^2 u_h)^T AR(u_h)dx|,$$

and

$$J_4 = |\sum_{T\in\mathcal{T}_h}\int_T (c(\omega u_h)^2 + u_h b\cdot\nabla(\omega^2 u_h)dx + \sum_{e\in\mathcal{E}_h}\int_e \langle u_h\rangle\, b\cdot[\omega^2 u_h]ds))|.$$

We see that J_1 is $|I_1|$ with u_h now being the LDG solution. Therefore,

$$J_1 \le Cd^{-2}||u_h||^2_{L_2(D_d)}.$$

After following similar manipulations as was done to simplify I_2, we get

$$\begin{aligned} J_2 &= |2\sum_{e\in\mathcal{E}_h}\int_e \langle P_A(\omega\nabla_h(\omega u_h) - \omega\nabla_h(\omega u_h)\rangle^T A[u_h]ds \\ &\quad +2\sum_{e\in\mathcal{E}_h^0}\int_e [A(P_A(\omega\nabla_h(\omega u_h) - \omega\nabla_h(\omega u_h))]\beta^T[u_h]ds|. \end{aligned}$$

Using, (2.3), (2.5), Lemma 2.4, and (2.10), we have

$$J_2 \leq \epsilon \sum_{T\in\mathcal{T}_h}\int_T \nabla(\omega u_h)^T A\nabla(\omega u_h)dx + Cd^{-2}||u_h||_{L_2(D_d)}.$$

Using the definitions of our lifting operators and the projection operator, as was done for I_3, we see that

$$\begin{aligned} J_3 &= |\sum_{e\in\mathcal{E}_h}\int_e \left\langle P_A(\omega^2 R(u_h)) - \omega^2 R(u_h)\right\rangle^T A[u_h]ds \\ &\quad +\sum_{e\in\mathcal{E}_h^0}\int_e [A(P_A(\omega^2 R(u_h)) - \omega^2 R(u_h))]\beta^T[u_h]ds|. \end{aligned}$$

Using the Cauchy-Schwarz inequality, (2.3), and (2.5), we have

$$\begin{aligned} J_3 \leq C\sum_{e\in\mathcal{E}_h} h^{-1/2}||u_h||_{L_2(S_e)}(h^{-1/2}||P_A(\omega^2 R(u_h)) - \omega^2 R(u_h)||_{L_2(S_e)} \\ + h^{1/2}||P_A(\omega^2 R(u_h)) - \omega^2 R(u_h)||_{H_h^1(S_e)}). \end{aligned}$$

Using the fact that P_A is linear, the definition of R, and that the supports of $r_{A,e}$, $l_{A,e}$ lie in S_e, we have

$$\begin{aligned} J_3 &\leq C\sum_{e\in\mathcal{E}_h} h^{-1/2}||u_h||_{L_2(S_e')}(h^{-1/2}||P_A(\omega^2 r_{A,e}([u_h])) - \omega^2 r_{A,e}([u_h])||_{L_2(S_e)} \\ &\quad +h^{1/2}||P_A(\omega^2 r_{A,e}([u_h])) - \omega^2 r_{A,e}([u_h])||_{H_h^1(S_e)}) \\ &+ C\sum_{e\in\mathcal{E}_h^0} h^{-1/2}||u_h||_{L_2(S_e')}(h^{-1/2}||P_A(\omega^2 l_{A,e}(\beta^T[u_h])) - \omega^2 l_{A,e}(\beta^T[u_h])||_{L_2(S_e)} \\ &\quad +h^{1/2}||P_A(\omega^2 l_{A,e}(\beta^T[u_h])) - \omega^2 l_{A,e}(\beta^T[u_h])||_{H_h^1(S_e)}). \end{aligned}$$

Here S_e' denotes S_e union with the triangles that share an edge with S_e. Finally, mimicking the argument for bounding I_3, we see that

$$\begin{aligned} J_3 &\leq Cd^{-2}||u_h||^2_{L_2(D_d)} + \epsilon\sum_{e\in\mathcal{E}_h}\int_{S_e}(\omega^2 r_{A,e}([u_h])^T A r_{A,e}([u_h])dx \\ &+ \epsilon\sum_{e\in\mathcal{E}_h^0}\int_{S_e}\omega^2 l_e(\beta^T[u_h])^T A l_e(\beta^T[u_h])dx. \end{aligned}$$

Furthermore, by (3.15),

$$J_3 \leq Cd^{-2}||u_h||^2_{L_2(D_d)} + C_1\epsilon\sum_e \frac{1}{h_e}\int_e [\omega u_h]^2 ds \tag{3.18}$$

where C_1 is independent of ϵ. Again, since J_4 is $|I_4|$ with u_h now being the LDG solution, we have

$$J_4 \equiv I_4 \leq Cd^{-2}||u_h||^2_{L_2(D_d)} + \epsilon \sum_{T\in\mathcal{T}_h} \int_T \nabla(\omega u_h)^T A \nabla(\omega u_h) dx$$
$$+\epsilon \sum_{e\in\mathcal{E}_h} \frac{1}{h_e} \int_e [\omega u_h]^2 ds.$$

Finally, taking ϵ sufficiently small to "kick back", we conclude that

$$J \leq CB(u_h, \omega^2 u_h) + Cd^{-2}||u_h||^2_{L_2(D_d)}.$$

This, of course, implies (3.3) for the LDG form, which in turn implies our theorem for the LDG form. We are left to show (3.15).

In bounding I_3, we showed that

$$(3.19) \quad |\sum_{e\in\mathcal{E}_h} \int_{S_e} (\omega^2 r_{A,e}([u_h])^T A r_{A,e}([u_h]) - r_{A,e}(\omega^2[u_h])^T A r_{A,e}([u_h]))dx|$$
$$\leq Cd^{-2}||u_h||^2_{L_2(D_d)} + \epsilon \sum_{e\in\mathcal{E}_h} \int_{S_e} \omega^2 (r_{A,e}([u_h]))^T A (r_{A,e}([u_h]))dx$$

where $\epsilon > 0$ is arbitrary. Following similar steps one can show

$$(3.20) |\sum_{e\in\mathcal{E}_h^0} \int_{S_e} (\omega^2 l_{A,e}(\beta^T[u_h])^T A l_{A,e}(\beta^T[u_h]) - l_{A,e}(\omega^2\beta^T[u_h])^T A l_{A,e}(\beta^T[u_h]))dx|$$
$$\leq Cd^{-2}||u_h||^2_{L_2(D_d)} + \epsilon \sum_{e\in\mathcal{E}_h^0} \int_{S_e} \omega^2 (l_{A,e}(\beta^T[u_h]))^T A (l_{A,e}(\beta^T[u_h]))dx.$$

By an inverse estimate, (2.5), and the arithmetic-geometric mean inequality, we have

$$(3.21) \quad |\sum_{e\in\mathcal{E}_h} \int_{S_e} r_{A,e}(\omega^2[u_h])^T A r_{A,e}([u_h])dx|$$
$$+|\sum_{e\in\mathcal{E}_h^0} \int_{S_e} l_{A,e}(\omega^2\beta^T[u_h])^T A l_{A,e}(\beta^T[u_h])dx|$$
$$= |\sum_{e\in\mathcal{E}_h} \int_e \omega[u_h]^T A \langle r_{A,e}([u_h])\rangle \, ds| + |\sum_{e\in\mathcal{E}_h^0} \int_e \omega\beta^T[u_h][\omega A l_{A,e}(\beta^T[u_h])]ds|$$
$$\leq C \sum_{e\in\mathcal{E}_h} \frac{1}{h_e} \int [\omega u_h]^2 ds + \epsilon \sum_{e\in\mathcal{E}_h} \int_{S_e} \omega^2 r_{A,e}([u_h])^T A r_{A,e}([u_h])dx$$
$$+\epsilon \sum_{e\in\mathcal{E}_h^0} \int_{S_e} \omega^2 l_{A,e}(\beta^T[u_h])^T A l_{A,e}(\beta^T[u_h])dx.$$

Using (3.19), (3.20), (3.21), the triangle inequality, and taking ϵ sufficiently small to "kick back" we arrive at (3.15).

Following similar techniques as above we can show (3.3) for the Modified BMMPR form. We omit the proof. □

Using approximation properties and Theorem 3.1, we have the following Corollary.

Corollary 3.2. Let x, D_0, D_d, u, and u_h be as in Theorem 3.1 then

$$|||u - u_h|||_{H_h^1(D_0)} \leq Ch^{r-1}|u|_{H^r(D_d)} + Cd^{-1}||u - u_h||_{L_2(D_d)}. \tag{3.22}$$

4. Pointwise Estimates

Now we are ready to state our pointwise error estimates.

Theorem 4.1. *Suppose $u \in W^{1,\infty}(\Omega)$ satisfies (2.1) and $u_h \in V_h$ satisfies (2.2) for any of the bilinear forms. Let $x \in \bar{\Omega}$ and s satisfy $0 \leq s \leq r-2, r \geq 2$. Then, there exists a constant C independent of x, u, u_h and h such that*

$$|(u - u_h)(x)| \leq Ch \log(1/h)^{\bar{s}} \inf_{\chi \in V_h} |||u - \chi|||_{W_h^{1,\infty}(\Omega),x,s} \tag{4.1}$$

where $\bar{s} = 0$ if $0 \leq s < r-2$, and $\bar{s} = 1$ if $s = r-2$.

Proof. Now that we have established Lemma 2.6 and Corollary 3.2, we can follow the techniques of the proof for Theorem 5.1 in [6]. The main difference is that their result does not handle hanging nodes. However, their proof can easily be modified to allow hanging nodes by using the fact that each element has at most K neighbors. Also, the norm appearing on the right hand side of our result has an extra term, but that term can be handled easily throughout their proof. □

Similarly, by following the proof of Theorem 5.2 in [6], we obtain gradient pointwise estimates.

Theorem 4.2. *Suppose $u \in W^{1,\infty}(\Omega)$ satisfies (2.1) and $u_h \in V_h$ satisfies (2.2) for any of the bilinear forms. Let $x \in \bar{\Omega}$ and s satisfy $0 \leq s \leq r-1, r \geq 2$. Then, there exists a constant C independent of x, u, u_h and h such that*

$$|\nabla_h(u - u_h)(x)| \leq C \log(1/h)^{\bar{\bar{s}}} \inf_{\chi \in V_h} |||u - \chi|||_{W_h^{1,\infty}(\Omega),x,s} \tag{4.2}$$

where $\bar{\bar{s}} = 0$ if $0 \leq s < r-1$, and $\bar{\bar{s}} = 1$ if $s = r-1$.

The results here carry over to higher dimensions, if we allow meshes that fit the boundary exactly. Of course, this is not practical in higher dimensions, but it can in principle be done. Let Ω' satisfy $\Omega \subset \Omega'$ and $dist(\partial\Omega, \partial\Omega') \leq Ch^2$. If we mesh the larger domain Ω' with simplices such that the simplicial domian contains Ω and then restrict the mesh to Ω, we will find our desired mesh.

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