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CONFORMING AND DIVERGENCE FREE STOKES ELEMENTS ON GENERAL TRIANGULAR MESHES

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ABSTRACT. We present a family of conforming finite elements for the Stokes problem on general triangular meshes in two dimensions. The lowest order case consists of enriched piecewise linear polynomials for the velocity and piecewise constant polynomials for the pressure. We show that the elements satisfy the inf-sup condition and converges optimally for both the velocity and pressure. Moreover, the pressure space is exactly the divergence of the corresponding space for the velocity. Therefore the discretely divergence free functions are divergence free pointwise. We also show how the proposed elements are related to a class of C^1 elements through the use of a discrete de Rham complex.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded polygonal domain. We consider conforming finite element approximations for the Stokes equation:

$$-\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1.1b}$$

$$\boldsymbol{u} = 0 \qquad \text{on } \partial\Omega. \tag{1.1c}$$

In (1.1a) $\boldsymbol{f}$ is a given $\boldsymbol{L}^2(\Omega) := [L^2(\Omega)]^2$ function and $\nu > 0$ is the effective viscosity. A detailed account of the notation used is given below. A pair of functions $(\boldsymbol{u}, p) \in \boldsymbol{V} \times W := \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ are defined to be a solution of (1.1) if there holds

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{1.2a}$$

$$(\operatorname{div} \boldsymbol{u}, q) = 0 \qquad \forall q \in W, \tag{1.2b}$$

where $L_0^2(\Omega)$ denotes the set of square integrable functions with vanishing mean.

We consider finite element methods that take the same form as (1.2). Namely, let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ be a pair of conforming finite element spaces with discretization parameter h. Then the finite element method reads: find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times W_h$ such that

$$\nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}) - (p_h, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{1.3a}$$

$$(\operatorname{div} \boldsymbol{u}_h, q) = 0 \qquad \forall q \in W_h. \tag{1.3b}$$

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The stability and the error estimates of the approximate pair $(\boldsymbol{u}_h, p_h)$ depends on the classical inf-sup condition

$$\sup_{\boldsymbol{v}\in \boldsymbol{V}_h\setminus\{0\}} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \ge \alpha \|q\|_{L^2(\Omega)} \qquad \forall q \in W_h, \tag{1.4}$$

where $\alpha > 0$ is a constant independent of the parameter h. If (1.4) is satisfied, then one may easily deduce the solvability of (1.3) as well as derive the quasi-optimal estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \le C \inf_{\boldsymbol{v}\in \boldsymbol{V}_h,\ q\in W_h} \big(\|\boldsymbol{u} - \boldsymbol{v}\|_{H^1(\Omega)} + \|p - q\|_{L^2(\Omega)}\big), \tag{1.5}$$

where the constant $C > 0$ depends on ν and α, but is independent of h.

In this paper we find a pair of spaces $\boldsymbol{V}_h \times W_h$ that satisfy the inf-sup condition (1.4) and in addition satisfy the following desirable property:

$$\{\boldsymbol{v} \in \boldsymbol{V}_h :\ (\operatorname{div} \boldsymbol{v}, q) = 0\ \forall q \in W_h\} \subset \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \operatorname{div} \boldsymbol{v} \equiv 0\}. \tag{1.6}$$

In other words, we find inf-sup stable spaces for the Stokes problem such that discretely divergence free functions are divergence free pointwise. In fact, our spaces satisfy $\operatorname{div} \boldsymbol{V}_h = W_h$.

Finite element spaces that do not satisfy (1.6) can lead to undesired instabilities in nonlinear problems; see for example [4, 17]. However on general meshes, most stable pairs (i.e., pairs satisfying (1.4)) in the literature do not satisfy (1.6), e.g., Taylor-Hood elements [15], the MINI element [1], and Bernardi-Raugel elements [5]; see the review paper [6] for a more comprehensive list of examples. On the other hand, the spaces $\boldsymbol{\mathcal{P}}_k - \mathcal{P}_{k-1}$ (with $\boldsymbol{\mathcal{P}}_k$ continuous and $\mathcal{P}_{k-1}$ discontinuous) are inf-sup stable and satisfy (1.6) provided certain restrictions of the polynomial degree and mesh hold. For example, Scott and Vogelius [21] proved that these elements are stable if $k \ge 4$ and the mesh does not contain singularly vertices. In [2, 26, 24] it was shown that the spaces $\boldsymbol{\mathcal{P}}_k - \mathcal{P}_{k-1}$ satisfy (1.6) for smaller values of k if the meshes were Hsieh-Clough-Tocher or Powell-Sabin triangulations. As far as we are aware, conforming finite element spaces that satisfy both (1.4) and (1.6) on general triangulations have not appeared in the literature.

However, there are *non-conforming* methods that are inf-sup stable and lead to exactly divergence free approximation (at least locally) for the Stokes problem. These methods include the classical Crouzeix–Raviart elements [11] and the Fortin–Soulie elements [12]. Another strategy to construct non-conforming methods with these properties is to modify $\boldsymbol{H}(\operatorname{div};\Omega)$ conforming elements so that they possess (weak) tangental continuity [18, 23, 25, 14]. The motivation behind this approach is the fact that classical $\boldsymbol{H}(\operatorname{div};\Omega)$ finite element spaces (e.g., RT and BDM) satisfy (1.4)–(1.6), and therefore, if they can be enriched with div-free elements that enforce weak continuity, then the end result is a convergent finite element for the Stokes problem satisfying (1.4)–(1.6). To be more precise, the local spaces constructed in [18, 23, 25, 14] are of the form

$$\boldsymbol{M}(T) + \mathbf{curl}\,(b_T Q(T)), \tag{1.7}$$

where $\boldsymbol{M}(T)$ is the local space corresponding to the $\boldsymbol{H}(\operatorname{div};\Omega)$ space, b_T is the triangle cubic bubble function and $Q(T)$ is some scalar space. For example, in [14] this scalar space in the lowest order case is defined as $Q(T) = \operatorname{span}\{b_{e_i}\}_{i=1}^3$ where $\{b_{e_i}\}_{i=1}^3$ denotes the quadratic edge bubbles. Since only divergence free functions are added in (1.7), the resulting space (1.7) still satisfies (1.4)–(1.6).

The results in the current paper are motivated by the finite element methods construction in [18, 14]. Namely, we also modify $\boldsymbol{H}(\operatorname{div};\Omega)$ conforming finite elements (locally) to enforce tangental continuity. However, enriching these spaces with only polynomials (as done in (1.7)) is not flexible enough to guarantee conformity. This is in large part to the relatively high polynomial degree of both b_T and $Q(T)$. For this reason, in this paper we instead enhance $\boldsymbol{H}(\operatorname{div};\Omega)$ elements with divergence free *rational functions* which seem to offer the correct flexibility to enforce (strong) continuity. We also mention that we use a non-standard $\boldsymbol{H}(\mathrm{div}\,;\Omega)$ base space $\boldsymbol{M}(T)$ in our construction, which as far as we are aware, has not appeared in the literature before.

In order to lessen the number of degrees of freedom, we also introduce reduced elements. The dimension of the reduced local velocity space $\boldsymbol{V}_R(T)$ restricted to a triangle T is as follows

$$\dim \boldsymbol{V}_R(T) = \begin{cases} \dim \boldsymbol{\mathcal{P}}_k(T) + 3 & \text{if } k = 1, \\ \dim \boldsymbol{\mathcal{P}}_k(T) + 5 & \text{if } k = 2, \\ \dim \boldsymbol{\mathcal{P}}_k(T) + 6 & \text{if } k \geq 3. \end{cases}$$

We note that the lowest-order element ($k = 1$) has the same dimension as the Bernardi-Raugel element [5] (the global dimension is the same as well).

Finally, we mention that there has been recent development in the construction of conforming, divergence free and stable elements for the Stokes problem on *rectangular grids*. These include the $Q_{k+1,k} \times Q_{k,k+1}$ elements [27, 16] as well as using splines [9]. However, it is not at all obvious how to extend these elements to triangular meshes.

The rest of the paper is organized as follows. After presenting some notation and preliminary results in Section 2, we present our finite element method in the lowest order case in Section 3. Here we define the local space and the associated degrees of freedom, and derive the approximation properties of the corresponding projection (Fortin) operator. We then proceed with the convergence analysis of the finite element method using the abstract results discussed above. In Section 4 we define the analogous higher order elements for any polynomial degree $k \geq 1$. Finally in Section 5 we describe some reduced elements that enjoy the same orders of convergence, but have less degrees of freedom.

2. Notation and Preliminaries

Given a set $D \subset \Omega$, we denote by $H^m(D)$ ($m \geq 0$) the Sobolev space consisting of all $L^2(D)$ functions whose distributional derivatives up to order m are in $L^2(\Omega)$, and $H_0^m(D)$ to denote the set of functions whose traces vanish up to order $m-1$ on ∂D. We then set the corresponding vector Sobolev spaces as $\boldsymbol{H}^m(D) = (H^m(D))^2$ and $\boldsymbol{H}_0^m(D) = (H_0^m(D))^2$, and define the space of square integrable with vanishing mean as $L_0^2(D)$.

The L^2 inner product over a two dimensional (resp., one dimensional) set D is denoted by $(\cdot,\cdot)_D$ (reps., $\langle\cdot,\cdot\rangle_D$). In the case $D = \Omega$ we set $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\partial\Omega}$. The curl of a scalar function is a vector given by

$$\mathbf{curl}\, v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right)^t,$$

where as the curl and divergence of a vector valued function $\boldsymbol{v} = (v_1, v_2)^t$ is defined, respectively by

$$\mathrm{div}\, \boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}, \qquad \mathrm{curl}\, \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

The corresponding Sobolev spaces of these two operators are then given by

$$\boldsymbol{H}(\mathrm{div}\,; D) = \left\{\boldsymbol{v} \in \boldsymbol{L}^2(D) :\ \mathrm{div}\, \boldsymbol{v} \in L^2(D)\right\},$$
$$\boldsymbol{H}(\mathrm{curl}\,; D) = \left\{\boldsymbol{v} \in \boldsymbol{L}^2(D) :\ \mathrm{curl}\, \boldsymbol{v} \in L^2(D)\right\},$$

and we also define

$$\boldsymbol{H}_0(\mathrm{div}\,; D) = \left\{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}\,; D) :\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial D} = 0\right\},$$

where $\boldsymbol{n}$ denotes the outward normal of the boundary ∂D.

For a given simplex S and $m \ge 0$, the vector-valued polynomials are defined as $\boldsymbol{\mathcal{P}}_m(S) = [\mathcal{P}_m(S)]^2$, where $\mathcal{P}_m(S)$ is the space of polynomials defined on S of degree less than or equal to m. We also set $\mathcal{P}_m(S)$ and $\boldsymbol{\mathcal{P}}_m(S)$ to be the empty set for any negative valued m. Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω [10, 7] with $h_T = \mathrm{diam}(T)$ for all $T \in \mathcal{T}_h$ and $h = \max_{T\in\mathcal{T}_h} h_T$. We define the patch of an edge e in $\mathcal{T}_h$ as

$$\omega(e) := \big\{T \in \mathcal{T}_h :\ \partial T \cap e \neq \emptyset\big\},$$

and we use the convention

$$\|v\|^2_{H^m(\omega(e))} = \sum_{T\in\omega(e)} \|v\|^2_{H^m(T)}.$$

Given $T \in \mathcal{T}_h$, we denote by $\boldsymbol{n}$ the outward unit normal of ∂T, by $\boldsymbol{t}$ the unit tangent of ∂T obtained by rotating $\boldsymbol{n}$ 90 degrees counterclockwise, and by $\{\lambda_i\}_{i=1}^3$ the three barycentric coordinates of T labeled such that λ_i vanishes on $e_i \subset \partial T$. We also denote by $\{x_i\}_{i=1}^3$ the three vertices of T with $\lambda_i(x_j) = \delta_{ij}$. The element bubble and edge bubbles are then respectively given by

$$b_T = \lambda_1\lambda_2\lambda_3 \in \mathcal{P}_3(T) \qquad b_{e_i} = \lambda_{i+1}\lambda_{i+2} \in \mathcal{P}_2(T) \pmod 3. \tag{2.1}$$

Due to their definitions, the element and edge bubbles satisfy the following properties:

$$b_T\big|_{\partial T} = 0, \qquad \frac{\partial b_T}{\partial \boldsymbol{n}_i}\Big|_{e_i} = \mathrm{a}_i b_{e_i}, \qquad b_{e_i}\big|_{\partial T\backslash e_i} = 0, \qquad b_{e_i}\big|_{e_i} > 0, \tag{2.2}$$

where

$$\mathrm{a}_i := -|\nabla\lambda_i|, \tag{2.3}$$

and $\boldsymbol{n}_i$ denotes the outward unit normal of e_i. We emphasize that $\mathrm{a}_i \neq 0$, as this property will be used frequently in the sequel. We also set the rational bubble functions as $(i = 1, 2, 3)$

$$\begin{aligned} &B_{e_i} = \frac{b_T b_{e_i}}{(\lambda_i + \lambda_{i+1})(\lambda_i + \lambda_{i+2})} && \text{for } 0 \le \lambda_i \le 1,\ 0 \le \lambda_{i+1}, \lambda_{i+2} < 1,\\ &B_{e_i}(x_{i+1}) = B_{e_i}(x_{i+2}) = 0 && \text{otherwise.} \end{aligned}$$

A few properties of the rational bubble functions are established in the following lemma.

Lemma 2.1. *There holds*

$$B_{e_i} \in C^1(\overline{T}) \cap W^{2,\infty}(T), \qquad B_{e_i}\big|_{\partial T} = 0, \qquad \nabla B_{e_i}(x_j) = 0\ (j = 1, 2, 3), \tag{2.4a}$$

$$\nabla B_{e_i}\big|_{\partial T\backslash e_i} = 0, \qquad \frac{\partial B_{e_i}}{\partial \boldsymbol{n}_i}\Big|_{e_i} = \mathrm{a}_i b_{e_i}, \qquad \nabla B_{e_i}\big|_{e_i} \in \boldsymbol{\mathcal{P}}_2(e_i). \tag{2.4b}$$

Proof. The property $B_{e_i} \in C^1(\overline{T})$ as well as the second and third properties have been shown in [10, p. 347–348]. To show the fourth property, we note that since b_T vanishes on ∂T,

$$\frac{\partial B_{e_i}}{\partial x_k}\bigg|_{\partial T} = \frac{\dfrac{\partial b_T}{\partial x_k} b_{e_i}}{(\lambda_i + \lambda_{i+1})(\lambda_i + \lambda_{i+2})}\Bigg|_{\partial T}. \tag{2.5}$$

Thus, since b_{e_i} vanishes on $\partial T\backslash e_i$, we obtain $\nabla B_{e_i}\big|_{\partial T\backslash e_i} = 0$. Moreover since λ_i vanishes on e_i, we have by (2.5) and (2.2),

$$\frac{\partial B_{e_i}}{\partial \boldsymbol{n}_i}\bigg|_{e_i} = \frac{\dfrac{\partial b_T}{\partial \boldsymbol{n}_i} b_{e_i}}{(\lambda_i + \lambda_{i+1})(\lambda_i + \lambda_{i+2})}\Bigg|_{e_i} = \frac{\partial b_T}{\partial \boldsymbol{n}_i}\bigg|_{e_i} = \mathrm{a}_i b_{e_i}.$$

Since $\frac{\partial B_{e_i}}{\partial \boldsymbol{t}}\big|_{\partial T} = 0$ and $\frac{\partial B_{e_i}}{\partial \boldsymbol{n}}\big|_{\partial T} \in \mathcal{P}_2(\partial T)$, we have $\nabla B_{e_i}\big|_{\partial T} \in \boldsymbol{\mathcal{P}}_2(\partial T)$.

Finally we show the property $B_{e_i} \in W^{2,\infty}(T)$. It is easy to see that B_{e_i} is well-behaved away from the vertices of T, so it suffices to show that the second derivatives of B_{e_i} are bounded at the vertices. Furthermore since the property $B_{e_i} \in W^{2,\infty}(T)$ is invariant through affine transformations, it is enough to consider the case when T is the unit triangle with vertices $(0,1)$, $(1,0)$ and $(0,0)$. The rational bubble is then given by

$$B_{e_1} = \frac{x_1 x_2^2 (1 - x_1 - x_2)^2}{(x_1 + x_2)(1 - x_2)}.$$

We study the behavior of B_{e_1} at the origin as the other vertices follow from the symmetry of the rational bubble functions. Writing $B_{e_1} = s(x_1,x_2)g(x_1,x_2)$ with $s(x_1,x_2) = \frac{x_1 x^2}{(x_1+x_2)}$ and $g(x_1,x_2) = \frac{(1-x_1-x_2)^2}{(1-x_2)}$, it suffices to show that $s \in W^{2,\infty}(T)$ since g is smooth at the origin. An easy calculation shows

$$\frac{\partial^2 s}{\partial x_1^2} = -\frac{2x_2^3}{(x_1+x_2)^3}, \qquad \frac{\partial^2 s}{\partial x_2^2} = \frac{2x_1^3}{(x_1+x_2)^3}, \qquad \frac{\partial^2 s}{\partial x_1 \partial x_2} = \frac{x_2^2(3x_1+x_2)}{(x_1+x_2)^3}.$$

Since $x_1, x_2 \geq 0$ in T we have

$$\Big|\frac{\partial^2 s}{\partial x_1^2}\Big| \leq \frac{2x_2^3}{(x_1+x_2)^3} \leq 2, \quad \text{and} \quad \Big|\frac{\partial^2 s}{\partial x_2^2}\Big| \leq \frac{2x_1^3}{(x_1+x_2)^3} \leq 2.$$

Similarly we obtain

$$\Big|\frac{\partial^2 s}{\partial x_1 \partial x_2}\Big| \leq \frac{3x_1x_2^2}{(x_1+x_2)^3} + \frac{x_2^3}{(x_1+x_2)^3} \leq 4.$$

It then follows that the second derivatives of s are bounded at the origin, and therefore $B_{e_1} \in W^{2,\infty}(T)$. ☐

Remark 2.2. Since $B_{e_i}\big|_T \in W^{2,\infty}(T)$, we clearly have $B_{e_i}|_T \in H^2(T)$.

Remark 2.3. Although the rational bubbles lie in $W^{2,\infty}(T)$, they are not $C^2(\overline{T})$ [10].

3. The finite element method in the lowest order case

3.1. **The local space.** In this section, we describe a finite element for the Stokes problem using enriched piecewise linear polynomials for the velocity and piecewise constants for the pressure. Essentially, we enrich $\boldsymbol{H}(\mathrm{div}\,;\Omega)$ elements with rational bubbles to obtain $\boldsymbol{H}^1(\Omega)$ approximations. First we describe the local space of the $\boldsymbol{H}(\mathrm{div}\,;\Omega)$ element.

For $T \in \mathcal{T}_h$ we define

$$\boldsymbol{M}_2(T) = \boldsymbol{\mathcal{P}}_1(T) + \mathrm{span}\big\{\mathbf{curl}\,(b_{e_i}\lambda_{i+1})\big\}_{i=1}^3. \tag{3.1}$$

The associated degrees of freedom of $\boldsymbol{M}_2(T)$ are given by

$$\boldsymbol{v}(x_i) \qquad \text{for all vertices } x_i, \tag{3.2a}$$

$$\big\langle \boldsymbol{v}\cdot\boldsymbol{n}_i, \kappa\big\rangle_{e_i} \qquad \text{for all } \kappa \in \mathcal{P}_0(e_i)\ (i=1,2,3). \tag{3.2b}$$

To see that the degrees of freedom (3.2) are unisolvent on $\boldsymbol{M}_2(T)$ we first notice that the sum in (3.1) is direct, and therefore the dimension of $\boldsymbol{M}_2(T)$ is $\dim \boldsymbol{\mathcal{P}}_1(T) + 3 = 9$ (proving that the sum is direct can easily be shown by using the techniques used below). Since there are a total of nine degrees of freedom given in (3.2), it suffices to show that if $\boldsymbol{v} \in \boldsymbol{M}_2(T)$ vanishes at the degrees of freedom, then $\boldsymbol{v} \equiv 0$.

First since $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_2(T)$ we have $\boldsymbol{v} \cdot \boldsymbol{n}\big|_{\partial T} = 0$. Writing $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{s}$ with $\boldsymbol{v}_0 \in \boldsymbol{\mathcal{P}}_1(T)$ and $\boldsymbol{s} = \sum_{i=1} \mathrm{d}_i \mathbf{curl}\,(b_{e_i}\lambda_{i+1})$ with $\mathrm{d}_i \in \mathbb{R}$, we then deduce that $\boldsymbol{s} \cdot \boldsymbol{n}\big|_{\partial T} \in \mathcal{P}_1(\partial T)$. Therefore by (2.1), we have for any $j = 1, 2, 3$,

$$\boldsymbol{s} \cdot \boldsymbol{n}_j\big|_{e_j} = \sum_{i=1}^{3} \mathrm{d}_i \frac{\partial(b_{e_i}\lambda_{i+1})}{\partial \boldsymbol{t}_j}\Big|_{e_j} = \mathrm{d}_j \frac{\partial(b_{e_j}\lambda_{j+1})}{\partial \boldsymbol{t}_j}\Big|_{e_j} = \mathrm{d}_j \frac{\partial(\lambda_{j+1}^2\lambda_{j+2})}{\partial \boldsymbol{t}_j}\Big|_{e_j} \in \mathcal{P}_1(e_j).$$

Noting $\lambda_{j+1} + \lambda_{j+2} = 1$ on e_j, it follows that

$$\begin{aligned}\boldsymbol{s} \cdot \boldsymbol{n}_j\big|_{e_j} &= \mathrm{d}_j \lambda_{j+1}\Big(2\lambda_{j+2}\frac{\partial \lambda_{j+1}}{\partial \boldsymbol{t}_j} + \lambda_{j+1}\frac{\partial \lambda_{j+2}}{\partial \boldsymbol{t}_j}\Big)\Big|_{e_j} \\ &= \mathrm{d}_j \lambda_{j+1}\frac{\partial \lambda_{j+1}}{\partial \boldsymbol{t}_j}\big(2 - 3\lambda_{j+1}\big)\Big|_{e_j} \in \mathcal{P}_1(e_j).\end{aligned}$$

We then conclude that $\mathrm{d}_j\lambda_{j+1}(2 - 3\lambda_{j+1})\big|_{e_j} \in \mathcal{P}_1(e_j)$, and therefore $\mathrm{d}_j = 0$. It then follows that $\boldsymbol{s} \equiv 0$ and therefore $\boldsymbol{v}_0 \cdot \boldsymbol{n}\big|_{\partial T} = 0$. This implies $\boldsymbol{v}_0 \equiv 0$ and so $\boldsymbol{v} \equiv 0$ as well. Thus the unisolvency of the degrees of freedom (3.2) is proved.

With the local space of the $\boldsymbol{H}(\mathrm{div}\,;\Omega)$ established we now describe the local space of the conforming velocity finite element for the Stokes problem. Set

$$\boldsymbol{V}(T) = \boldsymbol{M}_2(T) + \boldsymbol{Q}_2(T), \tag{3.3}$$

with

$$\boldsymbol{Q}_2(T) = \mathrm{span}\big\{\mathbf{curl}\,(B_{e_i})\big\}_{i=1}^{3}. \tag{3.4}$$

The associated degrees of freedom of $\boldsymbol{V}(T)$ are as follows:

$$\boldsymbol{v}(x_i) \qquad \text{for all vertices } x_i, \tag{3.5a}$$

$$\big\langle \boldsymbol{v}, \boldsymbol{\kappa}\big\rangle_{e_i} \qquad \text{for all } \boldsymbol{\kappa} \in \boldsymbol{\mathcal{P}}_0(e_i)\ (i = 1, 2, 3). \tag{3.5b}$$

Lemma 3.1. *There holds*

$$\boldsymbol{V}(T) = \boldsymbol{\mathcal{P}}_1(T) \oplus \boldsymbol{Q}_2(T), \tag{3.6}$$

$$\dim \boldsymbol{V}(T) = 12. \tag{3.7}$$

Furthermore, any function $\boldsymbol{v} \in \boldsymbol{V}(T)$ is uniquely defined by the degrees of freedom (3.5)*, and $\boldsymbol{V}(T)$ restricted to e_i is a subspace of $\boldsymbol{\mathcal{P}}_2(e_i)$ for i=1,2,3.*

Proof. It is clear from the definition of B_{e_i} that the sum in (3.3) is direct and therefore $\dim \boldsymbol{V}(T) = \dim \boldsymbol{M}_2(T) + 3 = 12$.

Next, since the number of degrees of freedom given in (3.5) is 12, to show unisolvency, it suffices to show that if $\boldsymbol{v} \in \boldsymbol{V}(T)$ vanishes at the degrees of freedom, then $\boldsymbol{v} \equiv 0$. To this end, we write $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{q}$ with $\boldsymbol{v}_0 \in \boldsymbol{M}_2(T)$ and $\boldsymbol{q} \in \boldsymbol{Q}_2(T)$. By Lemma 2.1, $\boldsymbol{q}$ vanishes at the vertices of T and $\boldsymbol{q} \cdot \boldsymbol{n}\big|_{\partial T} = 0$. Since these two types of degrees of freedom uniquely determine a function in $\boldsymbol{M}_2(T)$, it follows that $\boldsymbol{v}_0 \equiv 0$. Next, write $\boldsymbol{q} = \sum_{i=1}^{3} \mathrm{d}_i \mathbf{curl}\,(B_{e_i})$. Then by (3.5b), (2.2) and Lemma 2.1 we have

$$0 = \int_{e_i} \boldsymbol{q} \cdot \boldsymbol{t}_i\, ds = \mathrm{d_i} \int_{e_i} \frac{\partial B_{e_i}}{\partial \boldsymbol{n}_i}\, ds = \mathrm{a}_i \mathrm{d_i} \int_{e_i} b_{e_i}\, ds \quad \Longrightarrow \quad \mathrm{d_i} = 0.$$

It then follows that $\boldsymbol{v} \equiv 0$, and hence the degrees of freedom (3.5) are unisolvent on $\boldsymbol{V}(T)$.

Finally, the fact that $\boldsymbol{V}(T)$ restricted to the boundary ∂T is a subspace of $\boldsymbol{\mathcal{P}}_2(T)$ follows directly from the definition (3.3) and Lemma 2.1.

$\square$

3.2. **The global space and its approximation properties.** The degrees of freedom (3.5) naturally lead us to define the global space as

$$\boldsymbol{V}_h = \big\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) :\ \boldsymbol{v}\big|_T \in \boldsymbol{V}(T)\big\}, \tag{3.8}$$

and a projection $\widehat{\boldsymbol{\Pi}}_h : \boldsymbol{C}^0(\overline{\Omega}) \to \boldsymbol{V}_h$ defined locally by

$$\widehat{\boldsymbol{\Pi}}_h \boldsymbol{v}(x_i) = \boldsymbol{v}(x_i), \qquad \int_{e_i} \widehat{\boldsymbol{\Pi}}_h \boldsymbol{v}\, ds = \int_{e_i} \boldsymbol{v}\, ds \quad (i = 1, 2, 3). \tag{3.9}$$

Remark 3.2. Since the rational bubble functions satisfy $B_{e_i}\big|_T \in W^{2,\infty}(T)$, there holds the inclusion $\boldsymbol{V}_h \subset W^{1,\infty}(\Omega)$.

We also define the pressure space as the space consisting of piecewise constants

$$W_h = \big\{q \in L_0^2(\Omega) :\ q\big|_T \in \mathcal{P}_0(T)\big\}. \tag{3.10}$$

Note that by (3.9), we have

$$(\nabla \cdot (\boldsymbol{v} - \widehat{\boldsymbol{\Pi}}_h \boldsymbol{v}), q)_T = \big\langle (\boldsymbol{v} - \widehat{\boldsymbol{\Pi}}_h \boldsymbol{v}) \cdot \boldsymbol{n}, q\big\rangle_{\partial T} = 0 \qquad \forall q \in \mathcal{P}_0(T). \tag{3.11}$$

Thus, denoting by P_h the $L^2(\Omega)$ projection onto W_h, equation (3.11) is equivalent to the following commutative property:

$$\operatorname{div} \widehat{\boldsymbol{\Pi}}_h \boldsymbol{v} = P_h \operatorname{div} \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{C}^0(\overline{\Omega}) \cap \boldsymbol{H}_0^1(\Omega). \tag{3.12}$$

However, due to the first condition in (3.9), the operator $\widehat{\boldsymbol{\Pi}}_h$ is not well-defined on $\boldsymbol{H}_0^1(\Omega)$, and therefore some modifications are in order. To this end, we use the common approach of replacing $\boldsymbol{v}(x_i)$ in (3.9) with $\boldsymbol{\Pi}_S v(x_i)$, where $\boldsymbol{\Pi}_S : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{L}_h$ denotes the Scott-Zhang interplant [22] and $\boldsymbol{L}_h \subset \boldsymbol{H}_0^1(\Omega)$ is the linear Lagrange finite element space. This then leads to the definition of $\boldsymbol{\Pi}_h : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_h$ with

$$\boldsymbol{\Pi}_h \boldsymbol{v}(x_i) = \boldsymbol{\Pi}_S \boldsymbol{v}(x_i), \qquad \int_{e_i} \boldsymbol{\Pi}_h \boldsymbol{v}\, ds = \int_{e_i} \boldsymbol{v}\, ds \quad (i = 1, 2, 3). \tag{3.13}$$

It easily seen that the commutative property (3.12) holds for $\boldsymbol{\Pi}_h$ as well, i.e.,

$$\operatorname{div} \boldsymbol{\Pi}_h \boldsymbol{v} = P_h \operatorname{div} \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{3.14}$$

We now address the approximation properties of $\boldsymbol{\Pi}_h$. To this end, we first introduce the two auxiliary spaces

$$\boldsymbol{M}_h = \big\{\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{div}; \Omega) :\ v\big|_T \in \boldsymbol{M}_2(T)\ \forall T \in \mathcal{T}_h\big\}, \tag{3.15}$$

$$\boldsymbol{Q}_h = \big\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) :\ v\big|_T \in \boldsymbol{Q}_2(T)\ \forall T \in \mathcal{T}_h\big\}. \tag{3.16}$$

The associated projections of $\boldsymbol{M}_h$ and $\boldsymbol{Q}_h$ are then given respectively as $\boldsymbol{\Pi}_M : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{M}_h$, $\boldsymbol{\Pi}_Q : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{Q}_h$, defined locally by

$$\boldsymbol{\Pi}_M \boldsymbol{v}(x_i) = \boldsymbol{\Pi}_S \boldsymbol{v}(x_i), \qquad \int_{e_i} (\boldsymbol{\Pi}_M \boldsymbol{v}) \cdot \boldsymbol{n}_i\, ds = \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n}_i\, ds, \tag{3.17}$$

$$\int_{e_i} (\boldsymbol{\Pi}_Q \boldsymbol{v}) \cdot \boldsymbol{t}_i\, ds = \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{t}_i\, ds. \tag{3.18}$$

Following the arguments in Section 3.1, we see that these spaces and their corresponding projections are well-defined. Note that functions in $\boldsymbol{Q}_h$ vanish at the vertices of the mesh, and that their zeroth order normal moments vanish as well. It then follows from (3.17) that

$$\boldsymbol{\Pi}_M \boldsymbol{v}(x_i) + \boldsymbol{\Pi}_Q(\boldsymbol{I} - \boldsymbol{\Pi}_M)\boldsymbol{v}(x_i) = \boldsymbol{\Pi}_M \boldsymbol{v}(x_i) = \boldsymbol{\Pi}_S \boldsymbol{v}(x_i), \tag{3.19}$$

and

$$\int_{e_i} \big(\boldsymbol{\Pi}_M \boldsymbol{v} + \boldsymbol{\Pi}_Q(\boldsymbol{I} - \boldsymbol{\Pi}_M)\boldsymbol{v}\big)\cdot \boldsymbol{n}_i\, ds = \int_{e_i} \boldsymbol{\Pi}_M \boldsymbol{v}\cdot \boldsymbol{n}_i\, ds = \int_{e_i} \boldsymbol{v}\cdot \boldsymbol{n}_i, \tag{3.20}$$

where $\boldsymbol{I}$ denotes the identity operator on $\boldsymbol{H}_0^1(\Omega)$. Moreover, by (3.18) we have

$$\begin{aligned}&\int_{e_i} \big(\boldsymbol{\Pi}_M \boldsymbol{v} + \boldsymbol{\Pi}_Q(\boldsymbol{I} - \boldsymbol{\Pi}_M)\boldsymbol{v}\big)\cdot \boldsymbol{t}_i\, ds \\ &\qquad = \int_{e_i} \big(\boldsymbol{\Pi}_M \boldsymbol{v} + (\boldsymbol{I} - \boldsymbol{\Pi}_M)\boldsymbol{v}\big)\cdot \boldsymbol{t}_i\, ds = \int_{e_i} \boldsymbol{v}\cdot \boldsymbol{t}_i\, ds.\end{aligned} \tag{3.21}$$

It then follows from (3.19)–(3.21) and (3.13) that

$$\boldsymbol{\Pi}_h = \boldsymbol{\Pi}_M + \boldsymbol{\Pi}_Q(\boldsymbol{I} - \boldsymbol{\Pi}_M),$$

and therefore,

$$\boldsymbol{I} - \boldsymbol{\Pi}_h = (\boldsymbol{I} - \boldsymbol{\Pi}_Q)(\boldsymbol{I} - \boldsymbol{\Pi}_M). \tag{3.22}$$

Hence, the approximation properties of $\boldsymbol{\Pi}_h$ reduce to the stability estimates of $\boldsymbol{\Pi}_Q$ plus the approximation properties of $\boldsymbol{\Pi}_M$. We now address the first issue. Given $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, we write

$$\boldsymbol{\Pi}_Q \boldsymbol{v}\big|_T = \sum_{i=1}^{3} \mathrm{d}_i \,\mathbf{curl}\,(B_{e_i})\big|_T \qquad \text{with } \mathrm{d}_i \in \mathbb{R}.$$

By Lemma 2.1, we have

$$\boldsymbol{\Pi}_Q \boldsymbol{v}\cdot \boldsymbol{t}_j\big|_{e_j} = \mathrm{d}_j \frac{\partial B_{e_j}}{\partial \boldsymbol{n}_j}\Big|_{e_j} = \mathrm{a}_j \mathrm{d}_j b_{e_j}\big|_{e_j}. \tag{3.23}$$

Therefore by (3.18), we obtain

$$\mathrm{a}_j \mathrm{d}_j \int_{e_j} b_{e_j}\, dx = \int_{e_j} \boldsymbol{\Pi}_Q \boldsymbol{v}\cdot \boldsymbol{t}_j\, ds = \int_{e_j} \boldsymbol{v}\cdot \boldsymbol{t}_j\, ds,$$

and thus,

$$\mathrm{a}_j \mathrm{d}_j = \frac{1}{\int_{e_j} b_{e_j}\, ds} \int_{e_j} \boldsymbol{v}\cdot \boldsymbol{t}_j\, ds = \frac{6}{|e_j|} \int_{e_j} \boldsymbol{v}\cdot \boldsymbol{t}_j\, ds \le \frac{6}{|e_j|^{1/2}} \|\boldsymbol{v}\cdot \boldsymbol{t}_j\|_{L^2(e_j)}. \tag{3.24}$$

Hence by a scaling argument using the Piola transformation and (3.23)–(3.24), we obtain

$$\begin{aligned}\|\boldsymbol{\Pi}_Q \boldsymbol{v}\|_{L^2(T)} &\le C h_T^{1/2} \sum_{i=1}^{3} \big\|\boldsymbol{\Pi}_Q \boldsymbol{v}\cdot \boldsymbol{t}_i\big\|_{L^2(e_i)} \\ &= C h_T^{1/2} \sum_{i=1}^{3} \mathrm{a}_i \mathrm{d}_i \|b_{e_i}\|_{L^2(e_i)} \\ &\le C h_T^{1/2} \sum_{i=1}^{3} \mathrm{a}_i \mathrm{d}_i |e_i|^{1/2} \le C h_T^{1/2} \|\boldsymbol{v}\cdot \boldsymbol{t}\|_{L^2(\partial T)},\end{aligned} \tag{3.25}$$

where $C > 0$ is independent of h.

The arguments in [3] can be used to derive the approximation properties of $\boldsymbol{M}_h$ so we only sketch the main points. First, we introduce the operator $\boldsymbol{\Pi}_{M,0} : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{M}_h$, defined locally as

$$\boldsymbol{\Pi}_{M,0}\boldsymbol{v}(x_i) = 0, \qquad \int_{e_i} \boldsymbol{\Pi}_{M,0}\boldsymbol{v} \cdot \boldsymbol{n}_i \, ds = \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n}_i \, ds. \tag{3.26}$$

By (3.17) and (3.26), we have $\boldsymbol{I}-\boldsymbol{\Pi}_M = (\boldsymbol{I}-\boldsymbol{\Pi}_{M,0})(\boldsymbol{I}-\boldsymbol{\Pi}_S)$. Furthermore, by standard scaling arguments, we have $\|\boldsymbol{\Pi}_{M,0}\boldsymbol{v}\|_{L^2(T)} \le C\big(\|\boldsymbol{v}\|_{L^2(T)} + h_T\|\boldsymbol{v}\|_{H^1(T)}\big)$. It then follows that $\|\boldsymbol{v}-\boldsymbol{\Pi}_M\boldsymbol{v}\|_{L^2(T)} \le C\big(\|\boldsymbol{v}-\boldsymbol{\Pi}_S\boldsymbol{v}\|_{L^2(T)} + h_T\|\boldsymbol{v}-\boldsymbol{\Pi}_S\boldsymbol{v}\|_{H^1(T)}\big)$, and therefore by approximation properties of the Scott-Zhang operator and the inverse estimate, we deduce

$$\|\boldsymbol{v}-\boldsymbol{\Pi}_M\boldsymbol{v}\|_{H^m(T)} \le Ch_T^{s-m}\|\boldsymbol{v}\|_{H^s(\omega(T))} \quad (0 \le m \le s,\ 1 \le s \le 2). \tag{3.27}$$

Combining the decomposition (3.22) with (3.25), (3.27) and the trace inequality, we obtain

$$\begin{aligned}
\|\boldsymbol{v}-\boldsymbol{\Pi}_h\boldsymbol{v}\|_{L^2(T)} &\le \|\boldsymbol{v}-\boldsymbol{\Pi}_M\boldsymbol{v}\|_{L^2(T)} + Ch_T^{1/2}\|\boldsymbol{v}-\boldsymbol{\Pi}_M\boldsymbol{v}\|_{L^2(\partial T)} \\
&\le C\big(\|\boldsymbol{v}-\boldsymbol{\Pi}_M\boldsymbol{v}\|_{L^2(T)} + h_T\|\boldsymbol{v}-\boldsymbol{\Pi}_M\boldsymbol{v}\|_{H^1(T)}\big) \le Ch_T^s\|\boldsymbol{v}\|_{H^s(\omega(T))}.
\end{aligned}$$

With a further scaling argument we have the following lemma.

Lemma 3.3. *For any $\boldsymbol{v} \in \boldsymbol{H}^s(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ with $1 \le s \le 2$, there holds*

$$\|\boldsymbol{v}-\boldsymbol{\Pi}_h\boldsymbol{v}\|_{H^m(T)} \le Ch_T^{s-m}\|\boldsymbol{v}\|_{H^s(\omega(T))} \qquad (0 \le m \le 1). \tag{3.28}$$

3.3. **Convergence analysis.** To start the convergence analysis, we first verify that the inf-sup condition (1.4) holds as well as show that the discretely divergence free functions in $\boldsymbol{V}_h$ are divergence pointwise, that is, (1.6) holds. First, for given $q \in W_h \subset L_0^2(\Omega)$ there exists $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ such that [13]

$$C\|q\|_{L^2(\Omega)} \le \frac{(\operatorname{div}\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H^1(\Omega)}}.$$

It then follows from (3.14) and (3.28) that

$$C\|q\|_{L^2(\Omega)} \le \frac{(\operatorname{div}\boldsymbol{\Pi}_h\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \le C\frac{(\operatorname{div}\boldsymbol{\Pi}_h\boldsymbol{v}, q)}{\|\boldsymbol{\Pi}_h\boldsymbol{v}\|_{H^1(\Omega)}} \le \sup_{\boldsymbol{w}\in\boldsymbol{V}_h\setminus\{0\}} \frac{(\operatorname{div}\boldsymbol{w}, q)}{\|\boldsymbol{w}\|_{H^1(\Omega)}}.$$

Thus, the inf-sup condition holds. Furthermore it is easy to see from the definition of $\boldsymbol{V}_h$ and W_h that $\operatorname{div}\boldsymbol{V}_h \subset W_h$, from which we easily deduce $\operatorname{div}\boldsymbol{V}_h = W_h$. It then follows that (1.6) holds as well.

As is well known, since our spaces satisfy (1.6) we get estimates of the velocity which are independent of p. We omit the proof of the following theorem as it can be found in many places in the literature (e.g., [8, 6]).

Theorem 3.4. *Let $(\boldsymbol{u}, p)$ satisfy (1.1) and let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times W_h$ satisfy (1.3). We then have*

$$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\Omega)} \le \|\nabla(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u})\|_{L^2(\Omega)},$$

and

$$\|P_hp - p_h\|_{L^2(\Omega)} \le C\,\nu\|\nabla(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u})\|_{L^2(\Omega)}.$$

Consequently, by (3.28) and the Poincaré inequality there holds

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)} &\le Ch\|\boldsymbol{u}\|_{H^2(\Omega)}, \\
\|p-p_h\|_{L^2(\Omega)} &\le Ch\big(\nu\|\boldsymbol{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}\big).
\end{aligned}$$

3.4. **Characterization of divergence-free elements.** In this section, we discuss how the divergence-free functions of $\boldsymbol{V}_h$ can be explicitly characterized, and show the relation of this space with the C^1 singular Zienkiewicz finite element space [10]

$$Z_h = \big\{z \in H^2_0(\Omega) : \ z\big|_T \in Z(T)\big\}, \tag{3.29}$$

where

$$Z(T) = \mathcal{P}_3(T)\backslash \mathrm{span}\{b_T\} \oplus \mathrm{span}\big\{B_{e_i}\big\}_{i=1}^3. \tag{3.30}$$

The space Z_h consists of (reduced) Hermite polynomials enriched with rational bubble functions to enforce C^1 continuity across the interior edges of the mesh. The local space $Z(T)$ has dimension 12 whose degrees of freedom are

$$z(x_i), \nabla z(x_i) \qquad \text{for all vertices } x_i, \tag{3.31a}$$

$$\big\langle \partial z/\partial \boldsymbol{n}_i, \kappa\big\rangle_{e_i} \qquad \text{for all } \kappa \in \mathcal{P}_0(e_i)\ (i=1,2,3). \tag{3.31b}$$

We now show that the divergence-free functions in $\boldsymbol{V}_h$ can be written as the curl of functions in Z_h. Furthermore, we establish the commutative property

$$\mathbf{curl}\, I_h z = \boldsymbol{\Pi}_h \mathbf{curl}\, z, \tag{3.32}$$

where $I_h : H^2(\Omega) \to Z_h$ denotes the projection onto Z_h corresponding to the degrees of freedom (3.31); that is,

$$I_h z(x_i) = z(x_i), \quad \nabla I_h z(x_i) = \boldsymbol{\Pi}_S \nabla z(x_i), \quad \int_{e_i} \frac{\partial I_h z}{\partial \boldsymbol{n}_i}\, ds = \int_{e_i} \frac{\partial z}{\partial \boldsymbol{n}_i}\, ds \quad \forall z \in H^2(\Omega).$$

From the commuting property (3.32), we can then easily establish that the following de Rham complex is an exact sequence (i.e., the range of each map is the kernel of the following one):

$$\begin{array}{ccccccccc}
\mathbb{R} & \xrightarrow{\subset} & H^2(\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}^1(\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & 0 \\
 & & \big\downarrow I_h & & \big\downarrow \boldsymbol{\Pi}_h & & \big\downarrow P_h & & \\
\mathbb{R} & \xrightarrow{\subset} & Z_h & \xrightarrow{\mathbf{curl}} & \boldsymbol{V}_h & \xrightarrow{\mathrm{div}} & W_h & \longrightarrow & 0
\end{array} \tag{3.33}$$

We note that the sequence in the first row of (3.33) is exact provided the domain Ω is simply connected [13].

First, we claim that the curl operator maps Z_h to the space of divergence-free function of $\boldsymbol{V}_h$. Indeed, this follows by writing $\mathcal{P}_3(T)/\mathrm{span}\{b_T\} = \mathcal{P}_2(T) \oplus \mathrm{span}\{b_{e_i}\lambda_{i+1}\}_{i=1}^3$. Therefore, we have

$$\mathbf{curl}\, Z(T) = \mathbf{curl}\, \mathcal{P}_2(T) \oplus \mathrm{span}\{\mathbf{curl}\,(b_{e_i}\lambda_{i+1})\}_{i=1}^3 \oplus \mathrm{span}\{\mathbf{curl}\,(B_{e_i})\}_{i=1}^3 \subset \boldsymbol{V}(T).$$

Since $\mathbf{curl}\, Z_h \subset \boldsymbol{H}^1_0(\Omega)$, the claim is proved.

We also note that $\mathbf{curl}\,(I_h z)(x_i) = \boldsymbol{\Pi}_S \mathbf{curl}\,(z)(x_i) = \boldsymbol{\Pi}_h \mathbf{curl}\,(z)(x_i)$. Moreover, we have

$$\int_{e_i} \mathbf{curl}\,(I_h z)\cdot \boldsymbol{t}_i\, ds = \int_{e_i} \frac{\partial (I_h z)}{\partial \boldsymbol{n}_i}\, ds = \int_{e_i} \frac{\partial z}{\partial \boldsymbol{n}_i}\, ds = \int_{e_i} (\boldsymbol{\Pi}_h \mathbf{curl}\, z)\cdot \boldsymbol{t}_i\, ds,$$

and

$$\int_{e_i} \mathbf{curl}\,(I_h z)\cdot \boldsymbol{n}_i\, ds = \int_{e_i} \frac{\partial (I_h z)}{\partial \boldsymbol{t}_i}\, ds = \int_{e_i} \frac{\partial z}{\partial \boldsymbol{t}_i}\, ds = \int_{e_i} (\boldsymbol{\Pi}_h \mathbf{curl}\, z)\cdot \boldsymbol{n}_i\, ds.$$

Since $\mathbf{curl}\,(I_h z) \in \boldsymbol{V}_h$, it follows that the commutative property (3.32) holds.

Now suppose that $\boldsymbol{v} \in \boldsymbol{V}_h$ with $\operatorname{div} \boldsymbol{v} = 0$. It then follows from the first row of (3.33) that there exists $z \in H^2(\Omega)$ such that $\mathbf{curl}\, z = \boldsymbol{v}$. Then by (3.32) and the idempotency of $\boldsymbol{\Pi}_h$ there holds $\boldsymbol{v} = \boldsymbol{\Pi}_h \boldsymbol{v} = \boldsymbol{\Pi}_h(\mathbf{curl}\, z) = \mathbf{curl}\,(I_h z)$. It then follows that the diagram (3.33) is exact.

Remark 3.5. From the discussion above, we can deduce that the divergence-free functions in $\boldsymbol{M}_h$ (defined by (3.15)) can be written as the curl of reduced cubic Hermite functions, and the analogous exact de Rham complex holds:

$$\begin{array}{ccccccccc}
\mathbb{R} & \xrightarrow{\subset} & H(\mathrm{curl}\,;\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div}\,;\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & 0 \\
 & & \Big\downarrow \tilde{I}_h & & \Big\downarrow \boldsymbol{\Pi}_M & & \Big\downarrow P_h & & \\
\mathbb{R} & \xrightarrow{\subset} & \tilde{Z}_h & \xrightarrow{\mathbf{curl}} & \boldsymbol{M}_h & \xrightarrow{\mathrm{div}} & W_h & \longrightarrow & 0,
\end{array} \tag{3.34}$$

where $\tilde{Z}_h$ denotes the reduced cubic Hermite finite element space and $\tilde{I}_h$ the corresponding projection.

4. Higher Order Elements

The elements discussed above can be generalized to form a hierarchy of conforming finite elements of arbitrary order. For an integer $k \geq 1$, we set

$$\boldsymbol{V}(T) = \boldsymbol{M}_{k+1}(T) + \boldsymbol{Q}_{k+1}(T), \tag{4.1}$$

with

$$\boldsymbol{M}_{k+1}(T) = \boldsymbol{\mathcal{P}}_k(T) + \operatorname{span}\big\{\mathbf{curl}\,(b_{e_i}\lambda_{i+1}^k)\big\}, \tag{4.2}$$

$$\boldsymbol{Q}_{k+1}(T) = \sum_{i=1}^{3} \mathbf{curl}\,(B_{e_i} Q_{k-1}^{(i)}(T)), \tag{4.3}$$

and

$$Q_{k-1}^{(i)}(T) = \big\{q \in \mathcal{P}_{k-1}(T) :\ (q, B_{e_i} p)_T = 0 \ \ \forall p \in \mathcal{P}_{k-2}(T)\big\}. \tag{4.4}$$

In the case $k = 1$, we set $Q_{k-1}^{(i)}(T) = \mathcal{P}_0(T)$ so that we recover the local space discussed in Section 3. The degrees of freedom that uniquely determine a function in $\boldsymbol{V}(T)$ are then given as

$$v(x_i) \qquad \text{for all vertices } x_i, \tag{4.5a}$$

$$\big\langle \boldsymbol{v}, \boldsymbol{\kappa} \big\rangle_{e_i} \qquad \text{for all } \boldsymbol{\kappa} \in \boldsymbol{\mathcal{P}}_{k-1}(e_i)\ (i = 1, 2, 3), \tag{4.5b}$$

$$(\boldsymbol{v}, \boldsymbol{\rho})_T \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{N}_{k-1}(T), \tag{4.5c}$$

where

$$\boldsymbol{N}_{k-1}(T) = \boldsymbol{\mathcal{P}}_{k-2}(T) + \big\{\boldsymbol{w} \in \boldsymbol{\mathcal{P}}_{k-1}(T) :\ \boldsymbol{w} \cdot \boldsymbol{x} = 0\big\}$$

denotes the Nedelec space of index $k - 1$ [19].

We now prove the higher order analogue of Lemma 3.1.

Lemma 4.1. *There holds*

$$\boldsymbol{V}(T) = \boldsymbol{M}_{k+1}(T) \oplus \boldsymbol{Q}_{k+1}(T), \tag{4.6}$$

$$\dim \boldsymbol{V}(T) = \dim \boldsymbol{\mathcal{P}}_k(T) + 3(k + 1). \tag{4.7}$$

Moreover, the degrees of freedom (4.5) are unisolvent on $\boldsymbol{V}(T)$, and $\boldsymbol{V}(T)$ restricted to ∂T is a subspace of $\boldsymbol{\mathcal{P}}_{k+1}(\partial T)$.

Proof. First we show that $\boldsymbol{M}_{k+1} = \boldsymbol{\mathcal{P}}_k(T) \oplus \mathrm{span}\{b_{e_i}\lambda_{i+1}^k\}$. Suppose that $\boldsymbol{v} = \sum_{i=1}^3 \mathrm{d}_i b_{e_i}\lambda_{i+1}^k \in \boldsymbol{\mathcal{P}}_k(T)$ with $\mathrm{d}_i \in \mathbb{R}$. Then $\boldsymbol{v}\cdot\boldsymbol{n}_i\big|_{e_i} = \mathrm{d}_i \dfrac{\partial(b_{e_i}\lambda_{i+1}^k)}{\partial \boldsymbol{t}_i}\Big|_{e_i} \in \mathcal{P}_k(e_i)$ It then follows that

$$
\begin{aligned}
0 = \mathrm{d}_i \frac{\partial^{k+2}(b_{e_i}\lambda^{k+1})}{\partial \boldsymbol{t}_i^{k+2}}\Big|_{e_i} &= \frac{\mathrm{d}_i}{2}(k+1)(k+2)\frac{\partial^2 b_{e_i}}{\partial \boldsymbol{t}_i^2}\frac{\partial^k(\lambda_{i+1}^k)}{\partial \boldsymbol{t}_i^k}\Big|_{e_i} \\
&= \frac{\mathrm{d}_i}{2}k!(k+1)(k+2)\frac{\partial^2 b_{e_i}}{\partial \boldsymbol{t}_i^2}\Big(\frac{\partial \lambda_{i+1}}{\partial \boldsymbol{t}_i}\Big)^k\Big|_{e_i}
\end{aligned}
$$

Since $\frac{\partial^2 b_{e_i}}{\partial \boldsymbol{t}_i^2}\Big(\frac{\partial\lambda_{i+1}}{\partial \boldsymbol{t}_i}\Big)^k$ is a nonzero constant, it follows that $\mathrm{d}_i = 0$. It then follows that the direct sum (4.6) holds. Furthermore, it is clear that $\dim \boldsymbol{Q}_{k+1}(T) = 3k$ and $\dim \boldsymbol{S}_{k+1}(T) = 3$, and therefore the dimension count (4.7) follows from (4.6).

Now suppose that $\boldsymbol{v} \in \boldsymbol{V}(T)$ vanishes at all the degrees of freedom (4.5). Then to show unisolvency, it suffices to show that $\boldsymbol{v} \equiv 0$ since the number of degrees of freedom equals the dimension of $\boldsymbol{V}(T)$. Write $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{q}$ with $\boldsymbol{v}_0 \in \boldsymbol{M}_{k+1}(T)$ and $\boldsymbol{q} \in \boldsymbol{Q}_{k+1}(T)$. Noting that $\boldsymbol{q}\cdot\boldsymbol{n}\big|_{\partial T} = 0$, we see that $\boldsymbol{v}_0$ vanishes at the vertices of T and its normal components vanish on ∂T up to moments of degree $k-1$. Since $\boldsymbol{v}_0 \in \boldsymbol{\mathcal{P}}_{k+1}(T)$, we have $\boldsymbol{v}_0\cdot\boldsymbol{n}\big|_{\partial T} = 0$. By using the same arguments as above, we deduce $\boldsymbol{v}_0 \in \boldsymbol{\mathcal{P}}_k(T)$.

Next, we write $\boldsymbol{q} = \sum_{i=1}^3 \mathbf{curl}\,(B_{e_i}q_i)$ with $q_i \in Q_{k-1}^{(i)}(T)$. By (4.5c) and (4.4), we have

$$
\begin{aligned}
0 = (\boldsymbol{v},\boldsymbol{\rho})_T &= (\boldsymbol{v}_0,\boldsymbol{\rho})_T + \sum_{i=1}^3(\mathbf{curl}\,(B_{e_i}q_i),\boldsymbol{\rho})_T \\
&= (\boldsymbol{v}_0,\boldsymbol{\rho})_T - \sum_{i=1}^3(q_i, B_{e_i}\mathrm{curl}\,(\boldsymbol{\rho}))_T = (\boldsymbol{v}_0,\boldsymbol{\rho})_T \quad \forall\boldsymbol{\rho}\in\boldsymbol{N}_{k-1}(T).
\end{aligned}
$$

Here we have used the inclusion $\mathrm{curl}\,\boldsymbol{N}_{k-1}(T) \subset \boldsymbol{\mathcal{P}}_{k-2}(T)$. Since $\boldsymbol{v}_0\cdot\boldsymbol{n}$ vanishes on ∂T, it follows that $\boldsymbol{v}_0 \equiv 0$ [20]. Finally by (4.5c) and Lemma 2.1, we have

$$
0 = \big\langle \boldsymbol{v}\cdot\boldsymbol{t}_i, q_i\big\rangle_{e_i} = \big\langle \partial(B_i q_i)/\partial\boldsymbol{n}_i, q_i\big\rangle_{e_i} = \mathrm{a}_i\big\langle b_{e_i}q_i, q_i\big\rangle_{e_i}.
$$

Therefore $q_i = 0$ $(i=1,2,3)$ on e_i and hence we may write $q_i = \lambda_i p_i$ for some $p_i \in \mathcal{P}_{k-2}(T)$. But then by (4.4) we have $0 = (q_i, B_{e_i}p_i)_T = (p_i, B_{e_i}\lambda_i p_i)_T$. It then follows that $q_i \equiv 0$ and therefore $\boldsymbol{v} \equiv 0$. This completes the proof. $\square$

In the general case, the global spaces are defined as

$$
\begin{aligned}
\boldsymbol{V}_h &= \big\{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega):\ \boldsymbol{v}\big|_T \in \boldsymbol{V}(T)\big\}, \\
W_h &= \big\{q \in L_0^2(\Omega):\ q\big|_T \in \mathcal{P}_{k-1}(T)\big\}.
\end{aligned}
$$

It is easy to see that the corresponding projections $\boldsymbol{\Pi}_h$ and P_h satisfy the commutative property (3.14). Moreover, the following estimates can be shown by following the derivation of Lemma 3.3

$$
\|\boldsymbol{v} - \boldsymbol{\Pi}_h\boldsymbol{v}\|_{H^m(T)} \le Ch_T^{s-m}\|\boldsymbol{v}\|_{H^s(\omega(T))} \qquad 0\le m\le 2,\ m\le s\le k+1.
$$

Finally, we mention that divergence free functions in $\boldsymbol{V}_h$ can be written as the curl of functions belonging to a generalized Zienkiewicz finite element space. Indeed define

$$
Z(T) = \boldsymbol{\mathcal{P}}_{k+1}(T)\oplus\mathrm{span}\big\{b_{e_i}\lambda_{i+1}^k\big\}_{i=1}^3 + \mathrm{span}\big\{B_{e_i}Q_{k-1}^{(i)}(T)\big\}_{i=1}^3,
$$

and let $Z_h = \{z \in H_0^2(\Omega) :\ z\big|_T \in Z(T)\}$ be the corresponding global space. The degrees of freedom that uniquely determine functions in the local space of Z_h are

$$\begin{aligned}
&z(x_i),\ \nabla z(x_i) && \text{for all vertices } x_i,\\
&\langle z, \kappa\rangle_{e_i} && \text{for all } \kappa \in \mathcal{P}_{k-2}(e_i),\\
&(z, \rho)_T && \text{for all } \rho \in \mathcal{P}_{k-2}(e_i),\\
&\langle \partial z/\partial \boldsymbol{n}_i, \omega\rangle_{e_i} && \text{for all } \omega \in \mathcal{P}_{k-1}(e_i).
\end{aligned}$$

Following the arguments in the proof of Lemma 4.1, it is straightforward to show that these degrees of freedom are insolvent on $Z(T)$. Similar to the lowest order Zienkiewicz finite elements, the space $Z(T)$ consists of reduced Hermite-type elements plus $3k$ rational basis functions. We are not aware of any higher order generalization of the Zienkiewicz elements nor the reduced Hermite elements in the literature, although their practical value may be questionable.

5. Reduced Elements

In this section, we discuss how to construct reduced elements with smaller dimension. One plausible approach is to impose the condition that the tangental component of functions in $\boldsymbol{V}(T)$ (defined by (4.1)) are a subset of $\mathcal{P}_k(\partial T)$ when restricted to the boundary of T. The resulting local space has dimension that is exactly three less than $\boldsymbol{V}(T)$, i.e., the dimension is $\dim \boldsymbol{\mathcal{P}}_k(T) + 3k$. The degrees of freedom of this reduced space would then by the same as (4.5) except that the degrees of freedom (4.5b) is replaced by the $(k-1)$th moments of the normal component of $\boldsymbol{v}$ and the $(k-2)$th moments of the tangental component.

Here, we construct an alternative reduced space that has a smaller dimension than the one discussed above when $k \geq 2$. To describe the local space of these reduced elements we first need the following result.

Lemma 5.1. *Define*

$$\boldsymbol{s}_i := \mathbf{curl}\left(b_{e_i}\lambda_{i+1}^k + \mathrm{c}_i\lambda_{i+1}^{k-1}B_{e_i} + \lambda_{i+1}^{k-1}B_{e_{i+2}}\right), \tag{5.1}$$

where

$$\mathrm{c}_i := \left(\nabla\lambda_{i+2} - (k+1)\nabla\lambda_{i+1}\right)\cdot\nabla\lambda_i/\mathrm{a}_i^2. \tag{5.2}$$

Then $\boldsymbol{s}_i$ enjoys the following properties:

$$\operatorname{div}\boldsymbol{s}_i = 0, \qquad \boldsymbol{s}_i\cdot\boldsymbol{t}\big|_{\partial T} \in \mathcal{P}_k(\partial T), \tag{5.3a}$$

$$\boldsymbol{s}_i\cdot\boldsymbol{n}_j\big|_{e_j} = 0\ (i \neq j), \qquad \boldsymbol{s}_i\cdot\boldsymbol{n}_i\big|_{e_i} \in \mathcal{P}_{k+1}(e_i)\backslash\mathcal{P}_k(e_i). \tag{5.3b}$$

Proof. The identity $\operatorname{div}\boldsymbol{s}_i = 0$ is clear from the definition of $\boldsymbol{s}_i$. To show that $\boldsymbol{s}\cdot\boldsymbol{t}\big|_{\partial T} \in \mathcal{P}_k(\partial T)$, we employ Lemma (2.1) and (5.1) to obtain for any $e_j \subset \partial T$,

$$\begin{aligned}
\boldsymbol{s}_i\cdot\boldsymbol{t}_j\big|_{e_j} &= \frac{\partial(b_{e_i}\lambda_{i+1}^k)}{\partial\boldsymbol{n}_j} + \delta_{i,j}\mathrm{c}_i\mathrm{a}_i\lambda_{i+1}^{k-1}b_{e_i} - \delta_{i+2,j}\mathrm{a}_{i+2}\lambda_{i+2}^{k-1}b_{e_{i+2}}\\
&= \delta_{i,j}(k+1)\lambda_{i+1}^{k-1}b_{e_i}\frac{\partial\lambda_{i+1}}{\partial\boldsymbol{n}_j} + \lambda_{i+1}^{k+1}\frac{\partial\lambda_{i+2}}{\partial\boldsymbol{n}_j} + \delta_{i,j}\mathrm{c}_i\mathrm{a}_i\lambda_{i+1}^{k-1}b_{e_i} - \delta_{i+2,j}\mathrm{a}_{i+2}\lambda_{i+1}^{k-1}b_{e_{i+2}}\\
&= \delta_{i,j}\left[(k+1)\frac{\partial\lambda_{i+1}}{\partial\boldsymbol{n}_j} + \mathrm{c}_i\mathrm{a}_i\right]\lambda_{i+1}^{k-1}b_{e_i} - \delta_{i+2,j}\mathrm{a}_{i+2}\lambda_{i+1}^{k-1}b_{e_{i+2}} + \lambda_{i+1}^{k+1}\frac{\partial\lambda_{i+2}}{\partial\boldsymbol{n}_j}.
\end{aligned}$$

We note that if $j = i+1$, then $\boldsymbol{s}_i \cdot \boldsymbol{t}_j\big|_{e_j} = 0$. On the other hand, if $j = i$, then by (5.2), and since $\boldsymbol{n}_j = \nabla\lambda_j/\mathrm{a}_j$, we obtain

$$\begin{aligned}\boldsymbol{s}\cdot\boldsymbol{t}_j\Big|_{e_j} &= \Big((k+1)\frac{\partial\lambda_{i+1}}{\partial\boldsymbol{n}_j} + \mathrm{c}_i\mathrm{a}_i\Big)\lambda_{i+1}^{k-1}b_{e_i} + \lambda_{i+1}^{k+1}\frac{\partial\lambda_{i+2}}{\partial\boldsymbol{n}_j}\\ &= \Big((k+1)\frac{\partial\lambda_{i+1}}{\partial\boldsymbol{n}_i} + \mathrm{c}_i\mathrm{a}_i\Big)\lambda_{i+1}^{k}(1-\lambda_{i+1}) + \lambda_{i+1}^{k+1}\frac{\partial\lambda_{i+2}}{\partial\boldsymbol{n}_i}\\ &= \Big[-\Big((k+1)\frac{\partial\lambda_{i+1}}{\partial\boldsymbol{n}_i} + \mathrm{c}_i\mathrm{a}_i\Big) + \frac{\partial\lambda_{i+2}}{\partial\boldsymbol{n}_i}\Big]\lambda_{i+1}^{k+1} + \Big((k+1)\frac{\partial\lambda_{i+1}}{\partial\boldsymbol{n}_i} + \mathrm{c}_i\mathrm{a}_i\Big)\lambda_{i+1}^{k}\\ &= \Big((k+1)\frac{\partial\lambda_{i+1}}{\partial\boldsymbol{n}_i} + \mathrm{c}_i\mathrm{a}_i\Big)\lambda_{i+1}^{k} \in \mathcal{P}_k(e_i).\end{aligned}$$

When $j = i+2$ we have

$$\begin{aligned}\boldsymbol{s}\cdot\boldsymbol{t}_j\Big|_{e_j} &= \mathrm{a}_{i+2}\lambda_{i+1}^{k-1}b_{e_{i+2}} + \lambda_{i+1}^{k+1}\frac{\partial\lambda_{i+2}}{\partial\boldsymbol{n}_{i+2}}\\ &= \mathrm{a}_{i+2}\lambda_{i+1}^{k}(1-\lambda_{i+1}) + \mathrm{a}_{i+2}\lambda_{i+1}^{k+1} = \mathrm{a}_{i+2}\lambda_{i+1}^{k} \in \mathcal{P}_k(e_{i+2}).\end{aligned}$$

Finally, since the rational bubbles vanish on ∂T we have $\boldsymbol{s}_i\cdot\boldsymbol{n}\big|_{\partial T} = \frac{\partial(b_{e_i}\lambda_{i+1}^k)}{\partial\boldsymbol{t}}\big|_{\partial T}$. Since b_{e_i} vanishes on $\partial T\backslash e_i$, there holds $\boldsymbol{s}_i\cdot\boldsymbol{n}_j\big|_{e_j} = 0$ for $i = j$. On the other hand, on edge e_i, we have $\boldsymbol{s}_i\cdot\boldsymbol{n}_i\big|_{e_i} = \frac{\partial(\lambda_{i+1}^{k+1}\lambda_{i+2})}{\partial\boldsymbol{t}}\big|_{e_i} \in \mathcal{P}_{k+1}(e_i)\backslash\mathcal{P}_k(e_i)$. □

We define the local space of the reduced elements as follows:

$$\boldsymbol{V}_R(T) = \boldsymbol{M}_R(T) + \boldsymbol{Q}_R(T), \tag{5.4}$$

where

$$\boldsymbol{M}_R(T) = \boldsymbol{\mathcal{P}}_k(T) + \mathrm{span}\{\boldsymbol{s}_i\}_{i=1}^3, \tag{5.5}$$

and

$$\boldsymbol{Q}_R(T) = \begin{cases}\emptyset & \text{if } k=1,\\ \mathrm{span}\{\mathbf{curl}\,(B_{e_i})\}_{i=1}^2 & \text{if } k=2,\\ \mathrm{span}\{\mathbf{curl}\,(\lambda_{i+1}B_{e_i})\}_{i=1}^3 & \text{if } k\ge 3.\end{cases} \tag{5.6}$$

In (5.5), the functions $\boldsymbol{s}_i$ are defined in Lemma 5.1. It is easy to see that the summations in (5.4) are direct and

$$\dim\boldsymbol{V}_R(T) = \begin{cases}\dim\boldsymbol{\mathcal{P}}_k(T)+3 & \text{if } k=1,\\ \dim\boldsymbol{\mathcal{P}}_k(T)+5 & \text{if } k=2,\\ \dim\boldsymbol{\mathcal{P}}_k(T)+6 & \text{if } k\ge 3,\end{cases}$$

The degrees of freedom of $\boldsymbol{V}_R(T)$ are then

$$\boldsymbol{v}(x_i) \quad \text{for all vertices } x_i, \tag{5.7a}$$

$$\big\langle\boldsymbol{v}\cdot\boldsymbol{n}_i,\kappa\big\rangle_{e_i} \quad \text{for all } \kappa\in\mathcal{P}_{k-1}(e_i)\ (i=1,2,3), \tag{5.7b}$$

$$\big\langle\boldsymbol{v}\cdot\boldsymbol{t}_i,\omega\big\rangle_{e_i} \quad \text{for all } \omega\in\mathcal{P}_{k-2}(e_i)\ (i=1,2,3), \tag{5.7c}$$

$$(\boldsymbol{v},\nabla q)_T \quad \text{for all } q\in\mathcal{P}_{k-1}(T), \tag{5.7d}$$

$$(\boldsymbol{v},\mathbf{curl}\,(b_T^2 m))_T \quad \text{for all } m\in\mathcal{P}_{k-5}(T). \tag{5.7e}$$

Here we have used the convention that if $k \le 4$, then the degrees of freedom (5.7e) are omitted, and if $k=1$ then the degrees of freedom (5.7c) are omitted.

Lemma 5.2. *The degrees of freedom* (5.7) *are unisolvent on* $\boldsymbol{V}_R(T)$.

Proof. We prove the (harder) case $k \ge 3$ as the other cases can be handled similarly. We proceed by showing that if $\boldsymbol{v} \in \boldsymbol{V}_R(T)$ vanish at the degrees of freedom (5.7) then $\boldsymbol{v} \equiv 0$. Unisolvency then follows since the number of degrees of freedom in (5.7) and the dimension of $\boldsymbol{V}_R(T)$ match.

Write

$$\begin{aligned} &\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{q} && \boldsymbol{v}_0 = \bar{\boldsymbol{v}} + \boldsymbol{s}, && \bar{v} \in \boldsymbol{\mathcal{P}}_k(T), \\ &\boldsymbol{s} = \sum_{i=1}^{3} \mathrm{d}_i \boldsymbol{s}_i, && \boldsymbol{q} = \sum_{i=1}^{3} \mathrm{g}_i \boldsymbol{q}_i \in \boldsymbol{Q}_R(T), && \boldsymbol{q}_i = \mathbf{curl}\,(\lambda_{i+1} B_{e_i}), \end{aligned}$$

and $\mathrm{d}_i, \mathrm{g}_i \in \mathbb{R}$. By Lemmas 5.1 and 2.1 there holds $\boldsymbol{v}\cdot\boldsymbol{n}\big|_{\partial T} \in \mathcal{P}_{k+1}(\partial T)$. Therefore by (5.7a)–(5.7b), we have $\boldsymbol{v}\cdot\boldsymbol{n}\big|_{\partial T} = 0$. Hence, by (5.1) and Lemma 2.1 we obtain

$$0 = \boldsymbol{v}\cdot\boldsymbol{n}\big|_{e_j} = \boldsymbol{v}_0\cdot\boldsymbol{n}\big|_{e_j} + \mathrm{d}_j \frac{\partial(b_{e_j}\lambda_{j+1}^k)}{\partial \boldsymbol{t}_j}\Big|_{e_j}.$$

It then follows that $\mathrm{d}_j \partial(b_{e_j}\lambda_{j+1}^k)/\partial \boldsymbol{t}_j\big|_{e_j} \in \mathcal{P}_k(e_j)$, and therefore we conclude $\mathrm{d}_j = 0$ by using the same arguments found in the proof of Lemma 4.1. Thus, $\boldsymbol{v} = \bar{\boldsymbol{v}} + \boldsymbol{q}$.

Next by (5.7d), we have

$$0 = (\boldsymbol{v}, \nabla q)_T = (\bar{\boldsymbol{v}}, \nabla q)_T + \sum_{i=1}^{3} \mathrm{g}_i(\mathbf{curl}\,(\lambda_{i+1}B_{e_i}), \nabla q)_T = (\operatorname{div} \bar{\boldsymbol{v}}, q) \qquad \forall q \in \mathcal{P}_{k-1}(T)$$

since $\bar{\boldsymbol{v}}\cdot\boldsymbol{n}\big|_{\partial T} = 0$. It then follows that $\operatorname{div}\bar{\boldsymbol{v}} = 0$ and therefore $\bar{\boldsymbol{v}} = \mathbf{curl}\,(b_T r)$ for some $r \in \mathcal{P}_{k-2}(T)$. By (5.7c), Lemma 2.1 and (2.2), we have

$$\begin{aligned} 0 &= \big\langle \boldsymbol{v}\cdot\boldsymbol{t}_j, \omega\big\rangle_{e_j} = \big\langle \bar{\boldsymbol{v}}\cdot\boldsymbol{t}_j, \omega\big\rangle_{e_j} + \mathrm{a}_j\mathrm{g}_j\big\langle \lambda_{j+1}b_{e_j}, \omega\big\rangle_{e_j} \\ &= \big\langle \mathbf{curl}\,(b_T r)\cdot\boldsymbol{t}_j, \omega\big\rangle_{e_j} + \mathrm{a}_j\mathrm{g}_j\big\langle \lambda_{j+1}b_{e_j}, \omega\big\rangle_{e_j} \\ &= \mathrm{a}_j\big\langle (r + \mathrm{g}_j\lambda_{j+1})b_{e_j}, \omega\big\rangle_{e_j} \qquad \forall \omega \in \mathcal{P}_{k-2}(e_j). \end{aligned}$$

It then follows that $r + \mathrm{g}_j\lambda_{j+1}\big|_{e_j} = 0$ and therefore we may write $r = p_j\lambda_j - \mathrm{g}_j\lambda_{j+1}$ for some $p_j \in \mathcal{P}_{k-3}(T)$. Similarly, we have $r = p_{j+1}\lambda_{j+1} - \mathrm{g}_{j+1}\lambda_{j+2}$ for some $p_{j+1} \in \mathcal{P}_{k-3}(T)$. Then on edge e_{j+1} we have

$$p_j\lambda_j\big|_{e_{j+1}} = r\big|_{e_{j+1}} = -\mathrm{g}_{j+1}\lambda_{j+2}\big|_{e_{j+1}}.$$

From this identity, we conclude that $\mathrm{g}_{j+1} = 0$ and therefore $\boldsymbol{q} \equiv 0$ and r vanishes on ∂T. We can then write $\bar{\boldsymbol{v}} = \mathbf{curl}\,(b_T^2 m)$ for some $m \in \mathcal{P}_{k-5}(T)$, and hence the degree of freedom (5.7e) implies $\bar{\boldsymbol{v}} \equiv 0$. ∎

6. Conclusion

In this paper, we have developed a family of Stokes finite elements that produce conforming exactly divergence free approximations. We have exploited the corresponding smoothed de-Rham complex to make connections with H^2-conforming elements. We note that using complexes of function spaces have helped to develop conforming and symmetric elements for linear elasticity [3]. Our reduced elements seem to be computationally competitive. For example, the lowest order element has the same degrees of freedom as the Bernardi–Raugel element. We plan to test the computational advantage of these methods in the near future as well as develop the analogous three dimensional elements on general tetrahedral meshes.

References

1. D.N. Arnold, F. Brezzi, and M. Fortin, *A stable finite element for the Stokes equations*, Calcolo, 21:337–344, 1984.
2. D.N. Arnold and J. Qin, *Quadratic velocity/linear pressure Stokes elements*, in Advances in Computer Methods for Partial Differential Equations VII, ed. R. Vichnevetsky and R.S. Steplemen, 1992.
3. D.N. Arnold and R. Winther, *Mixed finite elements for elasticity*, Numer. Math., 92:401–419, 2002.
4. F. Auricchio, L. Beirão da Veiga, C. Lovadina and A. Reali *The importance of the exact satisfaction of the incompressibility constraint in nonlinear elasticity: mixed FEMs versus NURBS-based approximations*, Comp. Meth. Appl. Mech. and Eng., 199: 314–323, 2010.
5. C. Bernardi and G. Raugel, *Analysis of some finite elements for the Stokes problem*, Math. Comp., 44(169):71–79, 1985.
6. D. Boffi, F. Brezzi and M. Fortin, *Finite elements for the Stokes problem*, in *Mixed Finite Elements, Compatibility Conditions, and Applications*, Lectures given at the C.I.M.E. Summer School, Springer-Verlag, Berlin, 2008.
7. S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods* (third edition), Springer, 2008.
8. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Series in Computational Mathematics,15, Springer-Verlag, New York, 1991.
9. A. Buffa, C. de Falco, and G. Sangalli, *IsoGeometric Analysis: stable elements for the 2D Stokes equation*, Internat. J. Numer. Methods Fluids, 65(11):1407–1422, 2011.
10. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
11. M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations.I*, R.A.I.R.O Anal. Numer., 7: 33-75, 1973.
12. M. Fortin and M. Soulie, *A nonconforming piecewise quadratic finite element on triangles*. Int. J. Numer. Meth. Eng. 19: 505-520, 1983.
13. V. Girault and P.A. Raviart, *Finit eElement Methods for Navier-Stokes Equations, Theory and Algorithms*, Springer, Berlin Heidelberg New York, 1986.
14. J. Guzmán and M. Neilan, *A family of non-conforming elements for the Brinkman problem*, to appear in IMA J. Numer. Anal.
15. P. Hood and C. Taylor, *Numerical solution of the NavierStokes equations using the finite element technique*, Comput. Fluids, 1:1–28, 1973.
16. Y. Huang and S. Zhang, *A lowest order divergence-free finite element on rectangular grids*, Front. Math. China, 6(2):253–270, 2011.
17. A. Linke, *Collision in a cross-shaped domain A steady 2d NavierStokes example demonstrating the importance of mass conservation in CFD*, Comp. Meth. Appl. Mech. and Eng., 198: 3278–3286, 2009.
18. K.A. Mardal, X.-C. Tai and R. Winther, *A robust finite element method for Darcy-Stokes flow*, SIAM J. Numer. Anal. 40:1605–1631, 2002.
19. J.C. Nedelec, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 35:315–341, 1980.
20. J.C. Nedelec, *A new family of mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 50:57–81, 1986.
21. L.R. Scott and M. Vogelius, *Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials*, Math. Model. Numer. Anal., 9:11–43, 1985.
22. L.R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54(190):483–493, 1990.
23. X.-C. Tai and R. Winther *A discrete de Rham complex with enhanced smoothness*, Calcolo, 43:287–306.
24. J. Qin and S. Zhang, *Stability and approximability of the* $\mathcal{P}_1$–$\mathcal{P}_0$ *element for Stokes equations*, Int. J. Numer. Meth. Fluids, 54: 497-515, 2007.
25. X. Xie, J. Xu, and G. Xue, *Uniformly-stable finite element methods for Darcy-Stokes-Brinkman models*, J. Comput. Math. 26 (2008), pp. 437455.
26. S. Zhang, *On the P1 Powell-Sabin divergence-free finite element for the Stokes equations* J.Comput. Math. 26(3):456–470, 2008.
27. Z. Shangyou, *A family of* $Q_{k+1,k} \times Q_{k,k+1}$ *divergence-free finite elements on rectangular grids*, SIAM J. Numer. Anal.,47(3):2090–2107, 2009.
28. O.C. Zienkiewicz, *The finite element method in engineering science*, McGraw-Hill, London, 1971.

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