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Symmetric and Conforming Mixed Finite Elements for Plane Elasticity Using Rational Bubble Functions

Johnny Guzmán[1], **Michael Neilan**[2]

[1] Division of Applied Mathematics, Brown University, Providence, RI 02912;
e-mail: johnny_guzman@brown.edu
[2] Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260;
e-mail: neilan@pitt.edu

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Summary We construct stable, conforming and symmetric finite elements for the mixed formulation of the linear elasticity problem in two dimensions. In our approach we add three divergence free rational functions to piecewise polynomials to form the stress finite element space. The relation with the elasticity elements and a class of generalized C^1 Zienkiewicz elements is also discussed.

Key words finite elements, mixed method, elasticity, conforming, symmetric

1 Introduction

In this paper, we construct stable finite element pairs for the system of equations describing plane linear elasticity:

$$\operatorname{div}\sigma = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$A\sigma - \varepsilon(u) = 0 \qquad \text{in } \Omega, \tag{1.1b}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.1c}$$

Here, $\Omega \subset \mathbb{R}^2$ is a simply connected bounded polyhedral domain and $f \in L^2(\Omega;\mathbb{R}^2)$ is the given load. The unknown variables $\sigma \in \Sigma := H(\operatorname{div};\Omega;\mathbb{S})$ and $u \in V := L^2(\Omega;\mathbb{R}^2)$ represent the stress and displacement. The compliance tensor $A = A(x) : \mathbb{S} \to \mathbb{S}$ is assumed to be a bounded, symmetric and positive definite, and the linearized strain tensor is defined as $\varepsilon(u) := \frac{1}{2}\big(\nabla u + (\nabla u)^t\big)$. The pair $(\sigma, u) \in \Sigma \times V$ is to defined to be solutions provided

$$(A\sigma, \mu) + (u, \operatorname{div} \mu) = 0 \qquad \forall \mu \in \Sigma, \tag{1.2a}$$

$$(\operatorname{div} \sigma, w) = (f, w) \qquad \forall w \in V, \tag{1.2b}$$

where $(\cdot, \cdot)$ denotes the L^2 inner product over Ω. A detailed description of the notation is presented in the subsequent section.

Many mixed finite element methods have been developed for plane elasticity, and generally speaking, they can be grouped into two categories: methods that enforce the symmetry of the stress weakly, and methods that enforce the symmetry exactly (strongly). In the former category, the stress tensor is not necessarily symmetric, but rather orthogonal to anti-symmetric tensors up to certain moments. Weakly imposed stress symmetry methods also introduce a new variable into the formulation that approximates the anti-symmetric part of the gradient of u; see for example [18,21, 22,14,16,23,1,5,6,15,2]. On other hand, *exactly symmetric stress methods* have been much more difficult to construct. The first class of inf-sup stable methods were the so-called composite elements [20,4,3]. These elements approximate the displacements using discontinuous piecewise polynomials on an original grid and the stresses on a subgrid. Low order two dimensional elements were given by Johnson and Mercier [20] and generalized to any order by Arnold et al. [4]. Very recently a lower-order three dimensional element was devised by Ainsworth and Rankin [3]. In the past decade exact symmetry methods using polynomials on the same grid for the stresses and displacements have been devised by Arnold and Winther [8] and Arnold et al. [9]. It was also shown in those papers that vertex degrees of freedom are necessary for such methods if polynomials are used. Due to this requirement hybridization of the method cannot be done using standard techniques.

In this paper we construct exact symmetry elements for plane elasticity on general triangulations and without using a macro-element technique. Similar to the previous methods mentioned above, we simply use discontinuous piecewise polynomial approximations to approximate the displacement. For the stress approximation, we augment piecewise polynomials (locally) with *divergence free rational tensors*. In fact, for each triangle we add exactly three such tensors. The necessary inf-sup condition and optimal error estimates easily follow from the existence of a Fortin projection that commutes with the divergence operator. Along the way, we also develop corresponding H^2 elements for the biharmonic problem and show that all of the elements are related via an exact sequence. Finally, the boundary degrees of freedom (DOFs) of our stress elements are only edge based (i.e., no vertex degrees of freedom are needed), and therefore we can use hybrid techniques to obtain a symmetric positive-definite linear system for the Lagrange multipliers.

Our new elements are comparable to the composite elements mentioned above. There they augment standard piecewise polynomial spaces with other piecewise polynomials on a refined mesh. We instead, as mentioned before, augment with rational functions. In fact, the dimension of our finite element spaces are exactly the same as the composite elements given in [4]. We also construct a lower order stress element that has the same dimension as the Johnson-Mercier composite element [20], but the corresponding displacement space has smaller dimension. Both our elements and composite elements avoid vertex degrees of freedom. The reason we can avoid the vertex DOF requirement is that we both add tensors that are discontinuous at the vertices.

We mention that augmenting with rational functions was used by Zienkiewicz to construct conforming H^2-elements [24]. Very recently we used such rational functions to develop conforming, divergence free and inf-sup stable Stokes elements in two dimensions [17].

The rest of the paper is organized as follows. In Section 2 we provide the necessary notation that will be used throughout the paper as well as define the rational edge bubbles that will play a crucial part in the construction of the stress elements. We finish this section by deriving some properties of some divergence free rational functions. In Section 3 we define the local spaces of the stress and displacement, give the degrees of freedom, and provide two proofs of unisolvency. We also argue that lower order elements cannot be constructed. In Section 4 we define the global finite element spaces and show that they are inf-sup stable. In Section 5 we draw connections between the stress elements with a new class of Zienkiewicz-like elements. Section 6 is devoted to the convergence analysis of the mixed finite element method as well as its hybrid form. Finally in Section 7 we propose a lower order element using similar ideas as those found in [8].

2 Preliminaries

Given a set $D \subset \Omega$ and a vector space X, we denote by $L^2(D;X)$ the space of square integrable functions with domain D that take values in X. The Sobolev space $H^m(D;X)$ consists of all $L^2(D;X)$ functions whose distributional derivatives up to order m are in $L^2(D;X)$, and the space $H(\mathrm{div}\,; D;\mathbb{S})$ consists of all $L^2(D;\mathbb{S})$ functions whose divergence lies in $L^2(D;\mathbb{R}^2)$. Here, $\mathbb{S}$ denotes the space of all symmetric 2×2 tensors, and the divergence operator applied to a tensor is applied row-wise. We denote by $(\cdot,\cdot)_D$ the L^2 inner product over the domain D and use the convention $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$. Throughout the paper, the letter C will denote a generic positive constant that is independent of the discretization parameter h.

The curl of a scalar function p is defined as $\mathbf{curl}\, p = (-\partial p/\partial x_2, \partial p/\partial x_1)^t$ and the Airy stress function of p is defined as

$$J p = \begin{pmatrix} \dfrac{\partial^2 p}{\partial x_2^2} & -\dfrac{\partial^2 p}{\partial x_1 \partial x_2} \\ -\dfrac{\partial^2 p}{\partial x_1 \partial x_2} & \dfrac{\partial^2 p}{\partial x_1^2} \end{pmatrix}.$$

The following properties of the Airy stress function are well-known (cf. [8,7]), and they will be used frequently below

$$(Jq)n = \frac{\partial}{\partial s}\mathrm{curl}\, q, \qquad (Jq)n\cdot n = \frac{\partial^2 q}{\partial s^2}, \tag{2.1a}$$

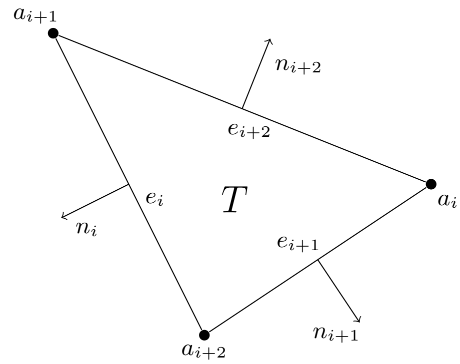

Fig. 1. A pictorial description of the notation.

$$(Jq)n \cdot t = -\frac{\partial^2 q}{\partial s \partial n}, \qquad \operatorname{div}(Jq) = 0. \tag{2.1b}$$

Let $\mathcal{T}_h$ be a shape regular triangulation of Ω with $h = \operatorname{diam}(T)\ \forall T \in \mathcal{T}_h$ and $h := \max_{T \in \mathcal{T}_h} h_T$. Given $T \in \mathcal{T}_h$, we denote by n the unit normal vector of ∂T, and by t the unit tangent vector of ∂T obtained by rotating n 90 degrees counterclockwise. The three vertices of T are denoted by $\{a_i\}_{i=1}^3$ and the three edges of T, $\{e_i\}_{i=1}^3$, are labeled such edge e_i does not contain vertex a_i. We denote by $\{\lambda_i\}_{i=1}^3$ the three barycentric coordinates of T labeled such that $\lambda_i\big|_{e_i} = 0$ and $\lambda_i(a_j) = \delta_{i,j}$. The unit outward normal of an edge e_i is denoted by n_i; that is, $n_i = n|_{e_i}$. We also set $t_i = t|_{e_i}$. We then have the following two well-known identities:

$$n_i = \mathrm{c}_i \nabla \lambda_i, \qquad t_i = -\mathrm{c}_i \mathbf{curl}\, \lambda_i, \tag{2.2}$$

where $\mathrm{c}_i := -|\nabla \lambda_i| < 0$.

Given a simplex S and an integer $m \geq 0$, the space of polynomials of degree m defined on S and with range X are denoted by $\mathbb{P}_m(S;X)$. In the case m is negative we set $\mathbb{P}_m(S;X)$ to be the emptyset.

The triangle and edge bubbles are then defined respectively as

$$b_T = \prod_{j=1}^{3} \lambda_j \in \mathbb{P}_3(T;\mathbb{R}), \quad b_i = \prod_{\substack{j=1 \\ j \neq i}}^{3} \lambda_j \in \mathbb{P}_2(T;\mathbb{R}).$$

By construction, the triangle and edge bubbles satisfy the following properties:

$$b_T\big|_{\partial T} = 0, \qquad \frac{\partial b_T}{\partial n_i}\Big|_{e_i} = \mathrm{c}_i b_i, \qquad b_i\big|_{\partial T \backslash e_i} = 0, \qquad b_i\big|_{e_i} > 0, \tag{2.3}$$

where $\partial b_T / \partial n_i = \nabla b_T \cdot n_i$. We define the *rational edge bubble functions* as $(i = 1,2,3)$

$$\begin{aligned} B_i &= \frac{b_T b_i}{(\lambda_i + \lambda_{i+1})(\lambda_i + \lambda_{i+2})} && \text{for } 0 \leq \lambda_i \leq 1,\ 0 \leq \lambda_{i+1}, \lambda_{i+2} < 1, \\ B_i(a_{i+1}) &= B_i(a_{i+2}) = 0 && \text{otherwise.} \end{aligned}$$

We state a few properties of the rational edge bubbles that were shown in [17] (also see [12]).

Lemma 1 *For any $i = 1,2,3$, there holds*

$$B_i \in C^1(\overline{T};\mathbb{R}) \cap W^{2,\infty}(T;\mathbb{R}), \qquad B_i\big|_{\partial T} = 0, \tag{2.4a}$$

$$\nabla B_i(x_j) = 0 \ (j = 1,2,3), \qquad \nabla B_i\big|_{\partial T\setminus e_i} = 0, \tag{2.4b}$$

$$\frac{\partial B_i}{\partial n_i}\Big|_{e_i} = \mathrm{c}_i b_i, \qquad \nabla B_i\big|_{e_i} = \nabla\lambda_i b_i \in \mathbb{P}_2(e_i;\mathbb{R}^2). \tag{2.4c}$$

The following Lemma is then a simple consequence of the above lemma and (2.1).

Lemma 2 *There holds*

$$(JB_i)n|_{e_j} = 0 \text{ for } j \neq i \quad \text{and} \quad (JB_i)n|_{e_i} \in \mathbb{P}_1(e_i;\mathbb{R}^2).$$

We will also need the following properties of the Airy stress function of the rational bubble functions.

Lemma 3 *Let $q_i = B_i p$ for some $p \in C^2(\overline{T};\mathbb{R})$ and $i \in \{1,2,3\}$. Then there holds*

$$Jq \in L^\infty(T;\mathbb{S}), \qquad (Jq)n\cdot n\big|_{\partial T} = 0, \tag{2.5a}$$

$$\int_{\partial T} \big((Jq)n\cdot t\big) w\, ds = \mathrm{c}_i \int_{\partial T} p b_i \frac{\partial w}{\partial s}\, ds. \tag{2.5b}$$

Proof The inclusion $Jq \in L^\infty(T;\mathbb{S})$ follows from the regularity result $B_i \in W^{2,\infty}(T)$ (cf. Lemma 1) and the definition of the Airy stress function. Next by (2.1) and since B_i vanishes on $\partial\Omega$ we have

$$(Jq)n\cdot n\big|_{\partial T} = \frac{\partial^2(B_i p)}{\partial s^2}\Big|_{\partial T} = 0.$$

Finally by (2.1), Lemma 1 and integration by parts (noting ∇B_i vanishes at the vertices of T), we have

$$\int_{\partial T} \big((Jq)n\cdot t\big) w\, ds = -\int_{\partial T} \frac{\partial^2(B_i p)}{\partial s\partial n} w\, ds = \int_{\partial T} \frac{\partial(B_i p)}{\partial n}\frac{\partial w}{\partial s}\, ds = \mathrm{c}_i \int_{\partial T} p b_i \frac{\partial w}{\partial s}\, ds.$$

Lemma 4 *Let $p_i = B_i\lambda_{i+1}$. Then there holds $(Jp_i)n\big|_{\partial T} \in \mathbb{P}_2(\partial T;\mathbb{R}^2)$ and*

$$\lim_{x\to a_i} (Jp_i)n_{i+1}\cdot n_{i+2}\big|_{e_{i+1}} = \lim_{x\to a_i} (Jp_i)n_{i+2}\cdot n_{i+1}\big|_{e_{i+2}} = 0, \tag{2.6a}$$

$$\lim_{x\to a_{i+2}} (Jp_i)n_i\cdot n_{i+1}\big|_{e_i} = \lim_{x\to a_{i+2}} (Jp_i)n_{i+1}\cdot n_i\big|_{e_{i+1}} = 0, \tag{2.6b}$$

$$\lim_{x\to a_{i+1}} (Jp_i)n_{i+2}\cdot n_i\big|_{e_{i+2}} = 0 \neq \lim_{x\to a_{i+1}} (Jp_i)n_i\cdot n_{i+2}\big|_{e_i}. \tag{2.6c}$$

Proof The inclusion $(Jp_i)n\big|_{\partial T} \in \mathbb{P}_2(\partial T;\mathbb{R}^2)$ follows from Lemma 1 and (2.1).

By Lemma 1 we have

$$\nabla p_i\big|_{\partial T\setminus e_i} = 0, \qquad \nabla p_i\big|_{e_i} = \nabla\lambda_i b_i \lambda_{i+1},$$

and therefore by (2.1) and (2.2),

$$\begin{aligned}
(Jp_i)n_{i+2}\cdot n_{i+1}\big|_{e_{i+2}} &= (Jp_i)n_{i+2}\cdot n_i\big|_{e_{i+2}} = 0,\\
(Jp_i)n_{i+1}\cdot n_{i+2}\big|_{e_{i+1}} &= (Jp_i)n_{i+1}\cdot n_i\big|_{e_{i+1}} = 0,\\
(Jp_i)n_i\cdot n_{i+1}\big|_{e_i} &= -(\mathrm{c}_i\mathrm{c}_{i+2})^{-1}(\mathbf{curl}\,\lambda_i\cdot\nabla\lambda_{i+1})\nabla(b_i\lambda_{i+1})\cdot\mathbf{curl}\,\lambda_i,\\
(Jp_i)n_i\cdot n_{i+2}\big|_{e_i} &= -(\mathrm{c}_i\mathrm{c}_{i+2})^{-1}(\mathbf{curl}\,\lambda_i\cdot\nabla\lambda_{i+2})\nabla(b_i\lambda_{i+1})\cdot\mathbf{curl}\,\lambda_i.
\end{aligned}$$

Clearly, we have

$$\lim_{x\to a_i}(Jp_i)n_{i+1}\cdot n_{i+2}\big|_{e_{i+1}} = \lim_{x\to a_i}(Jp_i)n_{i+2}\cdot n_{i+1}\big|_{e_{i+2}} = 0.$$

We also have

$$\begin{aligned}
&\lim_{x\to a_{i+2}}(Jp_i)n_i\cdot n_{i+1}\big|_{e_i}\\
&\quad= -\lim_{x\to a_{i+2}}(\mathrm{c}_i\mathrm{c}_{i+2})^{-1}(\mathbf{curl}\,\lambda_i\cdot\nabla\lambda_{i+1})(2b_i\nabla\lambda_{i+1}+\lambda_{i+1}^2\nabla\lambda_{i+2})\cdot\mathbf{curl}\,\lambda_i\\
&\quad= 0 = \lim_{x\to a_{i+2}}(Jp_i)n_{i+1}\cdot n_i\big|_{e_{i+1}},
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{x\to a_{i+1}}(Jp_i)n_i\cdot n_{i+2}\big|_{e_i}\\
&\quad= -\lim_{x\to a_{i+1}}(\mathrm{c}_i\mathrm{c}_{i+2})^{-1}(\mathbf{curl}\,\lambda_i\cdot\nabla\lambda_{i+2})(2b_i\nabla\lambda_{i+1}+\lambda_{i+1}^2\nabla\lambda_{i+2})\cdot\mathbf{curl}\,\lambda_i\\
&\quad= -(\mathrm{c}_i\mathrm{c}_{i+2})^{-1}(\mathbf{curl}\,\lambda_i\cdot\nabla\lambda_{i+2})^2.
\end{aligned}$$

We now claim that this last limit does not equal $0 = \lim_{x\to x_{i+1}}(Jp_i)n_{i+2}\cdot n_i\big|_{e_{i+2}}$. Indeed, if these two limits were equal then $(\mathbf{curl}\,\lambda_i\cdot\nabla\lambda_{i+2}) = 0$. Since $\nabla\lambda_{i+2}$ is orthogonal to t_{i+2} we must have that $\mathbf{curl}\,\lambda_i$ is parallel to the edge e_{i+2}. But $\mathbf{curl}\,\lambda_i$ is parallel to edge e_i, a contradiction. Thus $(\mathbf{curl}\,\lambda_i\cdot\nabla\lambda_{i+2}) \neq 0$, and the desired result (2.6c) immediately follows.

We end this section by stating a characterization result of divergence-free symmetric polynomial fields which will be important for unisolvency of our finite elements.

Lemma 5 *If* $\mu\in\mathbb{P}_k(T;\mathbb{S})$, $\mu n\cdot n\big|_{\partial T} = 0$ *and* $\operatorname{div}\mu = 0$, *then* $\mu = J(b_T r)$ *for some* $r\in\mathbb{P}_{k-1}(T;\mathbb{R})$.

Proof We recall that a symmetric matrix field $\mu \in H(\mathrm{div}\,; D; \mathbb{S})$ on a simply connected domain is divergence free if and only if $\mu = Jp$ for some scalar function $p \in H^2(D; \mathbb{R})$ which is unique up to addition of a linear polynomial [8]. Hence, we can assume that p vanishes at the vertices. Moreover, if $\mu \in \mathbb{P}_k(T; \mathbb{S})$ then $p \in \mathbb{P}_{k+2}(T; \mathbb{R})$. Using the identity $(Jp)n \cdot n = \frac{\partial^2 p}{\partial s^2}$ (see (2.1)) with the fact that $\mu n \cdot n\big|_{\partial T} = 0$ we see that p must vanish on ∂T and hence $p = b_T r$ for some $r \in \mathbb{P}_{k-1}(T; \mathbb{R})$.

3 The Local Finite Element Spaces

For an integer $k \geq 2$, we define the local space of the stress as

$$\Sigma(T) = \mathbb{P}_k(T; \mathbb{S}) + JQ(T), \tag{3.1}$$

where

$$Q(T) = \mathrm{span}\{\lambda_{i+1} B_i\}_{i=1}^{3}. \tag{3.2}$$

The local space of displacements consists of vector polynomials of degree $k-1$, namely,

$$V(T) = \mathbb{P}_{k-1}(T; \mathbb{R}^2). \tag{3.3}$$

The degrees of freedom that uniquely determine a function in $\Sigma(T)$ are given by

$$\big\langle \mu n_i, v \big\rangle_{e_i} \qquad \forall v \in \mathbb{P}_k(e_i; \mathbb{R}^2), \tag{3.4a}$$

$$(\mu, \rho)_T \qquad \forall \rho \in \varepsilon\big[\mathbb{P}_{k-1}(T; \mathbb{R}^2)\big] + J\big[b_T^2 \mathbb{P}_{k-4}(T; \mathbb{R})\big]. \tag{3.4b}$$

Since $\dim \varepsilon\big[\mathbb{P}_{k-1}(T; \mathbb{R}^2)\big] = \dim \mathbb{P}_{k-1}(T; \mathbb{R}^2) - 3 = k(k+1) - 3$ and $\dim J\big[b_T^2 \mathbb{P}_{k-4}(T; \mathbb{R})\big] = \dim \mathbb{P}_{k-4}(T; \mathbb{R}\big) = \frac{1}{2}(k-2)(k-3)$, we that there are exactly $6(k+1) + k(k+1) - 3 + \frac{1}{2}(k-2)(k-3) = \frac{3}{2}(k+2)(k+1) + 3$ degrees of freedom listed in (3.4).

From Lemma 2, we clearly see that $\mu n\big|_{\partial T} \in \mathbb{P}_2(\partial T; \mathbb{R}^2)$ for any $\mu \in JQ(T)$. Hence, for any $\mu \in \Sigma(T)$ there holds $\mu n\big|_{\partial T} \in \mathbb{P}_k(\partial T; \mathbb{R}^2)$ as long as $k \geq 2$.

Lemma 6 *The degrees of freedom* (3.4) *are unisolvent on* $\Sigma(T)$.

We provide two proofs of Lemma 6. The first uses similar arguments to those found in [17] and essentially uses the identities $\mathrm{div}\,(JB_i) = 0$ and $JB_i n \cdot n|_{\partial T} = 0$ to decouple the polynomial part and rational part of $\Sigma(T)$. The second proof, which we believe will be useful to derive three dimensional elements, exposes the fact that functions in $JQ(T)$ have a singularity at exactly one vertex (cf. Lemma 4) to prove unisolvency.

Proof (1) The sum in (3.1) is direct and therefore $\dim \Sigma(T) = \dim \mathbb{P}_k(T;\mathbb{S}) + 3 = \frac{3}{2}(k^2+3k+2)+3$ which is exactly the number of degrees of freedom given in (3.4). Thus, to show unisolvency, it suffices to show that if $\mu \in \Sigma(T)$ vanishes at the degrees of freedom (3.4), then μ is identically zero.

To show this, we write

$$\mu = \mu_0 + Jq \quad \text{with} \quad \mu_0 \in \mathbb{P}_k(T;\mathbb{S}) \quad \text{and} \quad q \in Q(T).$$

Since $(Jq)n\cdot n\big|_{\partial T} = 0$ and $\mu_0 \in \mathbb{P}_k(T;\mathbb{S})$, we have $\mu_0 n\cdot n\big|_{\partial T} = 0$ by (3.4a). Next, by (3.4) and since $\operatorname{div} Jq = 0$, we have

$$\int_T \operatorname{div}\mu_0 \cdot \kappa\, dx = \int_T \operatorname{div}\mu\cdot\kappa\, dx = -\int_T \mu : \varepsilon(\kappa)\, dx + \int_{\partial T}(\mu n)\cdot\kappa\, ds = 0$$

for all $\kappa \in \mathbb{P}_{k-1}(T;\mathbb{R}^2)$. It then follows that $\operatorname{div}\mu_0 = 0$ and since $\mu_0 n\cdot n\big|_{\partial T} = 0$, we may write $\mu_0 = J(b_T r)$ for some $r \in \mathbb{P}_{k-1}(T;\mathbb{R})$ (cf. Lemma 5).

Write $q = \sum_{i=1}^3 q_i\lambda_{i+1}B_i$ with $q_i \in \mathbb{R}$. Then by (3.4a), we deduce

$$\begin{aligned}0 &= \big\langle \mu n_i\cdot t_i, w\big\rangle_{e_i} = \big\langle J\big(b_T r + q\big)n_i\cdot t_i, w\big\rangle_{e_i}\\ &= -\big\langle \partial^2(b_T r + q)/\partial n_i\partial s_i, w\big\rangle_{e_i} = \mathrm{c}_i\big\langle b_i(r + q_i\lambda_{i+1}), \partial w/\partial s_i\big\rangle_{e_i} \qquad \forall w \in \mathbb{P}_k(e_i;\mathbb{R}).\end{aligned}$$

Since $k\ge 2$ and b_i is positive on e_i, it follows that $r + q_i\lambda_{i+1}\big|_{e_i} = 0$ and therefore there exists $p_i \in \mathbb{P}_{\max\{1,k-2\}}(T;\mathbb{R})$ such that $r+q_i\lambda_{i+1} = \lambda_i p_i$. By repeating the same argument on the edge e_{i+1}, we see that there also exists a $p_{i+1} \in \mathbb{P}_{\max\{1,k-2\}}(T;\mathbb{R})$ such that $r + q_{i+1}\lambda_{i+2} = \lambda_{i+1}p_{i+1}$. Therefore, on the edge e_{i+1}, we have

$$r\big|_{e_{i+1}} = -q_{i+1}\lambda_{i+2}\big|_{e_{i+1}} = \lambda_i p_i\big|_{e_{i+1}} = (1-\lambda_{i+2})p_i\big|_{e_{i+1}}.$$

From the identity $-q_{i+1}\lambda_{i+2}\big|_{e_{i+1}} = (1-\lambda_{i+2})p_i\big|_{e_{i+1}}$, we get $p_i\big|_{e_{i+1}} = 0$ and $q_{i+1} = 0$. Repeating the argument for all edges, we deduce that $q\equiv 0$ and $r\big|_{\partial T} = 0$. Hence we may write $r = b_T z$ for some $z \in \mathbb{P}_{k-4}(T)$. By (3.4b) we have $z\equiv 0$ and hence $\mu\equiv 0$. Thus, the degrees of freedom (3.4) are unisolvent on $\Sigma(T)$.

Proof (2) Again, we write $\mu = \mu_0 + Jq \in \Sigma(T)$ and show that if μ vanishes at the degrees of freedom (3.4), then $\mu \equiv 0$.

By the degrees of freedom (3.4a) and $\mu n\big|_{\partial T} \in \mathbb{P}_k(\partial T;\mathbb{R}^2)$ we have $\mu_0 n\big|_{\partial T} = -(Jq)n\big|_{\partial T}$. As before we write $q = \sum_{i=1}^3 q_i B_i\lambda_{i+1}$ with $q_i\in\mathbb{R}$. Since μ_0 is smooth on $\overline{T}$ and μn equals $-(Jq)n$ on ∂T we must have

$$\lim_{x\to a_{i+1}}(Jq)n_{i+2}\cdot n_i\big|_{e_{i+2}} = \lim_{x\to a_{i+1}}(Jq)n_i\cdot n_{i+2}\big|_{e_i},$$

and therefore by Lemma 4,

$$q_i\lim_{x\to a_{i+1}}(J(B_i\lambda_{i+1}))n_{i+2}\cdot n_i\big|_{e_{i+2}} = q_i\lim_{x\to a_{i+1}}(J(B_i\lambda_{i+1}))n_i\cdot n_{i+2}\big|_{e_i}.$$

Employing Lemma 4 once again we conclude that $q_i = 0$. Repeating this argument over all vertices we deduce that $q\equiv 0$. The rest of the proof proceeds as the previous one.

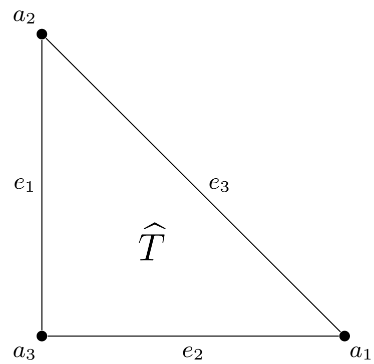

Fig. 2. The reference triangle $\widehat{T}$.

3.1 Remarks on lower order elements

A natural question is can we take $k = 1$ in definition (3.1) to derive lower order elements? Clearly $Q(T)$ must be modified in this case as the normal trace of $JQ(T)$ consist of polynomials of degree two. It is tempting to augment $\mathbb{P}_1(T;\mathbb{S})$ with the space spanned by $\{JB_i\}_{i=1}^3$. However, this construction will not work since this space will not be unisolvent using the degrees of freedom (3.4) (with $k = 1$). To see this, note that $Jb_T \in \mathbb{P}_1(T;\mathbb{S})$ and $(Jb_T)n = (JB_1 + JB_2 + JB_3)n$ on ∂T.

Another plausible way to formulate a lower order element is to construct $W^{2,\infty}$ rational functions q that have a singularity at exactly one vertex and satisfies $(Jq)n\big|_{\partial T} \in \mathbb{P}_1(\partial T;\mathbb{R}^2)$ (implying $\nabla q\big|_{\partial T} \in \mathbb{P}_2(\partial T;\mathbb{R}^2)$). Namely, we would like to derive functions that satisfy the conditions of Lemma 4 but decrease the polynomial degree by one. The proof of unisolvency would then follows the same lines as the second proof of Lemma 6. However, the following result essentially shows that it is impossible to construct such functions.

Lemma 7 *Let $\widehat{T}$ be the reference triangle with vertices $a_1 = (1,0)$, $a_2 = (1,0)$ and $a_3 = (0,0)$ (cf. Figure 2). Suppose that a function $q \in C^1(\overline{\widehat{T}}) \cap W^{2,\infty}(\widehat{T})$ (i) is smooth at vertices $a_3 = (0,0)$ and $a_2 = (0,1)$, and (ii) satisfies $\nabla q\big|_{\partial\widehat{T}} \in \mathbb{P}_2(\partial\widehat{T};\mathbb{R}^2)$. Then*

$$\lim_{x\to a_1} Jqn_2 \cdot n_3\big|_{e_2} = \lim_{x\to a_1} Jqn_3 \cdot n_2\big|_{e_3}, \tag{3.5}$$

where $n_2 = (0,-1)^T$ and $n_3 = (1,1)^T/\sqrt{2}$.

Proof Since $\nabla q \in \mathbb{P}_2(\partial\widehat{T};\mathbb{R}^2)$ on $\partial\widehat{T}$, we must have $q\big|_{\partial T} \in \mathbb{P}_3(\partial\widehat{T};\mathbb{R})$. Therefore since q is continuous, we may subtract a cubic polynomial p such that $(q-p)$ vanishes on the boundary of $\partial\widehat{T}$. We then set

$$\begin{aligned} B &= q - p - x_1x_2(1-x_1-x_2)\frac{\partial^2(q-p)}{\partial x_1\partial x_2}(0,0) \\ &= q - p - b_{\widehat{T}}\frac{\partial^2(q-p)}{\partial x_1\partial x_2}(0,0). \end{aligned}$$

Due to the properties of q (and since p is a smooth cubic polynomial), all of the properties of q hold for B as well. In particular,

– $B \in C^1(\overline{\widehat{T}}) \cap W^{2,\infty}(\widehat{T})$;
– B is smooth at vertices a_2 and a_3;
– $\nabla B\big|_{\partial\widehat{T}} \in \mathbb{P}_2(\partial\widehat{T};\mathbb{R})$;

In addition, we have

– $B\big|_{\partial\widehat{T}} = 0$;
– $\dfrac{\partial^2 B}{\partial x_1 \partial x_2}(0,0) = 0$;
– $\nabla B(a_i) = 0$ $(i = 1,2,3)$ (since $B\big|_{\partial\widehat{T}} = 0$ and $B \in C^1(\overline{\widehat{T}})$).

Define the quadratic polynomial $g(\tau) = \frac{\partial B}{\partial x_2}(\tau, 0)$ $(\tau \in [0,1])$. We then have by Taylor's Theorem,

$$\begin{aligned} g(\tau) &= g(0) + \tau g'(0) + \frac{\tau^2}{2} g''(0) \\ &= \frac{\partial B}{\partial x_2}(0,0) + \tau \frac{\partial^2 B}{\partial x_1 \partial x_2}(0,0) + \frac{\tau^2}{2} g''(0) = \frac{\tau^2}{2} g''(0). \end{aligned}$$

But since $0 = g(1) = \frac{1}{2} g''(0)$, we must have $g \equiv 0$ and therefore $\nabla B\big|_{e_2} = 0$. Similarly, repeating the same argument but with $g(\tau) = \frac{\partial B}{\partial x_1}(0,\tau)$, we obtain $\nabla B\big|_{e_1} = 0$.

Clearly we have $\frac{\partial^2 B}{\partial x_2^2}\big|_{e_1} = 0$, and since $\nabla B\big|_{e_1} = 0$ we have $\frac{\partial^2 B}{\partial x_1 \partial x_2}\big|_{e_1} = 0$ as well. In particular, we have

$$\frac{\partial^2 B}{\partial x_2^2}(0,1) = 0 \quad \text{and} \quad \frac{\partial^2 B}{\partial x_1 \partial x_2}(0,1) = 0 \tag{3.6}$$

Furthermore, since the tangental direction of edge e_3 is $(1,-1)/\sqrt{2}$, we have

$$\left(\frac{\partial^2 B}{\partial x_1^2} - 2 \frac{\partial^2 B}{\partial x_1 \partial x_2} + \frac{\partial^2 B}{\partial x_2^2} \right)\Big|_{e_3} = 0.$$

Combining this last identity with (3.6) we conclude that $D^2 B(0,1) = 0$.

Define $r(\tau) = \frac{\partial B}{\partial x_1}(1-\tau,\tau) + \frac{\partial B}{\partial x_2}(1-\tau,\tau) = \sqrt{2}\frac{\partial B}{\partial n_3}(1-\tau,\tau) \in \mathbb{P}_2([0,1],\mathbb{R})$. Then as before, we have

$$r(\tau) = r(0) + x r'(0) + \frac{\tau^2}{2} r''(0) = \frac{\tau^2}{2} r''(0),$$

and since $0 = r(1) = \frac{1}{2} r''(0)$ we obtain $r \equiv 0$. It then follows that $\nabla B\big|_{\partial\widehat{T}} = 0$ which implies that

$$\lim_{x \to a_1} JBn_2\big|_{e_2} = \lim_{x \to a_1} JBn_3\big|_{e_3} = 0.$$

Since

$$q = B + p + xy(1-x-y)\frac{\partial^2 (q-p)}{\partial x \partial y}(0,0),$$

the desired result (3.5) immediately follows.

4 Global Finite Element Spaces and the Fortin Projection

The global finite element spaces of the stress and displacements are given respectively by

$$\Sigma_h = \{\mu \in H(\operatorname{div}; \Omega; \mathbb{S}) : \ \mu|_T \in \Sigma(T) \ \forall T \in \mathcal{T}_h\},$$
$$V_h = \{v \in L^2(\Omega; \mathbb{R}^2) : \ v|_T \in V(T) \ \forall T \in \mathcal{T}_h\}.$$

Denote by $\Pi_h : H(\operatorname{div}; \Omega; \mathbb{S}) \cap L^p(\Omega; \mathbb{S}) \to \Sigma_h$ (where $p > 2$) the canonical projection associated with the degrees of freedom (3.4); that is, given a function $\mu \in H(\operatorname{div}; \Omega; \mathbb{S})$, $\Pi_h \mu \in \Sigma_h$ is uniquely determined (locally) by the following conditions:

$$\langle (\Pi_h \mu) n_i, v \rangle_{e_i} = \langle \mu n_i, v \rangle_{e_i} \qquad \forall v \in \mathbb{P}_k(e_i; \mathbb{R}^2), \tag{4.1a}$$
$$(\Pi_h \mu, \rho)_T = (\mu, \rho)_T \qquad \forall \rho \in \varepsilon\big[\mathbb{P}_{k-1}(T; \mathbb{R}^2)\big] + J\big[b_T^2 \mathbb{P}_{k-4}(T; \mathbb{R})\big]. \tag{4.1b}$$

By Lemma 6 Π_h is well-defined. Note that for any $v \in \mathbb{P}_{k-1}(T; \mathbb{R}^2)$, we have

$$\begin{aligned}\int_T \operatorname{div} \mu \cdot v \, dx &= -\int_T \mu : \varepsilon(v) \, dx + \int_{\partial T} \mu n \cdot v \, ds \\ &= -\int_T (\Pi_h \mu) : \varepsilon(v) \, dx + \int_{\partial T} (\Pi_h \mu) n \cdot v \, ds = \int_T \operatorname{div} \Pi_h \mu \cdot v \, dx.\end{aligned}$$

Thus, denoting by $P_h : L^2(\Omega; \mathbb{R}^2) \to V_h$ the L^2 projection onto V_h, we have the desirable commuting property

$$\operatorname{div} \Pi_h \mu = P_h \operatorname{div} \mu \qquad \forall \mu \in H^1(\Omega; \mathbb{S}). \tag{4.2}$$

Lemma 8 *For $\mu \in H^r(\Omega; \mathbb{S})$ $(r \geq 1)$, there holds*

$$\|\mu - \Pi_h \mu\|_{L^2(\Omega)} \leq C h^\ell \|\mu\|_{H^\ell(\Omega)} \quad \ell = \min\{k+1, r\}. \tag{4.3}$$

Proof The estimate can be used by standard scaling arguments using the Piola transform. We refer the reader to [17,8] for details.

Using standard arguments, we can derive the necessary inf-sup condition of the finite element pair $\Sigma_h \times V_h$ using the commuting property (4.2) and the estimate (4.3). For completeness we sketch this argument.

Given $w \in V_h \subset L^2(\Omega; \mathbb{R}^2)$, there exists $\mu \in H^1(\Omega; \mathbb{S})$ such that $\operatorname{div} \mu = w$ and $\|\mu\|_{H^1(\Omega)} \leq C \|w\|_{L^2(\Omega)}$. We then have

$$\begin{aligned}(\operatorname{div} \Pi_h \mu, w) &= (\operatorname{div} \mu, w) = \|w\|^2_{L^2(\Omega)} \\ &\geq C \|w\|_{L^2(\Omega)} \|\mu\|_{H^1(\Omega)} \geq C \|w\|_{L^2(\Omega)} \|\Pi_h \mu\|_{H(\operatorname{div}; \Omega)},\end{aligned}$$

where we have used the stability estimate $\|\Pi_h \mu\|_{L^2(\Omega)} \leq C \|\mu\|_{H^1(\Omega)}$ and the identity $\operatorname{div} \Pi_h \mu = w$. We then immediately obtain

$$\sup_{0 \neq \mu \in \Sigma_h} \frac{(\operatorname{div} \mu, w)}{\|\mu\|_{H(\operatorname{div}; \Omega)}} \geq C \|w\|_{L^2(\Omega)} \quad \forall w \in V_h, \tag{4.4}$$

where the constant $C > 0$ is independent of h.

5 Relation with C^1 Elements

In this section, we characterize divergence free elements of stress space with the use of a class of C^1 Zienkiewicz-like elements [24,12,17]. The local space of these elements are defined as ($k \ge 2$)

$$Z(T) = \mathbb{P}_{k+2}(T;\mathbb{R}) + Q(T), \tag{5.1}$$

where $Q(T)$ is given by (3.2). Clearly, we have

$$\dim Z(T) = \dim \mathbb{P}_{k+2}(T;\mathbb{R}) + \dim Q(T) = \frac{1}{2}k^2 + \frac{7}{2}k + 9.$$

The degrees of freedom that determine a function $z \in Z(T)$ are given by

$$\begin{aligned}
&z(a_i),\ \nabla z(a_i) && \text{for all vertices } a_i, && (5.2a)\\
&\langle z, \kappa\rangle_{e_i} && \forall \kappa \in \mathbb{P}_{k-2}(e_i), && (5.2b)\\
&(Jz, J(b_T^2\rho))_T && \forall \rho \in \mathbb{P}_{k-4}(T), && (5.2c)\\
&\langle \partial z/\partial n_i, \omega\rangle_{e_i} && \forall \omega \in \mathbb{P}_{k-1}(e_i). && (5.2d)
\end{aligned}$$

In the cases $k = 2$ and $k = 3$, the degree of freedoms (5.2c) are omitted. We remark that the family of generalized Zienkiewicz spaces presented here differs from the one constructed in [24,12,17]. In particular the local space (5.1) has $\frac{1}{2}(4k-6)$ less degrees of freedom than the local space in [17]. Furthermore, the elements presented here are expected to have better approximation properties since than those in [24, 12,17] since all of $\mathbb{P}_{k+2}(T;\mathbb{R})$ is contained in $Z(T)$ and not a subset of this space.

To show unisolvency of the degrees of freedom, write $z = z_0 + q$ with $z_0 \in \mathbb{P}_{k+2}(T;\mathbb{R})$ and $q \in Q(T)$, and suppose that z vanishes on (5.2). Since q vanishes on ∂T and ∇q vanishes at the vertices of T, it follows from (5.2a)–(5.2b) that $z_0 = b_T p$ for some $p \in \mathbb{P}_{k-1}(T;\mathbb{R})$. Then by (5.2d) we have

$$0 = \int_{e_i} \frac{\partial(z_0+q)}{\partial n_i}\omega\, ds = \mathrm{c}_i \int_{e_i} b_i(p + \lambda_{i+1} q_i)\omega\, ds \quad \forall \omega \in \mathbb{P}_{k-1}(e_i;\mathbb{R}), \tag{5.3}$$

with $q_i \in \mathbb{R}$ and $p \in \mathbb{P}_{k-1}(T;\mathbb{R})$. Using the same arguments in the first proof of Lemma 6, we deduce that $q \equiv 0$ and $p = b_T r$ with $r \in \mathbb{P}_{k-4}(T;\mathbb{R})$. Finally the degree of freedom (5.2c) implies $r = 0$ and therefore $z \equiv 0$.

Remark 1 Replacing the degree of freedom (5.2c) by $(z, b_T^2\rho)_T$ for all $\rho \in \mathbb{P}_{k-4}(T)$ still forms a unisolvent set for $Z(T)$. However, we use (5.2c) as it enables us to derive desirable commuting properties below.

The global space of the generalized Zienkiewicz element is defined as

$$Z_h = \left\{ z \in H_0^2(\Omega) :\ z\big|_T \in Z(T)\ \forall T \in \mathcal{T}_h \right\}.$$

Note that

$$JZ(T) = J\mathbb{P}_{k+2}(T;\mathbb{R}) + JQ(T) \subset \mathbb{P}_k(T;\mathbb{S}) + JQ(T) = \Sigma(T).$$

Furthermore, since

$$Jzn\big|_{\partial T} = -\frac{\partial}{\partial s}(\operatorname{curl} z)\big|_{\partial T} \quad \forall T \in \mathcal{T}_h$$

and $Z_h \subset C^1(\overline{\Omega})$, we have $JZ_h \subset H(\operatorname{div}; \Omega; \mathbb{S})$. It then follows that the Airy stress function maps Z_h to Σ_h.

To make further connections between the C^1 element and the stress space, we let $I_h : H_0^2(\Omega) \to Z_h$ denote the projection corresponding to the degrees of freedom (5.2). Then by (5.2) and (4.1b), there holds for all $v \in \mathbb{P}_k(e_i; \mathbb{R})$,

$$\begin{aligned}\int_{e_i} J(I_h z)n_i \cdot n_i v\, ds &= \int_{e_i} \frac{\partial^2 (I_h z)}{\partial s^2} v\, ds \\ &= \int_{e_i} I_h z \frac{\partial^2 v}{\partial s^2}\, ds + \frac{\partial (I_h z)}{\partial s} v\Big|_{a_{i+1}}^{a_{i+2}} - I_h z \frac{\partial v}{\partial s}\Big|_{a_{i+1}}^{a_{i+2}} \\ &= \int_{e_i} z \frac{\partial^2 v}{\partial s^2}\, ds + \frac{\partial z}{\partial s} v\Big|_{a_{i+1}}^{a_{i+2}} - z \frac{\partial v}{\partial s}\Big|_{a_{i+1}}^{a_{i+2}} \\ &= \int_{e_i} \frac{\partial^2 z}{\partial s^2} v\, ds = \int_{e_i} (Jzn_i \cdot n_i) v\, ds = \int_{e_i} (\Pi_h Jz) n_i \cdot n_i v\, ds.\end{aligned} \tag{5.4}$$

Similarly, we have by (5.2) and (4.1b),

$$\begin{aligned}\int_{e_i} J(I_h z)n_i \cdot t_i v\, ds &= -\int_{e_i} \frac{\partial^2 (I_h z)}{\partial s_i \partial n_i} v\, ds = \int_{e_i} \frac{\partial (I_h z)}{\partial n_i} \frac{\partial v}{\partial s_i}\, ds - \frac{\partial (I_h z)}{\partial n_i} v\Big|_{a_{i+1}}^{a_{i+2}} \\ &= \int_{e_i} \frac{\partial z}{\partial n_i} \frac{\partial v}{\partial s_i}\, ds - \frac{\partial z}{\partial n_i} v\Big|_{a_{i+1}}^{a_{i+2}} \\ &= -\int_{e_i} \frac{\partial^2 z}{\partial s_i \partial n_i} v\, ds = \int_{e_i} (Jz) n_i \cdot t_i v\, ds = \int_{e_i} (\Pi_h Jz) n_i \cdot t_i v\, ds.\end{aligned} \tag{5.5}$$

Continuing, we claim that

$$\int_{e_i} \operatorname{curl}(I_h z) \cdot v\, ds = \int_{e_i} \operatorname{curl} z \cdot v\, ds \qquad \forall v \in \mathbb{P}_{k-2}(e_i; \mathbb{R}^2). \tag{5.6}$$

Indeed, this identity can be derived by the following identities which follow from (5.2a), (5.2b), (5.2d) and integration by parts:

$$\begin{aligned}&\int_{e_i} \operatorname{curl}(I_h z) \cdot t_i w\, ds \\ &\quad = \int_{e_i} \frac{\partial (I_h z)}{\partial n_i} w\, ds = \int_{e_i} \frac{\partial z}{\partial n_i} w\, ds = \int_{e_i} \operatorname{curl} z \cdot t_i w\, ds \quad \forall w \in \mathbb{P}_{k-2}(e_i; \mathbb{R}),\end{aligned}$$

and

$$\int_{e_i} \operatorname{curl}(I_h z) \cdot n_i w\, ds$$

$$= \int_{e_i} \frac{\partial (I_h z)}{\partial s_i} w\, ds = \int_{e_i} \frac{\partial z}{\partial s_i} w\, ds = \int_{e_i} \operatorname{curl} z \cdot n_i w\, ds \quad \forall w \in \mathbb{P}_{k-2}(e_i;\mathbb{R}).$$

It then follows from (5.6) and (4.1b) that for any $w \in \mathbb{P}_{k-1}(T;\mathbb{R}^2)$, we have

$$\begin{aligned}\int_T J(I_h z) : \varepsilon(w)\, dx &= -\int_T \operatorname{div} J(I_h z) \cdot w\, dx + \int_{\partial T} J(I_h z) n \cdot w\, ds \\ &= \int_{\partial T} \frac{\partial}{\partial s}(\operatorname{curl}(I_h z)) \cdot w\, ds = -\int_{\partial T} \operatorname{curl}(I_h z) \cdot \frac{\partial w}{\partial s}\, ds \\ &= -\int_{\partial T} \operatorname{curl} z \cdot \frac{\partial w}{\partial s}\, ds = \int_T Jz : \varepsilon(w)\, dx,\end{aligned}$$

and therefore by (5.2c) and (4.1b)

$$\int_T J(I_h z) : \rho\, dx = \int_T \Pi_h(Jz) : \rho\, dx \qquad \forall \rho \in \varepsilon\big[\mathbb{P}_{k-1}(T;\mathbb{R}^2)\big] + J\big[b_T^2 \mathbb{P}_{k-4}(T)\big] \tag{5.7}$$

It follows from (5.4)–(5.7) and (4.1) that

$$J(I_h z) = \Pi_h J z. \tag{5.8}$$

Along with property (4.2) we have shown the following sequence commutes:

$$\begin{array}{ccccccccc} \mathbb{P}_1(\Omega;\mathbb{R}) & \xrightarrow{\subset} & H^2(\Omega;\mathbb{R}) & \xrightarrow{J} & H(\operatorname{div};\Omega;\mathbb{S}) & \xrightarrow{\operatorname{div}} & L^2(\Omega;\mathbb{R}^2) & \longrightarrow & 0 \\ & & \big\downarrow I_h & & \big\downarrow \Pi_h & & \big\downarrow P_h & & \\ \mathbb{P}_1(\Omega;\mathbb{R}) & \xrightarrow{\subset} & Z_h & \xrightarrow{J} & \Sigma_h & \xrightarrow{\operatorname{div}} & V_h & \longrightarrow & 0 \end{array} \tag{5.9}$$

We recall that the sequence in the first row in (5.9) is exact; that is, the range of each map is the null space of the succeeding map. In particular, every divergence free function in $H(\operatorname{div};\Omega;\mathbb{S})$ can be written as the Airy stress function of some $H^2(\Omega;\mathbb{R})$ function. It is also easy to see that the second row is exact as well. Indeed, suppose that $\mu \in \Sigma_h$ is divergence free. Then since $\Sigma_h \subset H(\operatorname{div};\Omega;\mathbb{S})$ we know there exists $z \in H^2(\Omega;\mathbb{R})$ (unique up to a linear function) such that $Jz = \mu$. We then have by (5.8) (and since Π_h is idempotent) $\mu = \Pi_h \mu = \Pi_h J z = J(I_h z)$. It then follows that both rows in the complex (5.9) are exact.

6 The Finite Element Method and its Hybrid Form

The finite element method will find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ satisfying

$$(A\sigma_h, \mu) + (u_h, \operatorname{div}\mu) = 0, \tag{6.1a}$$

$$(v, \operatorname{div}\sigma_h) = (f, v), \tag{6.1b}$$

for all $\mu \in \Sigma_h$ and $v \in V_h$. By the inf-sup condition (4.4), the discrete problem is well-posed. Furthermore using the Fortin projection (4.1) we can easily prove optimal order estimates of the method using standard arguments [10,8]. For completeness we give the argument.

We start with the error equations

$$(A(\sigma - \sigma_h), \mu) + (P_h u - u_h, \operatorname{div} \mu) = 0 \qquad \forall \mu \in \Sigma_h, \tag{6.2a}$$
$$(\operatorname{div}(\sigma - \sigma_h), v) = 0 \qquad \forall v \in V_h, \tag{6.2b}$$

where we recall that P_h denotes the L^2 projection onto V_h and we have used the fact $\operatorname{div} \Sigma_h \subset V_h$. By the second equation and (4.2) we obtain $\operatorname{div} \sigma_h = \operatorname{div} \Pi_h \sigma$ and therefore by standard properties of the L^2 projection and (4.2) we obtain $\|\operatorname{div} \sigma - \operatorname{div} \sigma_h\|_{L^2(\Omega)} = \|\operatorname{div} \sigma - P_h \operatorname{div} \sigma\|_{L^2(\Omega)} \le C h^m \|\operatorname{div} \sigma\|_{H^m(\Omega)}$ for $0 \le m \le k$ provided $\operatorname{div} \sigma \in H^m(\Omega; \mathbb{R}^2)$. We also have by (6.2a) that $(A(\sigma - \sigma_h), \sigma_h - \Pi_h \sigma) = (u_h - P_h u, \operatorname{div}(\sigma_h - \Pi_h \sigma)) = 0$, and therefore $\|A^{1/2}(\sigma - \sigma_h)\|^2_{L^2(\Omega)} = (A(\sigma - \sigma_h), \sigma - \Pi_h \sigma) \le \|A^{1/2}(\sigma - \sigma_h)\|_{L^2(\Omega)} \|A^{1/2}(\sigma - \Pi_h \sigma)\|_{L^2(\Omega)}$ We then have

$$\|A^{1/2}(\sigma - \sigma_h)\|_{L^2(\Omega)} \le \|A^{1/2}(\sigma - \Pi_h \sigma)\|_{L^2(\Omega)},$$

and therefore by Lemma 8 we obtain

$$\|\sigma - \sigma_h\|_{L^2(\Omega)} \le C h^m \|\sigma\|_{H^m(\Omega)} \quad 1 \le m \le k+1$$

provided $\sigma \in H^m(\Omega; \mathbb{S})$. Finally by the inf-sup condition (4.4) we obtain the following error estimate of the displacement:

$$\|P_h u - u_h\|_{L^2(\Omega)} \le C \|A^{1/2}(\sigma - \sigma_h)\|_{L^2(\Omega)} \le C h^m \|\sigma\|_{H^m(\Omega)},$$

and therefore by approximations properties of the L^2 projection,

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)} &\le C h^m \|\sigma\|_{H^m(\Omega)} + \|u - P_h u\|_{L^2(\Omega)} \\ &\le C\big(h^m \|u\|_{H^{m+1}(\Omega)} + h^m \|u\|_{H^m(\Omega)}\big) \le C h^m \|u\|_{H^{m+1}(\Omega)} \end{aligned}$$

with $1 \le m \le k$.

We should mention that we can improve the result $\|P_h u - u_h\|_{L^2(\Omega)}$ using a duality argument if we assume H^2-regularity. We omit the details.

In summary we have the following convergence result.

Theorem 1 *Let* $(\sigma, u) \in \Sigma \times V$ *satisfy* (1.2) *and* $(\sigma_h, u_h) \in \Sigma_h \times V_h$ *satisfy* (6.1). *Then there holds*

$$\begin{aligned} \|\operatorname{div} \sigma - \operatorname{div} \sigma_h\|_{L^2(\Omega)} &\le C h^m \|\operatorname{div} \sigma\|_{H^m(\Omega)} && 0 \le m \le k, \\ \|\sigma - \sigma_h\|_{L^2(\Omega)} &\le C h^m \|\sigma\|_{H^m(\Omega)} && 1 \le m \le k+1, \\ \|u - u_h\|_{L^2(\Omega)} &\le C h^m \|u\|_{H^{m+1}(\Omega)} && 1 \le m \le k. \end{aligned}$$

It might be advantageous to implement the hybrid form of the method instead. To do this one needs the space

$$M_h = \{m : m|_e \in \mathbb{P}_k(e;\mathbb{R}^2) \text{ for all } e \in \mathcal{E}_h, m|_{\partial\Omega} = 0\}.$$

Here $\mathcal{E}_h$ is the set of edges of the triangulation $\mathcal{T}_h$. We also need the non–conforming version of Σ_h.

$$\tilde{\Sigma}_h = \big\{\mu \in L^2(\Omega;\mathbb{S}) : \ \mu|_T \in \Sigma(T) \ \forall T \in \mathcal{T}_h\big\}.$$

The hybrid form will find $\sigma_h \in \tilde{\Sigma}_h$, $u_h \in V_h$ and $\lambda_h \in M_h$ that satisfies

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h} (A\sigma_h,\mu)_T + \sum_{T\in\mathcal{T}_h} (u_h, \mathrm{div}\mu)_T - \sum_{e\in\mathcal{E}_h^i} \big\langle \lambda_h, \mu n\big\rangle_e = 0,\\
&\sum_{T\in\mathcal{T}_h} (v, \mathrm{div}\sigma_h)_T = (f,v),\\
&\sum_{e\in\mathcal{E}_h^i} \big\langle m, \sigma_h n\big\rangle_e = 0,
\end{aligned}$$

for all $\mu \in \tilde{\Sigma}_h$, $v \in V_h$ and $m \in M_h$,

One can easily show that the σ_h and u_h resulting from the hybrid form will solve the original finite element method. Moreover, one can easily obtain a symmetric positive linear system involving only the Lagrange multiplier λ_h of the form

$$a_h(\lambda_h, m) = L(f) \quad \forall m \in M_h. \tag{6.3}$$

where a_h is a symmetric, coercive bilinear form and L is a bounded linear operator. Such a characterization was given by Cockburn and Gopalakrishan [13] for mixed methods applied to second order problems. Similar arguments will give us the characterization (6.3) in our setting. We omit the details.

7 A Low Order Element

In this section we construct a low order finite element pair that has the same number of degrees of freedom as the Johnson-Mercier composite element for the stress space, but has a smaller displacement space. To describe a reduced element, we introduce the space of infinitesimal rigid motions

$$RM(T) = \mathrm{span}\{(-x_2, x_1)^t\} + \mathbb{P}_0(T;\mathbb{R}^2).$$

We then define

$$\Sigma(T) = M(T) + JQ(T),$$

where

$$M(T) = \big\{\mu \in \mathbb{P}_2(T;\mathbb{S}) : \ \mathrm{div}\,\mu \in RM(T) \text{ and } \mu n_i \cdot n_i\big|_{e_i} \in \mathbb{P}_1(e_i;\mathbb{R})\big\},$$

and $Q(T)$ is defined by (3.2). The local space of the displacements are taken to be $RM(T)$. Since the dimension of $RM(T)$ is three, there are exactly six constraints imposed in the definition of $M(T)$. It then follows that $\dim M(T) \geq \dim \mathbb{P}_2(T;\mathbb{S}) - 6 = 12$ and therefore $\dim \Sigma(T) \geq 15$. To show that the dimension of $\Sigma(T)$ is 15, we define the 15 degrees of freedom

$$\left\langle \mu n_i \cdot n_i, v \right\rangle_{e_i} \qquad \forall v \in \mathbb{P}_1(e_i;\mathbb{R}), \tag{7.1a}$$

$$\left\langle \mu n_i \cdot t_i, w \right\rangle_{e_i} \qquad \forall w \in \mathbb{P}_2(e_i;\mathbb{R}). \tag{7.1b}$$

To see this is a unisolvent set, suppose $\mu \in \Sigma(T)$ vanishes at all the degrees of freedom. As before, we write $\mu = \mu_0 + Jq$. By the definition of $M(T)$, (7.1a) and since $(Jq)n \cdot n$ vanishes on ∂T, we have $\mu n \cdot n = 0$ on ∂T. Therefore by (7.1b), $\mu n\big|_{\partial T} = 0$. It then follows that for any $v \in RM(T)$,

$$\int_T \operatorname{div} \mu_0 \cdot v\, dx = \int_T (\mu n) \cdot v\, ds = 0.$$

Using the same arguments as above, we have $\mu \equiv 0$ and therefore the dimension of $\Sigma(T)$ is 15, and the degrees of freedom (7.1) form a unisolvent set.

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