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QUADRATURE AND SCHATZ'S POINTWISE ESTIMATES FOR FINITE ELEMENT METHODS*

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Abstract.

We investigate numerical integration effects on weighted pointwise estimates. We prove that local weighted pointwise estimates will hold, modulo a higher order term and a negative-order norm, as long as we use an appropriate quadrature rule. To complete the analysis in an application, we also prove optimal negative-order norm estimates for a corner problem taking into account quadrature. Finally, we present an example to show that our result is sharp.

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1 Introduction.

Weighted pointwise estimates obtained by Schatz, [9], greatly improve previous local W^1_∞ estimates. They show that the finite element approximation, in some cases, approximates the solution in a very sharp local sense. That is, the approximation error at a point x is more heavily influenced by the behavior of the solution near x rather than far from x. This has proven to be useful for superconvergence results [10] and pointwise a posteriori estimates [5]. We prove that these estimates are preserved, modulo a higher order term and a negative-order norm, if we use a quadrature rule of high enough order.

Let $\Omega \subset\subset R^N$ and consider the equation

$$Lu \equiv \sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega. \tag{1.1}$$

We assume f and a_{ij} are smooth and (a_{ij}) is uniformly elliptic in Ω.

If $\Omega_1 \subset\subset \Omega$, then u solves the local equation

$$A(u,v) = \int_{\Omega_1} fv\,dx, \quad \text{for all } v \in \mathring{H}^1(\Omega_1) \tag{1.2}$$

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where

$$A(w,v) = \int_\Omega \sum_{ij} a_{ij} \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j}.$$

Let $S^h_{r-1} \subset W^1_\infty(\Omega)$ be a one parameter family of finite element spaces. From now on $\Omega_1 \subset\subset \Omega$ will denote a fixed domain with the following properties. We assume that the family of meshes when restricted to Ω_1 is quasi-uniform and that each element intersecting Ω_1 is a simplex. If $\mathring{S}_{h,r-1}(\Omega_1)$ denotes those functions in S^h_{r-1} with compact support in the interior of Ω_1, then we require that $\mathring{S}_{h,r-1}(\Omega_1)$ be composed of continuous functions supported in Ω_1 such that their restriction to each simplex of our decomposition is a polynomial of degree at most $r-1$ (i.e. we consider Lagrange finite elements of degree $r-1$ in Ω_1).

The finite element solution $\bar{u}_h$ with exact quadrature will satisfy

$$A(u-\bar{u}_h, v) = 0, \quad \text{for all } v \in \mathring{S}_{h,r-1}(\Omega_1). \tag{1.3}$$

In Propositions 1.1–1.3 we shall review some known results. First we state the W^∞_1 estimates for the finite element approximation with exact quadrature found in [12].

Proposition 1.1. *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. If $t \geq 0$, there exists a constant C independent of h, u and $\bar{u}_h$ such that*

$$|u-\bar{u}_h|_{W^1_\infty(\Omega_0)} \leq C \inf_{\chi \in S^h_{r-1}} ||u-\chi||_{W^1_\infty(\Omega_1)} + C||u-\bar{u}_h||_{H^{-t}(\Omega_1)}.$$

Applying the techniques in [12], one can prove local W^∞_1 estimates for the finite element approximation with numerical quadrature, let us denote it by u_h. Quadrature rules employed will be precisely defined in Section 2.

Proposition 1.2. *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and $t \geq 0$. If a quadrature rule of order $2(r-1)-2+q$ ($q \geq 0$) is used to compute u_h, then there exists a constant C independent of h, u, and u_h such that*

$$\begin{aligned} |u-u_h|_{W^1_\infty(\Omega_0)} &\leq C \inf_{\chi \in S^h_{r-1}} ||u-\chi||_{W^1_\infty(\Omega_1)} + C||u-u_h||_{H^{-t}(\Omega_1)} \\ &\quad + Ch^{r-1+q}\log(1/h)(||u||_{W^r_\infty(\Omega_1)} + ||f||_{W^{r-1+q}_\infty(\Omega_1)}). \end{aligned} \tag{1.4}$$

The case $q=0$ is Corollary 5.1 [12]. Following that proof, one can easily generalize this result to $q>0$. The first term of the right hand side of (1.4) can be bounded using the Bramble-Hilbert lemma, to get $\inf_{\chi \in S^{r-1}_h} ||u-\chi||_{W^1_\infty(\Omega_1)} \leq Ch^{r-1}|u|_{W^r_\infty(\Omega_1)}$. Therefore, if $q>0$ one, in some sense, preserves the local estimates, modulo a higher order term and a negative-order norm. In the case $q=0$, the last term in the right hand side of (1.4) is of the same order as the typical order of the first term. Quadrature rules of order $2(r-1)-2$ ($q=0$) are used in [4] to prove H^1 error estimates.

Now we compare these estimates to the sharper weighted pointwise estimates of Schatz. In the case of exact quadrature we have (Theorem 1.2 [9]):

PROPOSITION 1.3. *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and consider $x \in \Omega_0$. Let $0 \le s \le r-1$, u solve* (1.2) *and $\bar{u}_h$ satisfy* (1.3). *If $t \ge 0$, there exists a C independent of h, u, and $\bar{u}_h$ such that*

$$|\nabla(u-\bar{u}_h)(x)| \le C\left(\log\frac{1}{h}\right)^{\bar{\bar{s}}} \inf_{\chi\in S^h_{r-1}} ||u-\chi||_{W^1_\infty(\Omega_1),x,s} + C||u-\bar{u}_h||_{H^{-t}(\Omega_1)}.$$

Here $\bar{\bar{s}} = 0$ if $0 \le s < r-1$ and $\bar{\bar{s}} = 1$ if $s = r-1$.

The weighted norm is defined as

$$||v||_{W^1_\infty(\Omega_1),x,s} = ||\sigma_x^s v||_{L_\infty(\Omega_1)} + ||\sigma_x^s \nabla v||_{L_\infty(\Omega_1)}$$

where $\sigma_x(y) = h/(|x-y|+h)$. Note that if $y = x$, then $\sigma_x^s(y) = 1$. On the other hand, if $|y-x| = O(1)$, then $\sigma_x^s(y) = O(h^s)$. If $s = 0$, we get Proposition 1.1. The improvement comes when $s > 0$.

We now state the main result of this note which is the corresponding weighted pointwise estimates with numerical quadrature.

THEOREM 1.4. *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and consider $x \in \Omega_0$. Let $0 \le s \le r-1$, u solve* (1.2) *and u_h satisfy* (2.1) *where we use a quadrature rule of order $2(r-1)-2+q$ with $q \ge s$. If $t \ge 0$, there exists a C independent of h, x, u, and u_h such that*

$$|\nabla(u-u_h)(x)| \le C\left(\log\frac{1}{h}\right)^{\bar{\bar{s}}} \inf_{\chi\in S^h_{r-1}} ||u-\chi||_{W^1_\infty(\Omega_1),x,s} + C||u-u_h||_{H^{-t}(\Omega_1)}$$

$$+ C\left(\log\frac{1}{h}\right) h^{r-1+q}(||u||_{W^r_\infty(\Omega_1)} + ||f||_{W^{r-1+q}_\infty(\Omega_1)}). \tag{1.5}$$

Here $\bar{\bar{s}} = 0$ if $0 \le s < r-1$ and $\bar{\bar{s}} = 1$ if $s = r-1$.

If $q > s$, we preserve the weighted pointwise estimates, modulo a higher order term and a negative-order norm. In the case $q = s$, the third term in the right hand side of (1.5) is of the same order, modulo a logarithmic factor, as $\sigma_x^s(y)\nabla(u-\chi)(y)$ for $|y-x| = O(1)$; however, closer to x the local structure of Schatz's results are preserved.

In the next section we describe the quadrature rules that we consider. In Section 3 we prove Theorem 1.4. In Section 4 we complete the picture for an application by estimating $||u-u_h||_{H^{-t}(\Omega)}$ in a polygonal domain with refinements at the corners. Finally, in Section 5 we show that Theorem 1.4 is sharp.

2 Quadrature.

Let the simplex $\hat{T}$ denote a reference element, and assume we are using a quadrature rule that approximates $\int_{\hat{T}} g dx$:

$$Q_{\hat{T}}(g) = \sum_i \hat{w}_l g(\hat{b}_l),$$

where the $\hat{w}_l > 0$ and $\hat{b}_l \in \hat{T}$. Q is of order k if $Q_{\hat{T}}(p) = \int_{\hat{T}} p dx$ for all polynomials p of degree less then or equal to k, but fails to integrate a polynomial of degree

$k+1$ exactly. We know that $Q_{\hat{T}}$ induces a quadrature rule for any simplex T,

$$Q_T(g) = \sum_i w_l g(b_l).$$

Here $w_l = J(R_T)\hat{w}_l$ and $b_l = R_T(\hat{b}_l)$ where $R_T : \hat{T} \to T$ is our standard affine map. We define the error of our quadrature in $\hat{T}$ and T as

$$E_{\hat{T}}(\hat{g}) = Q_{\hat{T}}(\hat{g}) - \int_{\hat{T}} \hat{g} d\hat{x},$$
$$E_T(g) = Q_T(g) - \int_T g dx.$$

Here $\hat{g}(\hat{x}) = g(R_T(\hat{x}))$. Notice that $E_T(g) = J(R_T)E_{\hat{T}}(\hat{g})$. Let us suppose that we use this type of quadrature in Ω_1. Then, our finite element approximation u_h will satisfy

$$A(u-u_h, v) = F(v), \quad \forall v \in \mathring{S}_{h,r-1}(\Omega_1) \tag{2.1}$$

where $F = F_1 + F_2$,

$$F_1 = \sum_T F_1^T(v), \quad F_1^T(v) = E_T\left(\sum_{ij} a_{ij} \frac{\partial u_h}{\partial x_i} \frac{\partial v}{\partial x_j}\right),$$

and

$$F_2(v) = \sum_T F_2^T(v), \quad F_2^T(v) = E_T(fv).$$

3 Main result.

Now we prove Theorem 1.4.

Proof. From now on set $e = u - u_h$. Let us consider $y \in \Omega_0$. Let $\Omega_0 \subset\subset \Omega_2 \subset\subset \Omega_1$. By Theorem 1.2 in [9], there exists a C independent of y such that

$$\begin{aligned} |e(y)| + |\nabla e(y)| \le C\left(\log\frac{1}{h}\right)^{\bar{\bar{s}}} \inf_\chi ||u-\chi||_{W^1_\infty(\Omega_2),y,s} \\ + C||e||_{H^{-t}(\Omega_2)} + C\left(\log\frac{1}{h}\right)|||F|||_{-1,\Omega_2} \end{aligned} \tag{3.1}$$

where $\bar{\bar{s}} = 0$ if $0 \le s < r-1$ and $\bar{\bar{s}} = 1$ if $s = r-1$. Here

$$|||F|||_{-1,G} = \sup_{\substack{\psi \in \mathring{W}^1_1(G) \\ ||\psi||_{W^1_1(G)}=1}} F(\psi).$$

First we multiply (3.1) by $\sigma_x^s(y)$, and take the supremum over $y \in \Omega_0$. Then, by noting that $\sigma_x(y)\sigma_y(z) \le 2\sigma_x(z)$ and $\sigma_x(y) \le 1$, we obtain

$$\begin{aligned} ||e||_{W^1_\infty(\Omega_0),x,s} \le C\left(\log\frac{1}{h}\right)^{\bar{\bar{s}}} \inf_\chi ||u-\chi||_{W^1_\infty(\Omega_2),x,s} \\ + C||e||_{H^{-t}(\Omega_2)} + C\left(\log\frac{1}{h}\right)|||F|||_{-1,\Omega_2}. \end{aligned} \tag{3.2}$$

By using the Bramble-Hilbert lemma (see Corollary 5.1 in [12]), we see that

$$|||F_1|||_{-1,\Omega_2} \leq Ch^{r-1+q}||u_h||_{W_\infty^{r-1,h}(\Omega_2)}.$$

The broken norm is defined as $||v||_{W_\infty^{r-1,h}(G)} = \sup_T ||v||_{W_\infty^{r-1}(T\cap G)}$ for $G \subset \Omega$.

A slight modification of Theorem 4.1.5 in [4] (which uses the Bramble-Hilbert lemma) shows that

$$|||F_2|||_{-1,\Omega_2} \leq Ch^{r-1+q}||f||_{W_\infty^{r-1+q}(\Omega_2)}.$$

Therefore, we have that

$$|||F|||_{-1,\Omega_2} \leq h^{r-1+q}(||u_h||_{W_\infty^{r-1,h}(\Omega_2)} + ||f||_{W_\infty^{r-1+q}(\Omega_2)}). \tag{3.3}$$

By the triangle inequality and inverse estimates, we get

$$||u_h||_{W_\infty^{r-1,h}(\Omega_2)} \leq Ch^{2-r}||e||_{W_\infty^1(\Omega_2)} + C||u||_{W_\infty^r(\Omega_2)}. \tag{3.4}$$

After observing that $h^s \leq C\sigma_x^s(z)$ for $z \in \Omega_2$, and combining (3.2), (3.3) and (3.4), we find that for all M

$$\begin{aligned} ||e||_{W_\infty^1(\Omega_0),x,s} \leq\, & C\left(\log\frac{1}{h}\right)^{\bar{\bar{s}}} \inf_\chi ||u-\chi||_{W_\infty^1(\Omega_2),x,s} + C||e||_{H^{-t}(\Omega_2)} \\ & + C\left(\log\frac{1}{h}\right)h^{r-1+q}(||u||_{W_\infty^r(\Omega_2)} + ||f||_{W_\infty^{r-1+q}(\Omega_2)}) \\ & + C\left(\log\frac{1}{h}\right)h^{1+q-s}||e||_{W_\infty^1(\Omega_2),x,s}. \end{aligned} \tag{3.5}$$

If we apply (3.5) M times on a sequence of nested domains and then apply (3.2) and (3.3), we get that

$$\begin{aligned} ||e||_{W_\infty^1(\Omega_0),x,s} \leq\, & C\left(\log\frac{1}{h}\right)^{\bar{\bar{s}}} \inf_\chi ||u-\chi||_{W_\infty^1(\Omega_1),x,s} + C||e||_{H^{-t}(\Omega_1)} \\ & + C\left(\log\frac{1}{h}\right)h^{r-1+q}(||u||_{W_\infty^1(\Omega_1)} + ||f||_{W_\infty^{r-1+q}(\Omega_2)}) \\ & + C\left(\left(\log\frac{1}{h}\right)h\right)^M ||u_h||_{W_\infty^{r-1,h}(\Omega_1)}. \end{aligned}$$

Applying an inverse estimate, we observe that

$$||u_h||_{W_\infty^{r-1,h}(\Omega_1)} \leq Ch^{-(r-1)-t-N/2}||u_h||_{H^{-t}(\Omega_1)}.$$

By the triangle inequality $||u_h||_{H^{-t}(\Omega_1)} \leq ||e||_{H^{-t}(\Omega_1)} + ||u||_{H^{-t}(\Omega_1)}$. Choosing M large enough we arrive at

$$\begin{aligned} ||u-u_h||_{W_\infty^1(\Omega_0),x,s} \leq\, & C\left(\log\frac{1}{h}\right)^{\bar{\bar{s}}} \inf_\chi ||u-\chi||_{W_\infty^1(\Omega_1),x,s} \\ & + C\left(\log\frac{1}{h}\right)^{r-1+q} (||u||_{W_\infty^r(\Omega_1)} + ||f||_{W_\infty^{r-1+q}(\Omega_1)})h \\ & + C||u-u_h||_{H^{-t}(\Omega_1)}. \end{aligned}$$

Our result now follows by noting that $|\nabla(u-u_h)(x)| \leq ||u-u_h||_{W_\infty^1(\Omega_0),x,s}$. ∎

For various problems we can use standard duality arguments to find bounds for $||u-\bar{u}_h||_{H^{-t}(\Omega_1)}$ which will be better then h^{r-1}. However, we need to keep in mind that u_h is the FEM solution with numerical quadrature. Therefore, in the next section we give an application that guarantees the optimal negative-order norm estimate taking into account numerical quadrature.

4 Negative-order norm estimates with quadrature.

Banerjee and Osborn [3] proved negative-order norm estimates with numerical quadrature in one dimension. We extend their result to a problem on a polygonal domain in two dimensions assuming we have appropriate refinements near the corners. This was done for the L_2-norm in [8]. Our proof follows the same lines.

Let Ω be a polygonal domain. Let $Vtx = x_1, x_2, x_3, \dots, x_q$ be the set of vertices. We introduce some weighted norm spaces that the solution belongs to, as in [2].

Definition 4.1. *Let m be a positive integer, $a \in R$ and define $\rho(x) = dist(x, Vtx)$. Then for $G \subset \Omega$ define the weighted space*

$$K_a^m(G) = \left\{ u \in L_2^{loc}(G),\ \rho^{|\alpha|-a-1} D^\alpha u \in L_2(G) \right\}.$$

This space is equipped with the norm

$$||u||^2_{K_a^m(G)} = \sum_{|\alpha|\le m} ||\rho^{|\alpha|-a-1} D^\alpha u||^2_{L_2(G)}.$$

Now we state a result about existence and uniqueness in plane polygonal domains for (1.1). This is a simple consequence of the results in [7] and [6].

Lemma 4.1. *Let m be a non-negative integer. There exists a $\eta > 0$ such that for every $0 < \beta < \eta$ and every $f \in K^m_{\beta-2}(\Omega)$ there exists a unique $u \in K^{m+2}_\beta(\Omega)$ satisfying* (1.1) *and $u = 0$ on $\partial\Omega$ with the bound*

$$||u||_{K^{m+2}_\beta(\Omega)} \le C||f||_{K^m_{\beta-2}(\Omega)}$$

where C is independent of f and u.

Proof. Following a similar argument as was done for Laplace's equation in Theorem 2.6.1 in [7], we have that there exists a $\eta > 0$ such that for every $|\beta| < \eta$ and $f \in K^m_{\beta-2}(\Omega)$ there exists a $u \in K^{m+2}_\beta(\Omega)$. By Theorem 1.4.1 in [7] we have that there exists a C independent of u and f such that

$$||u||_{K^{m+2}_\beta(\Omega)} \le C(||f||_{K^m_{\beta-2}(\Omega)} + ||u||_{L_2(\Omega)}).$$

Using the weak form of the PDE and the uniform ellipticity condition we have

$$||\nabla u||^2_{L_2(\Omega)} \le C \int_\Omega |fu| dx \le C \left(\int_\Omega \rho^2 f^2 dx \right)^{1/2} \left(\int_\Omega \rho^{-2} u dx \right)^{1/2}.$$

Since $u \in \mathring{H}^1(\Omega)$, we have by Lemma 6.6.1 in [6] that

$$\left(\int_\Omega \rho^{-2} u^2 dx\right)^{1/2} \leq C||\nabla u||_{L_2(\Omega)}.$$

Furthermore, since $\beta > 0$, we have that $(\int_\Omega \rho^2 f^2 dx)^{1/2} \leq C(\int_\Omega \rho^{2(1-\beta)} f^2)^{1/2} \leq C||f||_{K^m_{\beta-2}(\Omega)}$. This shows that $||\nabla u||_{L_2(\Omega)} \leq C||f||_{K^m_{\beta-2}(\Omega)}$. The result now follows since $||u||_{L_2(\Omega)} \leq C||\nabla u||_{L_2(\Omega)}$. □

If we are solving Laplace's equation, then $\eta = \frac{\pi}{\alpha}$ where α is the largest interior angle. More generally, η is a computable number which depends on the local frozen coefficient problems on each vertex. One can prove a more precise statement. In that case, one would have to define a norm that is weighted differently near each vertex. For simplicity we considered the present setting.

For the following we choose $\beta \leq 1$ and, of course, $0 < \beta < \eta$. Now we use the mesh refinement condition in [1], [8] and [2]. Let h_T be the mesh size of the element T, set $h = \max_T h_T$, and $d_T = dist(T, Vtx)$. Then we require

$$h_T \leq \begin{cases} Chd_T^{((r-1)-\beta)/(r-1)} & \text{if } d_T > 0 \\ Ch^{(r-1)/\beta} & \text{if } d_T = 0. \end{cases}$$

We let S^h_k denote the Lagrange finite element space of order k on Ω. We can show as in [8] that the following lemma holds.

Lemma 4.2. *Let $w \in K^m_\beta(\Omega)$. If $k \geq m-1$ we have*

$$||\nabla(w - w_I)||_{L_2(\Omega)} \leq Ch^{m-1}||w||_{K^m_\beta(\Omega)} \tag{4.1}$$

where $w_I \in S^h_k$ is the continuous interpolant of w.

By the work in [8] we have the following.

Lemma 4.3. *Let $u_h \in S^h_{r-1}$ be our FEM approximation with quadrature of order at least $2(r-1)-2$. Then*

$$||\nabla(u - u_h)||_{L_2(\Omega)} \leq Ch^{r-1}||u||_{K^r_\beta(\Omega)}.$$

This next lemma corresponds to Lemma 6.2 in [3]. We give a proof since it is slightly different.

Lemma 4.4. *Suppose that we are using a quadrature rule that is of order $r-2+q$ and l is chosen such that $r-1+q > 2/l$. If $v \in P_q(T)$, then*

$$\left|F^T_2(v)\right| \leq meas(T)^{1/l-1/2} h_T^{r-1+q}||f||_{W_l^{r-1+q}(T)}||v||_{H^q(T)}.$$

Here $P_q(T)$ denotes the space of polynomials of degree less than or equal to q.

Proof. We have

$$F^T_2(v) = E_T(fv) = J(R_T)E(\hat{f}\hat{v}) \tag{4.2}$$

where $\hat{T}$ is the reference element and R_T is the affine map from $\hat{T}$ to T.

For $\hat{\psi} \in W_l^{r-1+q}(\hat{T})$, we then have

$$E_{\hat{T}}(\hat{\psi}) \leq C|\hat{\psi}|_{L_\infty(\hat{T})} \leq C||\hat{\psi}||_{W_l^{r-1+q}(\hat{T})}$$

where we used imbedding theorems in the last inequality. By the Bramble-Hilbert lemma, we have

$$E_{\hat{T}}(\hat{\psi}) \leq C|\hat{\psi}|_{W_l^{r-1+q}(\hat{T})}.$$

Setting $\hat{\psi} = \hat{f}\hat{v}$, we get

$$E_{\hat{T}}(\hat{f}\hat{v}) \leq C(|\hat{f}|_{W_l^{r-1+q}(\hat{T})}|\hat{v}|_{L_\infty(\hat{T})} + ... + |\hat{f}|_{W_l^{r-1}(\hat{T})}|\hat{v}|_{W^q_\infty(\hat{T})}).$$

If we use the equivalence of norms in finite dimensional space, we obtain

$$E_{\hat{T}}(\hat{f}\hat{v}) \leq C(|\hat{f}|_{W_l^{r-1+q}(\hat{T})}|\hat{v}|_{L_2(\hat{T})} + ... + |\hat{f}|_{W_l^{r-1}(\hat{T})}|\hat{v}|_{H^q(\hat{T})}).$$

Scaling back to the physical element we get that

$$\begin{aligned} &E_{\hat{T}}(\hat{f}\hat{v}) \\ &\leq Ch_T^{r-1+q}J(R_T)^{-1/2-1/l}(|f|_{W_l^{r-1+q}(T)}|v|_{L_2(T)} + ... + |f|_{W_l^{r-1}(T)}|v|_{H^q(T)}). \end{aligned}$$

After using (4.2) we arrive at our result. □

Following similar arguments we can bound F_1^T (see Lemma 6.1 in [3]).

LEMMA 4.5. *Suppose that we are using a quadrature rule of order $r-2+q$. If $v \in P_q(T)$ then*

$$F_1^T(v) \leq Ch_T^{r-1+q}||u_h||_{H^{r-1}(T)}||v||_{H^q(T)}.$$

Now we can state and prove our main result of this section.

THEOREM 4.6. *Let u solve* (1.1) *with $u = 0$ on $\partial\Omega$. Let $u_h \in S^h_{r-1}$ be the FEM solution with a quadrature rule of order* $\max(2(r-1)-2, r-2+q)$ *with $1 \leq q \leq r-1$. Then*

$$||u-u_h||_{H^{-(q-1)}(\Omega)} \leq Ch^{r-1+q}. \tag{4.3}$$

PROOF. We know by a duality argument (see Problem 4.1.3 [4])

$$||u-u_h||_{H^{-(q-1)}(\Omega)} \leq C \sup_{\substack{g\in H^{q-1}(\Omega) \\ ||g||_{H^{q-1}(\Omega)}=1}} (||\nabla(u-u_h)||_{L_2(\Omega)}||\nabla(\phi-\phi_I)||_{L_2(\Omega)} + F(\phi_I))$$

where ϕ satisfies $L\phi = g$ and vanishes on the boundary and $\phi_I \in S^h_q$ is the continuous interpolant of ϕ. By Lemma 4.2, Lemma 4.1 and the fact that $||g||_{K^{q-1}_{\beta-2}(\Omega)} \leq ||g||_{H^{q-1}(\Omega)}$, we observe that

$$||\nabla(\phi-\phi_I)||_{L_2(\Omega)} \leq Ch^q. \tag{4.4}$$

Therefore, after using this fact and Lemma 4.3, we have that

$$||u - u_h||_{H^{-(q-1)}(\Omega)} \leq Ch^{r-1+q} + C \sup_{\substack{g \in H^{q-1}(\Omega) \\ ||g||_{H^{q-1}(\Omega)}=1}} F(\phi_I).$$

We first bound F_2. By Lemma 4.4 we have

$$F_2(\phi_I) \leq \sum_T h_T^{r-1+q} ||\phi_I||_{H^q(T)} ||f||_{W_l^{r-1+q}(T)} meas(T)^{1/l-1/2}.$$

For $d_T > 0$, using approximation properties of ϕ_I and the definition of h_T, we get

$$h_T^{r-1+q} ||\phi_I||_{H^q(T)} \leq h^{r-1+q} d_T^{(r-1-\beta)(1+q/(r-1))} ||\phi||_{H^{q+1}(T)}.$$

It is clear that $q - \beta \leq (r-1-\beta)(1+q/(r-1))$. Since $d_T \leq \rho(x) \ \ \forall x \in T$, we have

$$h_T^{r-1+q} ||\phi_I||_{H^q(T)} \leq h^{r-1+q} ||\phi||_{K_\beta^{q+1}(T)}.$$

Now assume $d_T = 0$. One can show that $||\phi - \phi_I||_{H^1(T)} \leq ||\phi||_{W_1^2(T)}$ (see [11]). Also, since $d_T = 0$ we have that $||\phi||_{W_1^2(T)} \leq h_T^\beta ||\phi||_{K_\beta^2(T)}$. Therefore, using these inequalities, an inverse inequality and the triangle inequality, we get

$$h_T^{r-1+q} ||\phi_I||_{H^q(T)} \leq Ch_T^r ||\phi||_{K_\beta^2(T)}.$$

Since $h_T \leq h^{(r-1)/\beta} \leq h^{r-1} (\beta \leq 1)$, we have that

$$h_T^{r-1+q} ||\phi_I||_{H^q(T)} \leq h^{(r-1)+q} ||\phi||_{K_\beta^{q+1}(T)}$$

where we have used that $1 \leq q \leq r-1$ and $r \geq 2$. Finally, using the generalized Hölder inequality, we get that

$$F_2(\phi_I) \leq h^{r-1+q} ||\phi||_{K_\beta^{q+1}(\Omega)} ||f||_{W_l^{r-1+q}(\Omega)} meas(\Omega)^{1/2-1/l}. \tag{4.5}$$

Now we bound $F_1(\phi_I)$. Using Lemma 4.5

$$F_1(\phi_I) \leq \sum_T h_T^{r-1+q} ||u_h||_{H^{r-1}(T)} ||\phi_I||_{H^q(T)}.$$

We employ the triangle inequality to get

$$\begin{aligned} F_1(\phi_I) \leq & \sum_T h_T^{r-1+q} ||u_I||_{H^{r-1}(T)} ||\phi_I||_{H^q(T)} \\ & + \sum_T h_T^{r-1+q} ||u_h - u_I||_{H^{r-1}(T)} ||\phi_I||_{H^q(T)}. \end{aligned}$$

Using inverse estimates, the triangle inequality and Lemmas 4.2 and 4.3, we get

$$\begin{aligned}\sum_T h_T^{r-1+q} || u_h - u_I||_{H^{r-1}(T)} ||\phi_I||_{H^q(T)} &\le C \sum_T h_T^{1+q} ||u_h - u_I||_{H^1(T)} ||\phi_I||_{H^q(T)} \\ &\le C||u_h - u_I||_{H^1(\Omega)} \left(\sum_T \left(h_T^{1+q} ||\phi_I||_{H^q(T)}\right)^2\right)^{1/2} \\ &\le Ch^{r-1} \left(\sum_T \left(h_T^{1+q} ||\phi_I||_{H^q(T)}\right)^2\right)^{1/2}.\end{aligned}$$

Now by considering two separate cases ($d_T > 0$ and $d_T = 0$), and using arguments as above in bounding F_2, we get

$$\left(\sum_T \left(h_T^{1+q} ||\phi_I||_{H^q(T)}\right)^2\right)^{1/2} \le Ch^{1+q} ||\phi||_{K_\beta^{q+1}(\Omega)}.$$

Therefore, we have

$$\sum_T h_T^{r-1+q} ||u_h - u_I||_{H^{r-1}(T)} ||\phi_I||_{H^q(T)} \le Ch^{r+q} ||\phi||_{K_\beta^{q+1}(\Omega)}.$$

Next, we bound $\sum_T h_T^{r-1+q} ||u_I||_{H^{r-1}(T)} ||\phi_I||_{H^q(T)}$.

If $d_T > 0$,

$$\begin{aligned} h_T^{r-1+q} || u_I||_{H^{r-1}(T)} ||\phi_I||_{H^q(T)} &\le Ch_T^{r-1+q} ||u||_{H^r(T)} ||\phi||_{H^{q+1}(T)} \\ &\le h^{r-1+q} d_T^{r-1-\beta} ||u||_{H^r(T)} d_T^{q(r-1-\beta)/(r-1)} ||\phi||_{H^{q+1}(T)} \\ &\le h^{r-1+q} ||u||_{K_\beta^r(T)} ||\phi||_{K_\beta^{q+1}(T)}.\end{aligned}$$

In the first inequality we used approximation properties of u_I and ϕ_I. In the second inequality we used the definition of h_T. Finally, in the third inequality we used that $q(r-1-\beta)/(r-1) \ge q - \beta$.

If $d_T = 0$,

$$\begin{aligned} h_T^{r-1+q} || u_I||_{H^{r-1}(T)} ||\phi_I||_{H^q(T)} &\le h_T^2 ||u_I||_{H^1(T)} ||\phi_I||_{H^1(T)} \\ &\le h_T^2 ||u||_{K_\beta^2(T)} ||\phi||_{K_\beta^2(T)} \\ &\le h^{2(r-1)/\beta} ||u||_{K_\beta^2(T)} ||\phi||_{K_\beta^2(T)} \\ &\le h^{r-1+q} ||u||_{K_\beta^r(T)} ||\phi||_{K_\beta^{q+1}(T)}.\end{aligned}$$

In the first inequality we used an inverse estimate. For the second inequality we used an argument as was done to bound F_2. In the third inequality we used the definition of h_T. We used that $1 \le q \le r-1$, $r \ge 2$ and $\beta \le 1$ in the last inequality.

Therefore, we have that

$$\sum_T h_T^{r-1+q} ||u_I||_{H^{r-1}(T)} ||\phi_I||_{H^q(T)} \le h^{r-1+q} ||u||_{K^r_\beta(\Omega)} ||\phi||_{K^{q+1}_\beta(\Omega)}.$$

We conclude that

$$F_1(\phi_I) \le Ch^{r-1+q} ||\phi||_{K^{q+1}_\beta(\Omega)}. \tag{4.6}$$

Finally, using (4.5), (4.6) and Lemma 4.1 we arrive at our conclusion. □

5 Sharpness of result.

In order to prove the sharpness of Theorem 1.4, we need to state a corollary to this result with $q = s$ (see [9]).

Corollary 5.1. *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and let $x \in \Omega_0$. Let u solve* (1.2) *and let u_h satisfy* (2.1) *where we use a quadrature rule of order* $2(r-1)-2+s$. *Let $\gamma \le r-1+s$. Suppose that $\sum_{r \le |\alpha| \le \gamma} |D^\alpha u(x)| = 0$, then*

$$|\nabla(u-u_h)(x)| \le C \log\left(\frac{1}{h}\right) h^\gamma \tag{5.1}$$

provided that

$$||u-u_h||_{H^{-t}(\Omega_1)} \le C_1 h^\gamma \quad \textit{for some } t. \tag{5.2}$$

Here C is independent of h, x, u, and u_h.

Let now $\Omega = (-1, 1)$ and consider the problem

$$\begin{aligned} -((x^{r-1+s-1}+2)u'(x))' &= f(x) \quad x \in \Omega, \\ u(-1) = u'(1) &= 0. \end{aligned} \tag{5.3}$$

Suppose that u is a linear function with slope one in an interval I containing $x = 0$. Suppose also that we have a uniform mesh of mesh size h and that $x = 0$ is always a mesh point. Suppose further that we are using elements of polynomial order $r-1$ to approximate u. Let us first assume that we use a quadrature rule of order $2(r-1)-2+s$ with $1 \le s \le r-1$. For this problem we can easily show that $||u-u_h||_{H^{-(s-1)}(\Omega)} \le Ch^{r-1+s}$. As we have shown in higher dimensions, Corollary 5.1, we have superconvergence on I. More precisely, $||(u-u_h)'||_{L_\infty(I)} \le C \log(1/h) h^{r-1+s}$.

However, as we shall now show, if we use a quadrature rule of order $2(r-1)-2+s-1$ then we no longer have a superconvergence result of this order. This would show that are results are sharp.

For simplicity let us suppose that we integrate the right hand side ($\int_\Omega fvdx$) exactly. Suppose we use a quadrature rule of order $2(r-1)-2+s-1$ for the left hand side. We show that the error in I can not be of order h^γ if $\gamma > 2(r-1)-2+s-1$. To this end, let $T=(0,h)$. We conveniently choose a continuous v in the following way: $v(x)=0$ if $x<0$, $v(x)=1$ if $x>h$ and $v(x)=(x/h)^{r-1}$ on T. Since $v' \equiv 0$ outside of T,

$$Q_T(au_h'v') = \int_\Omega fvdx$$

where $a(x) = x^{r-1+s-1}+2$. Of course, the exact solution will satisfy

$$\int_T au'v'dx = \int_\Omega fvdx.$$

Therefore, for this v, we have the relationship

$$\int_T au'v'dx - Q_T(au'v') = Q_T(a(u_h-u)'v'). \tag{5.4}$$

Now we investigate the left hand side of (5.4). Note that $\int_T 2u'v' = Q_T(2u'v')$ since $2u'v'$ is polynomial of degree $r-2 \leq 2(r-1)-2+s-1$ on T. Since $u'(x)=1$ and $v'(x)=(r-1)(1/h)(x/h)^{r-2}$, we get after a change of variables that

$$\begin{aligned}&\int_T au'v'dx - Q_T(au'v') \\ &\qquad = (r-1)h^{r-1+s-1}\left(\int_0^1 \hat{x}^{2(r-1)-2+s}d\hat{x} - Q(\hat{x}^{2(r-1)-2+s})\right).\end{aligned}$$

Of course, since we are using a quadrature rule of order $2(r-1)-2+s-1$, we have that

$$\int_0^1 \hat{x}^{2(r-1)-2+s}d\hat{x} - Q(\hat{x}^{2(r-1)-2+s}) = C_2 \neq 0.$$

Therefore, for the left hand side in (5.4),

$$\int_T au'v'dx - Q_T(au'v') = C_2(r-1)h^{r-1+s-1}.$$

On the other hand, if $||(u-u_h)'||_{L_\infty(T)} \leq Ch^\gamma$ for $\gamma > r-1+s-1$, then for the right hand side in (5.4),

$$Q_T(a(u_h-u)'v') \leq Ch^\gamma ||av'||_{L_\infty(T)} Q_T(1) \leq Ch^\gamma.$$

Which leads to a contradiction. Therefore, $(u-u_h)'$ is at most $O(h^{r-1+s-1})$ on I. This, of course, shows that Corollary 5.1 is sharp, and in turn, implies that Theorem 1.4 is sharp.

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REFERENCES

1. I. Babuška, *Finite element method for domains with corners*, Computing, 6 (1970), pp. 264–273.
2. C. Bacuta, V. Nistor and L. T. Zikatanov, *Improving the rate of convergence of 'higher order finite elements' on polygons and domains with cusps*, Numer. Math., 100 (2005), pp. 165–184.
3. U. Banerjee and J. E. Osborn, *Estimation of the effect of numerical integration in finite element eigenvalue approximation*, Numer. Math., 56 (1990), pp. 735–762.
4. P. G. Ciarlet, *The Finite Element Methods for Elliptic Problems*, North-Holland, Amsterdam, 1978.
5. W. Hoffmann, A. H. Schatz, L. B. Wahlbin and G. Wittum, *Asymptotically exact a posteriori error estimators for the pointwise gradient error on each element on irregular grids. I. A smooth problem and globally quasi-uniform meshes*. Math. Comput., 70 (2001), pp. 897–909.
6. V. A. Kozlov, V. G. Maz'ya and J. Rossman, *Elliptic Boundary Value Problems in Domains with Point Singularities*, Mathematical Surveys and Monographs, vol. 52, Amer. Math. Soc., Providence, Rhode Island, 1997.
7. V. A. Kozlov, V. G. Maz'ya and J. Rossman. *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations*, Mathematical Surveys and Monographs, vol. 85, Amer. Math. Soc., Providence, Rhode Island, 2000.
8. G. Raugel, *Rèsolution Numèrique de Problémes Elliptiques Dans des Domaines Avec Coins*, Thesis, University of Rennes, 1978.
9. A. H. Schatz, *Pointwise error estimates and asymptotic expansion inequalities for the finite element method on irregular grids: Part II. Interior estimates*, SIAM J. Numer. Anal., 38 (2000), pp. 1269–1293.
10. A. H. Schatz, *Perturbation of forms and error estimates for the finite element method at a point, with an application to improved superconvergence error estimates for subspaces that are symmetric with respect to a point*, SIAM J. Numer. Anal., 42 (2005), pp. 2342–2365.
11. A. H. Schatz, V. Thomée and W. L. Wendland, *Mathematical Theory of Finite and Boundary Element Methods*, Birkhäuser, Basel, 1990.
12. A. H. Schatz and L. B. Wahlbin, *Interior maximum-norm estimates for finite element methods, Part II*, Math. Comput., 64 (1995), pp. 907–928.