Homework #10 Solutions

#10.3.3

Analyze the long-term behavior of the map $x_{n+1} = \frac{r x_n}{(1+x_n^2)}$, where $r>0$.

Solution:

The fixed points are given by solutions of the equation

$$x = \frac{rx}{1+x^2}$$

$$\Rightarrow x_1^* = 0 \quad \text{or} \quad x_{2,3}^* = \pm\sqrt{r-1}.$$

Therefore, if $r>1$ we have three fixed points otherwise there are two.
Let $f(x) = \frac{rx}{1+x^2}$ we have that

$$f'(0) = r \quad \text{and} \quad f'(\pm\sqrt{r-1}) = \frac{2}{r} - 1.$$

Therefore, the fixed point 0 is unstable if and only if $r>1$ while the points $x_{2,3}^* = \pm\sqrt{r-1}$ are stable if and only if $r>1$.

We will analyze the existence of non-trivial periodic orbits in two cases.

Case 1:

If $r \leq 1$ then $|f(x)| = \left|\frac{rx}{1+x^2}\right| \leq r|x| \leq |x|$, with equality if and only if $x=0$. Therefore,

$$|f^n(x)| = |f(f^{n-1}(x))| \leq |f^{n-1}(x)|$$

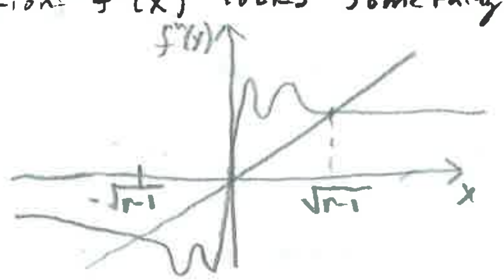
Continuing by induction we have $\forall x \neq 0$ that

$$|f^n(x)| < |x|.$$

Therefore, if $r<1$ there are no periodic orbits. However, since $x_n = f^n(x_0)$ satisfies $|x_n| < |x_{n+1}|$ it follows that all sequences are monotone decreasing. Consequently $\lim_{n\to\infty} x_n = 0$. Therefore, there is no Chaos in this regime.

Case 2:

If $r>1$, the nonexistence of periodic orbits is less clear. Graphically, the function $f^n(x)$ looks something like:



Therefore no periodic orbits exist.

10.3.4

Consider the quadratic map $x_{n+1} = x_n^2 + c$.

a.) Find and classify all the fixed points as a function of c.

Solution:

Solving the equation

$$x^2 - x + c = 0$$

we have that fixed points correspond to:

$$x^*_{1,2} = \frac{1 \pm \sqrt{1-4c}}{2}.$$

These points exist if and only if $1-4c>0 \Rightarrow c < 1/4$.

Letting $f(x) = x^2 + c$ we have that:

$$f'(x^*_{1,2}) = 1 \pm \sqrt{1-4c}$$

Therefore, when these points exist one is stable and the other is unstable.

b.) Find the values of c at which the fixed points bifurcate.

Solution:

At $c = 1/4$ there is a saddle-node bifurcation.

c.) For what values of c is there a stable 2-cycle?

Solution:

$f(f(x)) = (x^2+c)^2 + c = x^4 + 2x^2c + c^2 + c$. Dividing, we have that:

$$\begin{array}{r|l} & x^2 + x + c + 1 \\ \hline x^2 - x + c & x^4 + 2x^2c - x + c^2 + c \\ & x^4 - x^3 + x^2c \\ \hline & x^3 + x^2c \\ & x^3 - x^2 + cx \\ \hline & (c+1)x^2 - (c+1)x + c(c+1) \end{array}$$

Therefore, period 2-orbits satisfy

$$x^2 + x + c + 1 = 0$$

$$\Rightarrow x^*_{3,4} = \frac{-1 \pm \sqrt{1-4(c+1)}}{2}.$$

Therefore, period 2-cycles exist if $c < -3/4$. To analyze stability we know that

$$\frac{d}{dx}(f(f(x^*_3))) = f'(x^*_4)f'(x^*_3) = (-1+\sqrt{1-4(c+1)})(-1-\sqrt{1-4(c+1)}) = 4(c+1).$$

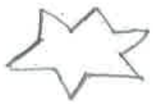
#11.4.1

Find the box dimension of the von-Koch snowflake.

Solution:

S_0:  $N_1 = 3$

$\varepsilon_1 = 1$

S_1:  $N_2 = 12 = 4 \cdot 3$

$\varepsilon_2 = 1/3$

$$\Rightarrow N_n = 3 \cdot (4^n)$$

$$\varepsilon_n = \left(\frac{1}{3}\right)^n$$

Therefore, $d = \lim_{n\to\infty} \ln\left(\frac{3\cdot 4^n}{3^n}\right) = \lim_{n\to\infty}\left(\frac{\ln(3)}{\ln(3^n)} + \frac{\ln(4)}{\ln(3)}\right) = \frac{\ln(4)}{\ln(3)}$

$\blacksquare$

11.4.2

Find the box dimension of the Sierpinski carpet.

Solution:

$$\varepsilon_n = \left(\frac{1}{3}\right)^n$$

$$N_n = (8)^n$$

$$\Rightarrow d = \lim_{n\to\infty} \frac{\ln(8^n)}{\ln(3^n)} = \frac{\ln(8)}{\ln(3)}$$

$\blacksquare$

Therefore, the two-cycle is stable if

$$-\tfrac{5}{4} < c < -\tfrac{3}{4}.$$

#10.3.7

Consider the decimal shift map on $I = [0,1]$ defined by

$$x_{n+1} = 10x_n \pmod 1$$

b.) Find all fixed points

Solution:

Any $x \in I$ with decimal expansion $x = .aaaa\ldots$, wher $a \in \{0,1,\ldots,9\}$.

c.) Show that the map has periodic orbits of all periods, but all of them are unstable.

Solution:

Any $x \in I$ with a repeating decimal expansion is a periodic orbit. Let $f(x) = 10x \bmod 1$. Since $f'(x) = 10$ it follows that all periodic orbits are unstable.

d.) Show that the map has infinitely many aperiodic orbits.

Solution:

The periodic orbits correspond to $\mathbb{Q} \cap I$. Consequently, the aperiodic orbits correspond to $(\mathbb{R} \setminus \mathbb{Q}) \cap I$, i.e. the set of irrational numbers in $[0,1]$ which is infinite.

e.) Show that the map has sensitive dependence on initial conditions.

Solution:

The Lyaponov exponent is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln(10) = \ln(10).$$