Homework #9 Solutions

#9.2.1

a.) For the Lorenz equations, show that the characteristic equation for the eigenvalues of the Jacobian matrix at C^+, C^- is
$$\lambda^3+(\sigma+b+1)\lambda^2+(r+\sigma)b\lambda+2b\sigma(r-1)=0.$$

Solution:

The Lorenz equations are given by:
$$\begin{cases}\dot{x}=\sigma(y-x)\\ \dot{y}=rx-y-xz\\ \dot{z}=xy-bz\end{cases}$$

Therefore at the fixed point $C^+=(x^*,y^*,z^*)$ the Jacobian is given by:
$$J(C^+)=\begin{pmatrix}-\sigma & \sigma & 0\\ r-z^* & -1 & -x^*\\ y^* & x^* & -b\end{pmatrix}$$

$$\Rightarrow \det(\lambda I-J)=(\lambda+\sigma)((\lambda+1)(\lambda+b)+x^{*2})+\sigma(-(r-z^*)(\lambda+b)+x^*y^*)$$

Consequently, the characteristic equation is given by:
$$(\lambda+\sigma)(\lambda^2+(b+1)\lambda+b+x^{*2})+\sigma(z^*\lambda-r\lambda-rb+z^*b+x^*y^*)=0$$
$$\Rightarrow \lambda^3+(b+1)\lambda^2+(b+x^{*2})\lambda+\sigma\lambda^2+\sigma(b+1)\lambda+\sigma(b+x^{*2})$$
$$+\sigma z^*\lambda-\sigma r\lambda-\sigma rb+\sigma bz^*+\sigma x^*y^*=0$$
$$\Rightarrow \lambda^3+(\sigma+b+1)\lambda^2+(b+x^{*2}+\sigma b+\sigma+\sigma z^*-\sigma r)\lambda+\sigma(b+x^{*2})-\sigma rb+\sigma bz^*$$
$$+\sigma x^*y^*=0.$$

Substituting the values of x^*, y^*, z^* we have that
$$\lambda^3+(\sigma+b+1)\lambda^2+(b+br-b+\sigma b+\sigma+\sigma r-\sigma-\sigma r)\lambda+\sigma b+\sigma br-\sigma b-\sigma rb$$
$$+\sigma br-\sigma b+\sigma br-\sigma b=0$$
$$\Rightarrow \lambda^3+(\sigma+b+1)\lambda^2+(r+\sigma)b\lambda+2\sigma b(r-1)=0.$$

b.) By seeking solutions of the form $\lambda=i\omega$, where $\omega\in\mathbb{R}$, show that there is a pair of pure imaginary eigenvalues when $r=r_H=\sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right)$. Explain why we need to assume $\sigma>b+1$.

Solution:

Letting $\lambda=i\omega$ we have that:
$$-i\omega^3-(\sigma+b+1)\omega^2+i(r+\sigma)b\omega+2\sigma b(r-1)=0$$
$$\Rightarrow \omega^2=(r+\sigma)b \quad\text{and}\quad (\sigma+b+1)\omega^2=2\sigma b(r-1)$$
$$\Rightarrow (\sigma+b+1)(r+\sigma)b=2\sigma b(r-1) \Rightarrow r=\sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right).$$

c.) Find the third eigenvalue.

Solution:

Dividing we have that

$$\begin{array}{r} \lambda + (\sigma+b+1) \\ \lambda^2+w^2 \overline{\smash{\big)}\, \lambda^3+(\sigma+b+1)\lambda^2+(r+\sigma)b\lambda+2\sigma b(r-1)} \\ \underline{\lambda^3 + 0\lambda^2 + w^2\lambda} \\ (\sigma+b+1)\lambda^2+[(r+\sigma)b\lambda-w^2]\lambda+2\sigma b(r-1) \\ \underline{(\sigma+b+1)\lambda^2 + 0\lambda + (\sigma+b+1)} \\ 0 \end{array}$$

$$\Rightarrow \lambda+(\sigma+b+1)=0$$

$$\Rightarrow \lambda=-(\sigma+b+1)$$

#9.2.4

Show that the z-axis is an invariant line for the Lorenz equations.

Solution:

Suppose $x_0, y_0 = 0$ then the solution to the Lorenz equations with initial conditions $x(0)=x_0, y(0)=y_0, z(0)=z_0$ satisfies:

$$\dot{x}(t)=0$$
$$y(t)=0$$
$$z(t)=z_0 e^{-bt}$$

#9.3.8

Consider the following system in polar coordinates

$$\dot{r}=r(1-r^2)$$
$$\dot{\theta}=1$$

Let D be the disk $x^2+y^2 \leq 1$.

a.) Is D an invariant set.

Solution:

Yes.

b.) Does D attract an open set of intial conditions?

Solution:

Yes.

c.) Is D an attractor?

Solution:

No, it is not a minimal set.

d.) Repeat part C for the circle $x^2+y^2=1$.

Solution:

Yes, its basin of attraction is all of $\mathbb{R}^2$.

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#9.4.2

Consider the map

$$x_{n+1} = \begin{cases} 2x_n, & 0 \le x_n \le \frac{1}{2} \\ 2-2x_n & \frac{1}{2} \le x_n \le 1 \end{cases}$$

b.) Find all fixed points and classify there stability

Solution:

Fixed points are $x=0, \frac{2}{3}$ which are both unstable.

c.) Show that the map has a period-2 orbit. Is it stable or unstable.

Solution:

Period n-orbits correspond to fixed points of the map graphed below:

Clearly, there exist an infinite number of periodic orbits but they are all unstable.

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