Homework #8 Solutions

#8.1.5

Prove that at any zero-eigenvalue bifurcation in two dimensions, the null-clines always intersect tangentially.

proof:

Consider the system

$$\dot{x} = f(x, y, \nu)$$
$$\dot{y} = g(x, y, \nu)$$

and assume that a zero-eigenvalue bifurcation occurs at ν^* with fixed point (x^*, y^*). Therefore,

$$\det(J(x^*, y^*)) = f_x g_y - f_y g_x = 0$$
$$\Rightarrow \nabla f \cdot (g_y, -g_x) = 0$$

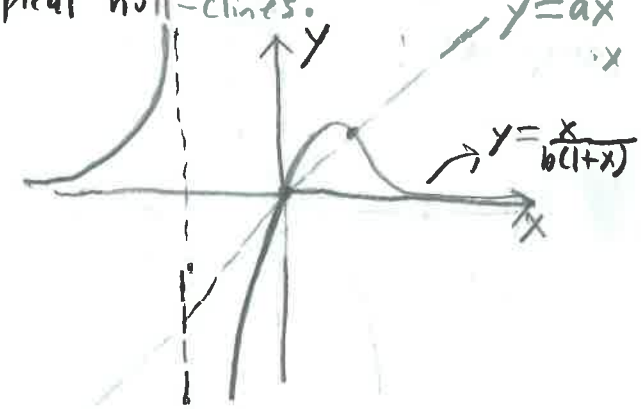
Therefore, ∇f and $(g_y, -g_x)$ are orthogonal which implies ∇f and ∇g are parallel at (x^*, y^*). Consequently, since ∇f and ∇g are the outward normals to the contours of f and g it follows that null-clines intersect tangentially.

#8.1.7

Find and classify all bifurcations for the system $\dot{x} = y - ax$, $\dot{y} = -by + \frac{x}{1+x}$.

Solution:

The fixed points for this system are $(0,0)$ and $\left(\frac{1-ab}{ab}, \frac{1-ab}{b}\right)$. Lets draw the typical null-clines:



$a>0, b>0$

We can see that the fixed point will pass through the origin so we expect a transcritical bifurcation. $J(0,0) = \begin{pmatrix} -a & 1 \\ 1 & -b \end{pmatrix}$ which has eigenvalues

$$2\lambda_{1,2} = -a-b \pm \sqrt{(a+b)^2 - 4ab + 4}$$
$$= -(a+b) \pm \sqrt{(a-b)^2 + 4}$$

We see that when $ab = 1$ the two fixed points intersect and the origin changes stability from a stable node to a saddle. Therefore, the bifurcation is transcritical at $ab = 1$.

#8.1.11

Show that the system

$$\dot{u}=a(1-u)-uv^2,\quad \dot{v}=uv^2-(a+k)v$$

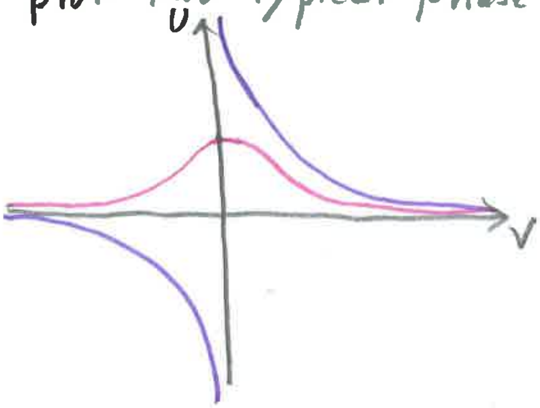
has a saddle node bifurcation at $k=-a\pm\frac{1}{2}\sqrt{a}$.

Solution:

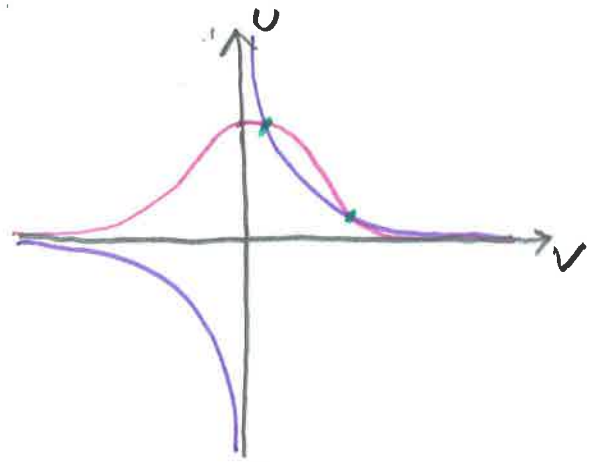
The null-clines for this system are given by

$$u=\frac{a+k}{v} \quad \text{and} \quad u=\frac{a}{a+v^2}$$

Lets plot two typical phase planes.



No fixed points



2 fixed points.

We can see that a saddle-node bifurcation will occur. The v coordinate of the fixed points are given by

$$v=\frac{a\pm\sqrt{a^2-4(k+a)(a^2+ka)}}{2(k+a)},$$

Therefore, the saddle-node bifurcation occurs when

$$a^2-4(k+a)(a^2+ka)=0$$

$$\Rightarrow a-4(k+a)^2=0$$

$$\Rightarrow k=\pm\frac{\sqrt{a}}{2}-a$$

#8.2.1

Consider the biased van der Pol oscillator $\ddot{x}+\mu(x^2-1)\dot{x}+x=a$. Find the curves in (μ, a) space at which a Hopf bifurcation occurs.

Solution:

We rewrite the system as:

$$\dot{x}=v$$
$$\dot{v}=-\mu(x^2-1)v-x+a$$

Which has the fixed point $x=a$, $v=0$. The Jacobian is given by:

$$J(a,0)=\begin{pmatrix} 0 & 1 \\ -a & -\mu(a^2-1) \end{pmatrix}$$

The eigenvalues are given by:

$$2\lambda_{1,2}=-\mu(a^2-1)\pm\sqrt{\mu^2(a^2-1)^2-4a^2}.$$
$$=-\mu(a^2-1)\pm\sqrt{\mu^2a^4-2\mu^2a^2+\mu^2-4a^2}.$$

We can see that a Hopf-bifurcation occurs when $a=\pm 1$ or $\mu=0$.

#8.2.8

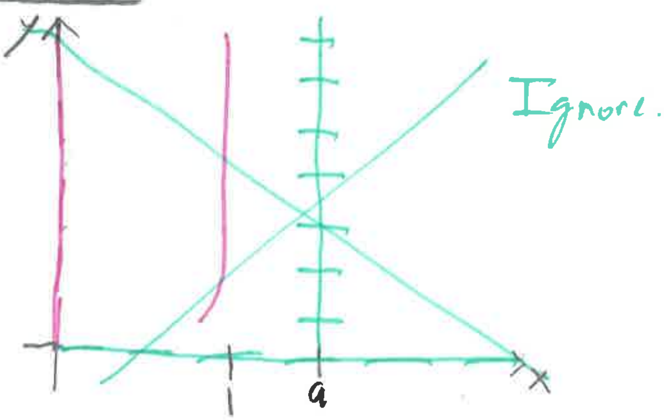
Consider the system

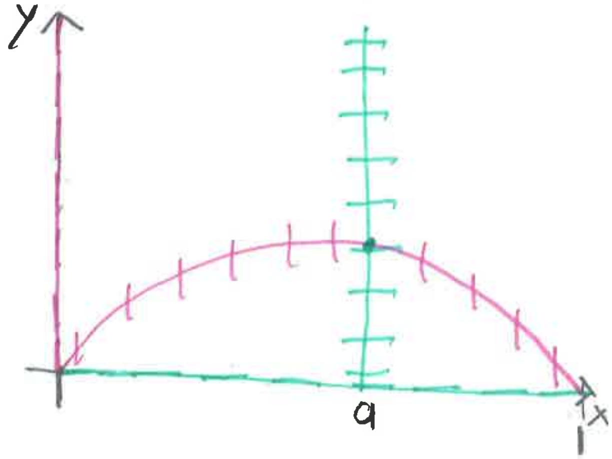
$$\dot{x}=x[x(1-x)-y],$$
$$\dot{y}=y(x-a),$$

where $x,y\geq 0$ and $a\geq 0$.

a) Sketh the null-clines in the first quadrant.

Solution:





b.) Show that the fixed points are $(0,0)$, $(1,0)$ and $(a, a-a^2)$.

Solution:

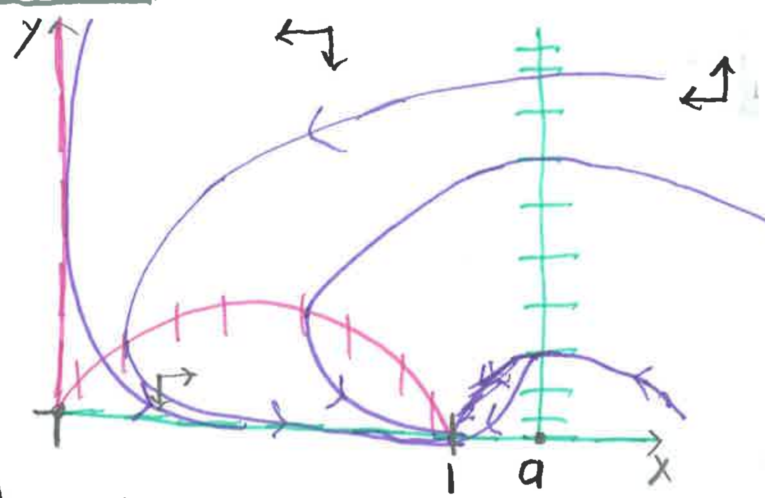
$J(0,0)=\begin{bmatrix}0 & 0\\ 0 & -a\end{bmatrix}$, but we can see from the null-clines that $(0,0)$ is a saddle.

$J(1,0)=\begin{bmatrix}-1 & 1\\ 0 & 1-a\end{bmatrix}$; This point is a saddle if $a<1$ otherwise it is a stable node.

$J(a, a-a^2)=\begin{bmatrix}a-2a^2 & -a\\ a-a^2 & 0\end{bmatrix}$; $2\lambda_{1,2}=a-2a^2 \pm \sqrt{(a-2a^2)^2-4a(a-a^2)} = a-2a^2 \pm \sqrt{a(1-8a)}$.

c. Sketch the phase portrait for $a>1$ and show the predators go extinct.

Solution:



The predators go extinct.

d.) Show that a Hopf bifurcation occurs at $a_c = 1/2$. Is it super or subcritical?

Solution:

The eigenvalues at the fixed point $(a, a-a^2)$ are pure imaginary when $a = 1/2$ so there is a Hopf-bifurcation at $a_c = 1/2$. Clearly this is a super-critical Hopf-bifurcation.

f.) Sketch all the topologically different phase portraits for $a<1$.

Solution: