Homework #6

#7.2.9

For each of the following systems, decide whether it is a gradient system. If so, find V sketh the phase portrait and the equipotentials of V.

a.) $\dot{x}=y+x^2y$, $\dot{y}=-x+2xy$.

Solution:

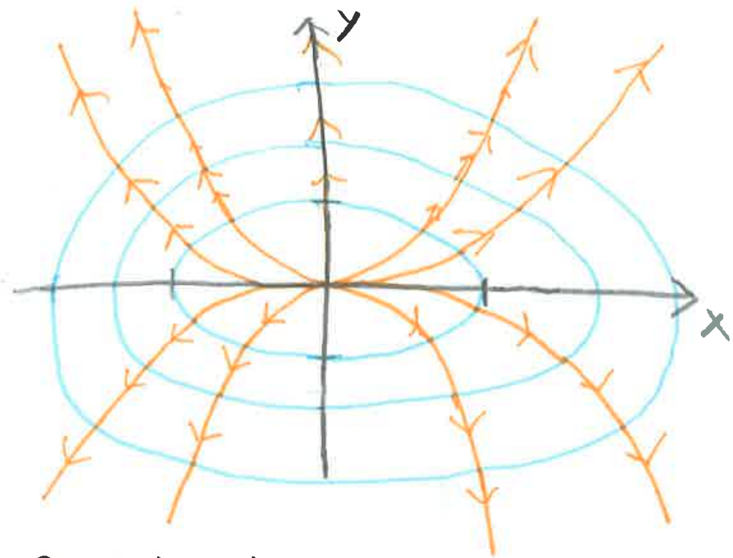
$$\frac{\partial}{\partial y}\dot{x}=1+x^2,\ \frac{\partial}{\partial x}\dot{y}=-1+2x.$$

Therefore, this cannot be a gradient system.

b.) $\dot{x}=2x$, $\dot{y}=8y$

Solution:

This is a gradient system with $V=x^2+4y^2$. The phase portrait is given below:

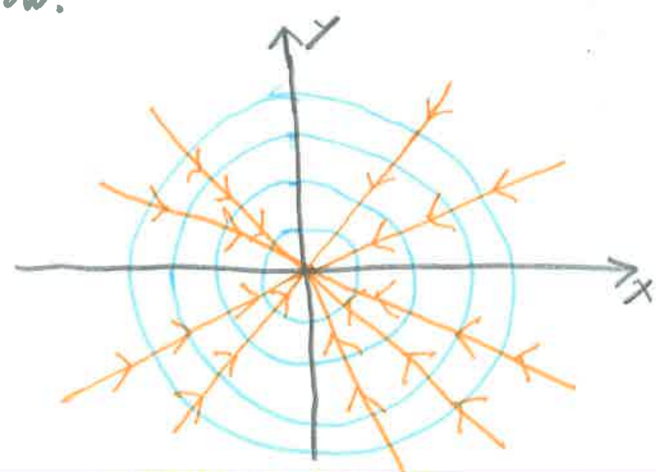


Trajectories are orthogonal to equipotentials.

c.) $\dot{x}=-2xe^{x^2+y^2}$, $\dot{y}=-2ye^{x^2+y^2}$

Solution:

This is a gradient system with $V=-e^{x^2+y^2}$. The phase portrait is graphed below:



#7.2.10

Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.

Solution:

Let $x(t), y(t)$ solve the governing equations and define $V = ax^2 + by^2$. Then,

$$\frac{dV}{dt} = 2ax\dot{x} + 2by\dot{y} = 2xy(a-b) - 2(ax^4 + by^4).$$

Consequently, setting $a = b = 1$ we have that

$$\frac{dV}{dt} = -2(x^4 + y^4) < 0.$$

Therefore, V is a Liapunov function and there can be no closed orbits. ■

#7.3.3

Show that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution.

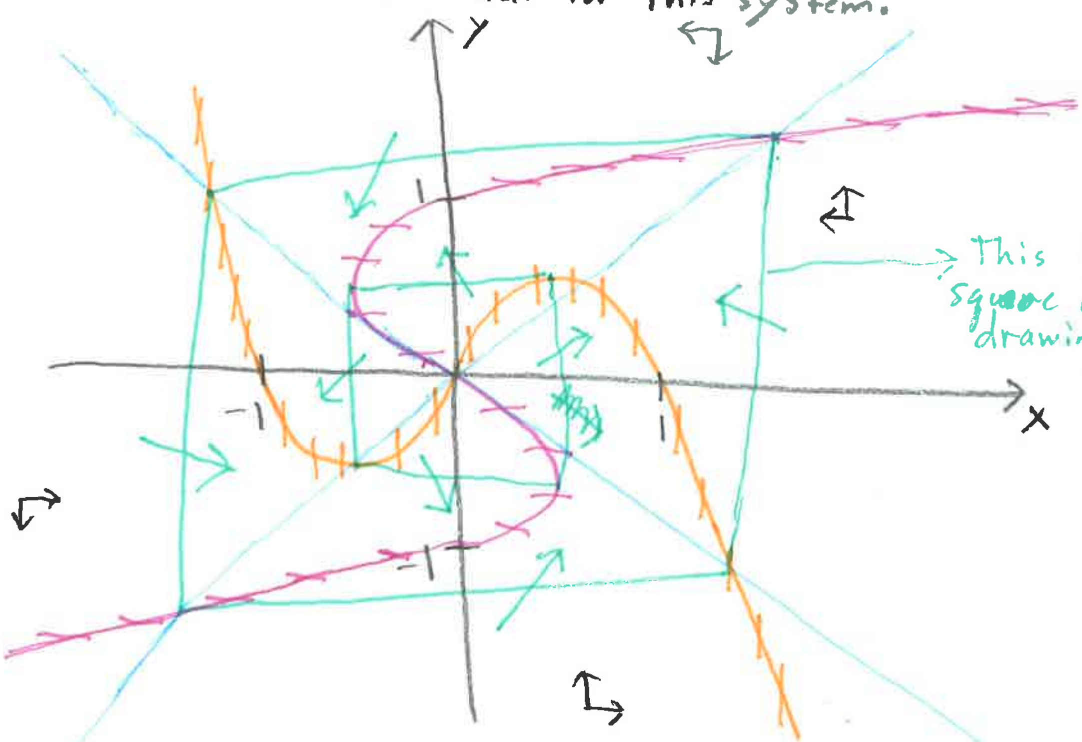
Solution:

The null-clines for this system are given by:

$y = x - x^3$

$x = y^3 - y$

Lets sketch the vector fields for this system:



This is actually a square in a more precise drawing.

From symmetry we know there is a square connecting points on the null-clines. This square is found by looking at the intersections of the curves $y = x$ and $y = -x$. The region enclosed by the green curves is a trapping region with no fixed points. ■

#7.3.4

Consider the system

$$\dot{x} = x(1-4x^2-y^2) - \tfrac{1}{2}y(1+x), \quad \dot{y} = y(1-4x^2-y^2) + 2x(1+x).$$

a.) Show that the origin is an unstable fixed point.

Solution:

The Jacobian at $(0,0)$ is given by

$$J(0,0) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & 1 \end{pmatrix}$$

which has eigenvalues $\lambda_{1,2} = 1 \pm i\sqrt{2}$. Therefore the origin is an unstable spiral.

b.) Show that all trajectories approach the ellipse $4x^2+y^2=1$ as $t \to \infty$

Solution:

Let $V = (1-4x^2-y^2)^2$. Then,

$$\begin{aligned}
\frac{dV}{dt} &= 2(1-4x^2-y^2)(-8x\dot{x}-2y\dot{y}) \\
&= 2|V|^{1/2}\left(-8x\cdot\left(x|V|^{1/2} - \tfrac{1}{2}y(1+x)\right) - 2y\left(y|V|^{1/2} + 2x(1+x)\right)\right) \\
&= 4|V|^{1/2}\left(-4x^2|V|^{1/2} + 4xy(1+x) - 4y^2|V|^{1/2} - 4xy(1+x)\right) \\
&= -16V(x^2+y^2) \\
&< 0.
\end{aligned}$$

Therefore, since $V \geq 0$ and $V=0$ if and only if $1-4x^2-y^2=0$ it follows that $\lim_{t\to\infty} V(x(t),y(t)) = 0$. Therefore, all trajectories approach the ellipse $4x^2+y^2=1$.

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#7.5.5

Consider the system $\ddot{x}+\nu(|x|-1)\dot{x}+x=0$. Find the approximate period of the limit cycle for $\nu \gg 1$

Solution:

Let $F(x)$ be the piecewise function defined by

$$F(x)=\begin{cases} \frac{x^2}{2}-x, & x>0, \\ -\frac{x^2}{2}-x, & x<0, \end{cases}$$

Then it follows that

$$\frac{d}{dt}(\dot{x}+\nu F(x))=-x.$$

If we let $v=\dot{x}+\nu F(x)$ then we have the system:

$$\dot{x}=v-\nu F(x)$$
$$\dot{y}=-x$$

Rescaling by $y=\frac{1}{\nu}v$ we have the following system:

$$\dot{x}=\nu(y-F(x))$$
$$\dot{y}=-\frac{1}{\nu}x.$$

For large ν the system spends most of its time on the null-cline $y=F(x)$. The period can be approximated by

$$T=2\int_0^{T/2} dt=2\int_{x_0}^{x_1}\frac{dt}{dx}dx=2\int_{x_0}^{x}F'(x)\frac{dt}{dy}=2\int_{x_0}^{x_1}-\frac{(|x|-1)\nu}{x}dx.$$

$$\Rightarrow T=O(\nu).$$

$\blacksquare$