# Homework #5 Solutions.

## #6.3.9

Consider the system $\dot{x}=y^3-4x$, $\dot{y}=y^3-y-3x$.

a.) Find all the fixed points and classify them.

Solution:

The fixed points satisfy the equation

$$0=y^3-4x$$
$$0=y^3-y-3x$$

The null-clines are then given by

$$y=(4x)^{1/3}$$
$$x=\frac{y^3-y}{3}$$

The null-clines intersect at the point:

$$y^3=\frac{4y^3-4y}{3}$$
$$\Rightarrow y^3-4y=0$$
$$\Rightarrow y=0,\pm 2$$

The fixed points are then

$$(0,0),(2,2),(-2,-2)$$

The Jacobian matrix is then

$$J=\begin{pmatrix}-4 & 3y^2\\ -3 & 3y^2-1\end{pmatrix}$$

(0,0)

$$J|_{(0,0)}=\begin{pmatrix}-4 & 0\\ -3 & -1\end{pmatrix}.$$

Therefore, (0,0) is a stable fixed point.

(2,2)

$$J|_{(2,2)}=\begin{pmatrix}-4 & 12\\ -3 & 11\end{pmatrix}$$

$$\Rightarrow \det(J|_{2,2})=-8<0$$

Therefore (2,2) is a saddle node.

$(-2,2)$

From odd symmetry we have that $J(2,2) = -J(-2,-2)$
$\Rightarrow \det(J(2,2)) = \det(J(-2,-2))$.

Therefore $(-2,2)$ is a saddle point as well.

$\blacksquare$

b.) Show that the line $y=x$ is invariant.

Solution:

$$\dot{y}-\dot{x} = x-y = -(y-x)$$

Therefore, if we let $z = y-x$ it follows that
$$\dot{z} = -z$$
which has a fixed point at $z=0 \Rightarrow y-x=0$ for all $t$ if $x(0)=y(0)$.

$\blacksquare$

c.) Show that $|x(t)-y(t)| \to 0$ as $t\to\infty$ for all trajectories.

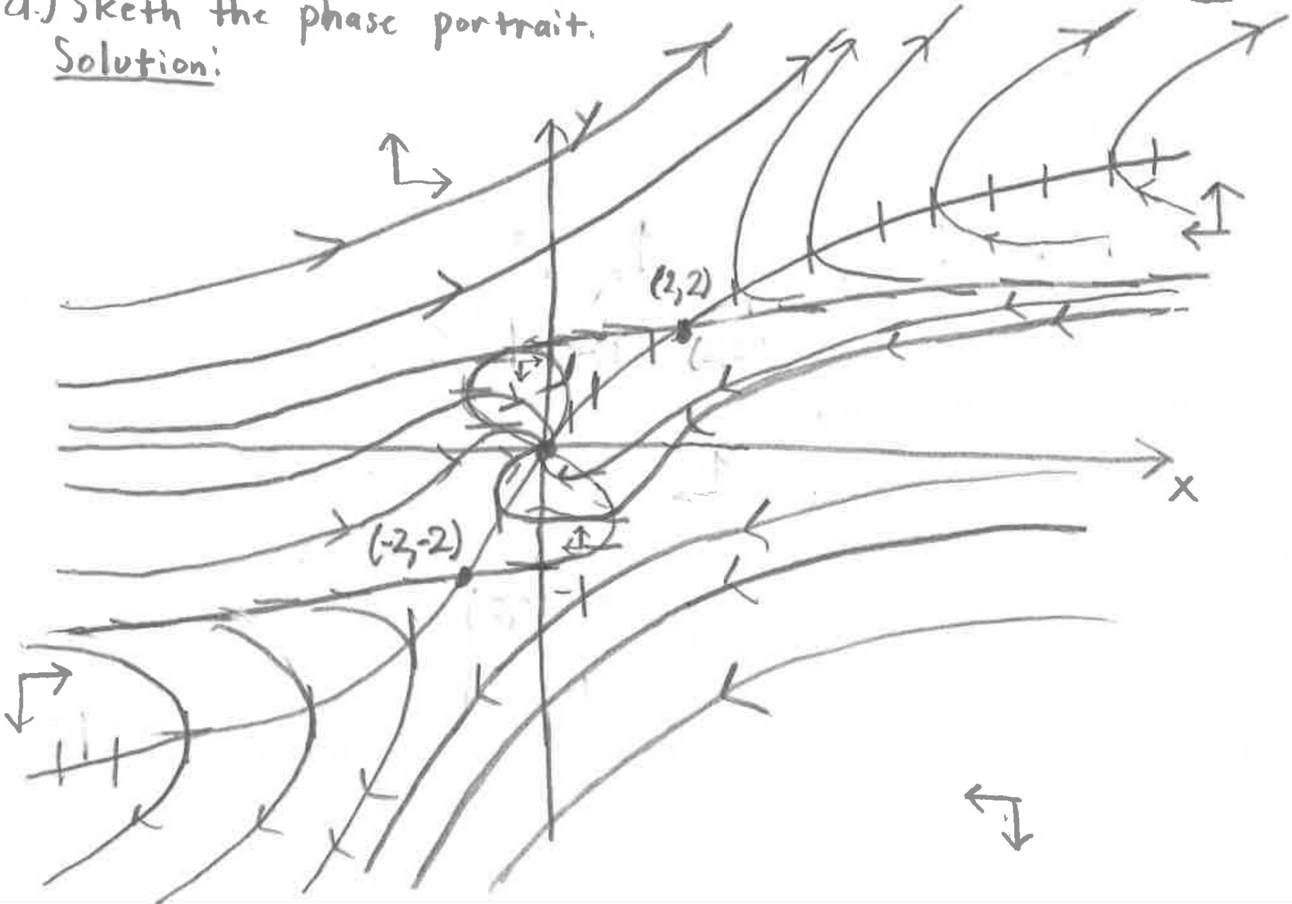
Solution:

For the one-dimensional system $\dot{z}=-z$, $z=0$ is a stable fixed point.
Consequently, $\lim_{t\to\infty} |x(t)-y(t)| \to 0$.

$\blacksquare$

d.) Sketh the phase portrait.

Solution:



#6.3.14

Classify the fixed point at the origin for the system $\dot{x} = -y + ax^3$ and $\dot{y} = x + ay^3$.

Solution:

Let $r^2 = x^2 + y^2$. Then,

$$\dot{r} = \frac{x\dot{x} + y \cdot \dot{y}}{r} = \frac{a(x^4 + y^4)}{r}$$

Therefore, $(0,0)$ is a stable fixed point if $a<0$, unstable if $a>0$ and a linear center if $a=0$.

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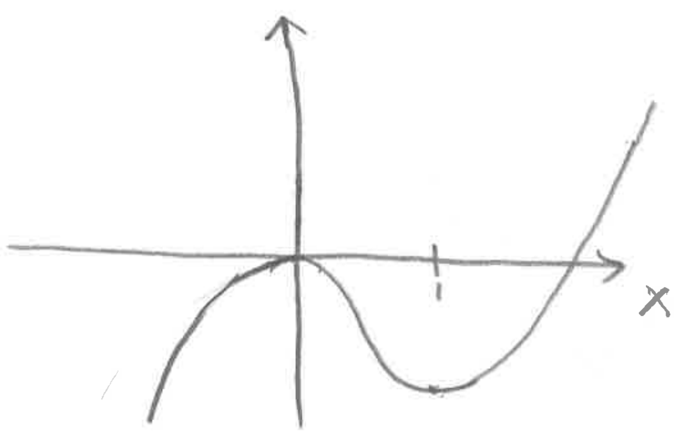
#6.5.2

Consider the system $\ddot{x} = x - x^2$.

a.) Find and classify the equilibrium points.

Solution:

This is a conservative system with potential $V(x) = -\frac{x^2}{2} + \frac{x^3}{3}$ which is plotted below.

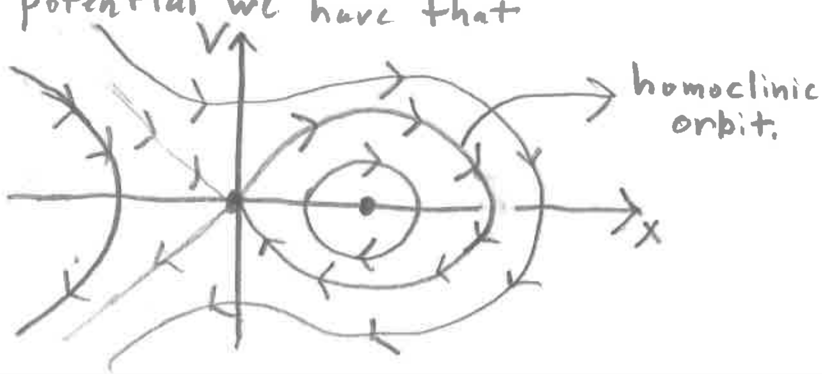



Therefore, $x=1$ is a nonlinear center and $x=0$ is a saddle.

b.) Sketh the phase portrait.

Solution:

From the potential we have that



c.) Find an equation for the homoclinic orbit,

Solution:

The enery is given by $E=\frac{1}{2}v^2-\frac{x^2}{2}+\frac{x^3}{3}$. At $x=0$, $v=0$ – the start of the homoclinic orbit – we have that $E=0$. Consequenrly the homoclinic orbit is given by the curve:

$$v=\pm\sqrt{x^2-\frac{2}{3}x^3},\quad x>0.$$

$\blacksquare$

# #6.8.6

A closed orbit in the phase plane encircles S saddles, N nodes, F spirals, and C centers. Show that $N+F+C=1+S$.

Solution:

The index of the curve is 1 so we have that

$$N+F+C-S=1$$
$$\Longrightarrow N+F+C=1+S.$$

$\blacksquare$

# #6.8.13

Consider a smooth vector field $\dot{x}=f(x,y)$, $\dot{y}=g(x,y)$ and let C be a simple closed curve in the plane that does not pass through any fixed points. Let $\phi=\tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)$.

a.) Show that

$$d\phi=\frac{f\,dg-g\,df}{f^2+g^2}$$

Solution:

$$d\phi=\frac{1}{1+g^2/f^2}\cdot\frac{f\,dg-g\,df}{f^2}=\frac{f\,dg-g\,df}{f^2+g^2}$$

b.) Derive the formula

$$I_c=\frac{1}{2\pi}\oint_C\frac{f\,dg-g\,df}{f^2+g^2}$$

Solution:

$$I_c=\frac{1}{2\pi}\oint d\phi=\frac{1}{2\pi}\oint\frac{f\,dg-g\,df}{f^2+g^2}$$

$\blacksquare$