Homework #4

#5.2.1

Consider the system

$$\dot{x} = 4x - y$$
$$\dot{y} = 2x + y$$

a.) Write the system $\dot{\vec{x}} = A\vec{x}$. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, and find the eigenvalues and eigenvectors, of A.

Solution:

Clearly,

$$\dot{\vec{x}} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \vec{x}.$$

The characteristic polynomial $p(\lambda)$ is given by:

$$p(\lambda) = \det \begin{pmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{pmatrix} = \lambda^2 - 5\lambda + 6.$$

The eigenvalues satisfy:

$$p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2) = 0$$

which implies that $\lambda_1 = 3$, $\lambda_2 = 2$ are the eigenvalues. The eigenvectors are solutions of the equation

$$(\lambda_{1,2} I - A)\vec{x}_{1,2} = 0.$$

Case 1:

$\lambda_1 = 3$

$$\Rightarrow \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \vec{x} = 0$$

$$\Rightarrow \vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Case 2:

$\lambda_2 = 2$

$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$$

$$\Rightarrow x_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

b.) Find the general solution of the system.

Solution:

From linearity it follows that

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

c.) Classify the fixed point at the origin.

Solution:

Since both eigenvalues are positive the origin is an unstable fixed point.

d.) Solve the system subject to the initial condition $(x_0, y_0) = (3, 4)$.

Solution:

Setting $t=0$ we get the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore, the solution curve is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\blacksquare$

#5.2.13

The motion of a damped harmonic oscillator is described by $m\ddot{x}+b\dot{x}+kx=0$ where $b>0$.

a.) Rewrite the equation as a two-dimensional linear system.

Solution:

Setting $v=\dot{x}$ we have the system:

$$m\dot{v}=-bv-kx$$
$$\dot{x}=v.$$

b.) Classify fixed points at the origin and sketch the phase portrait.

Solution:

The only fixed point is $v=0$, $x=0$. If we let $\alpha=b/m$ and $\beta=k/m$ then the matrix for this system is:

$$\begin{pmatrix}\dot{v}\\ \dot{x}\end{pmatrix}=\begin{pmatrix}-\alpha & -\beta\\ 1 & 0\end{pmatrix}\begin{pmatrix}v\\ x\end{pmatrix}.$$

The eigenvalues satisfy

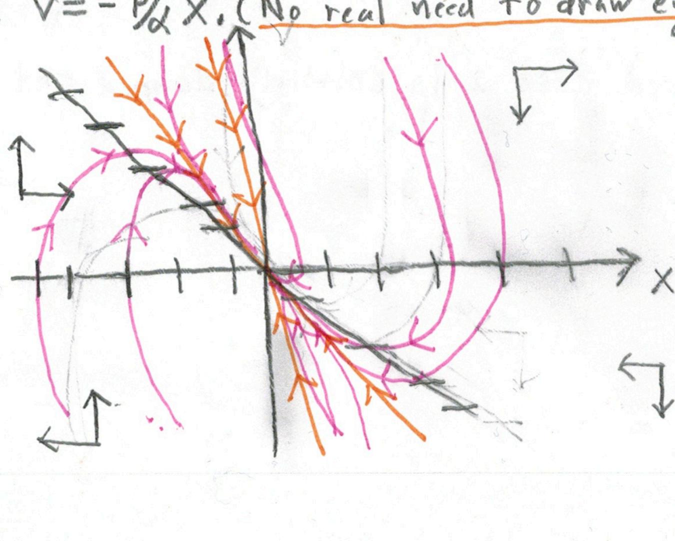
$$\frac{\beta}{\lambda_2}+\lambda_2=-\alpha,\quad \lambda_1=\frac{\beta}{\lambda_2}$$

Therefore,

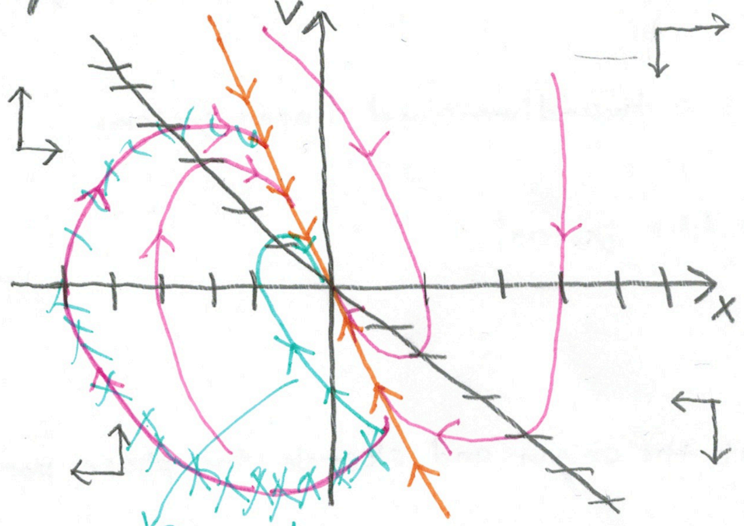
$$\lambda_{1,2}=\frac{-\alpha\pm\sqrt{\alpha^2-4\beta}}{2},$$

Note that if $\alpha^2-4\beta>0$ then $\sqrt{\alpha^2-4\beta}>\alpha$. Therefore we have three cases to consider:

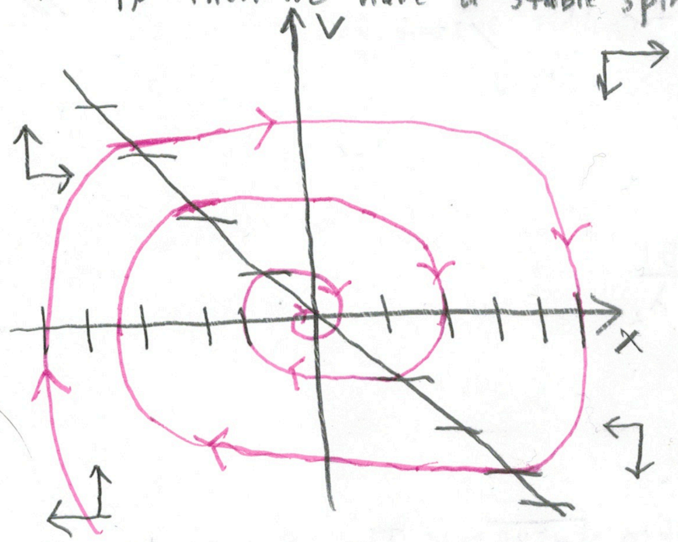
1. If $\alpha^2-4\beta>0$ the origin is a stable node. The null-clines are simply $v=0$ and $v=-\beta/\alpha\, x$. (No real need to draw eigenvectors)



2. If $\alpha^2 = 4\beta$ then there is only one eigenvector and we have a degenerate stable node:



3. If $\alpha^2 < 4\beta$ then we have a stable spiral.



C.) How do your results relate to standard notions of overdamped, critically damped, and underdamped vibrations.

Solution:

Case 1 is overdamped, case 2 is critically damped and case 3 is underdamped.

#5.3.4

Analyze $\dot{R} = aR + bJ$, $\dot{J} = -bR - aJ$.

Solution:

In matrix notation we have that

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}$$

The characteristic polynomial is given by:

$$\lambda^2 + (b^2 - a^2) = p(\lambda).$$

Therefore, the eigenvalues are given by:

$$\lambda = \pm\sqrt{a^2 - b^2}.$$

Case 1:

If $a^2 > b^2$ then the origin is a saddle point. Lets draw a phase portrait in this case. The null-clines are given by:

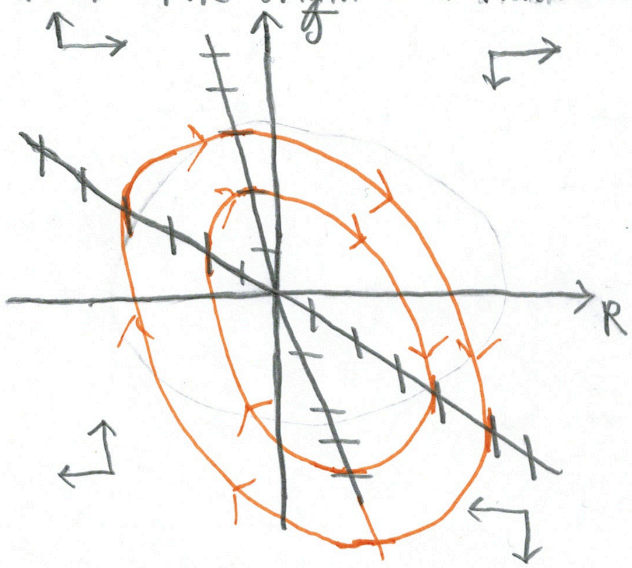
$$J = -\frac{a}{b}R \quad (\dot{R} = 0),$$

$$J = -\frac{b}{a}R \quad (\dot{J} = 0)$$

In this case either Juliet falls for Romeo while he despises or Romeo falls in love with her while she despises him.

Case 2:

If $a^2 < b^2$ the origin is a linear center.



In this case Romeo and Juliet switch between states of love and hate.

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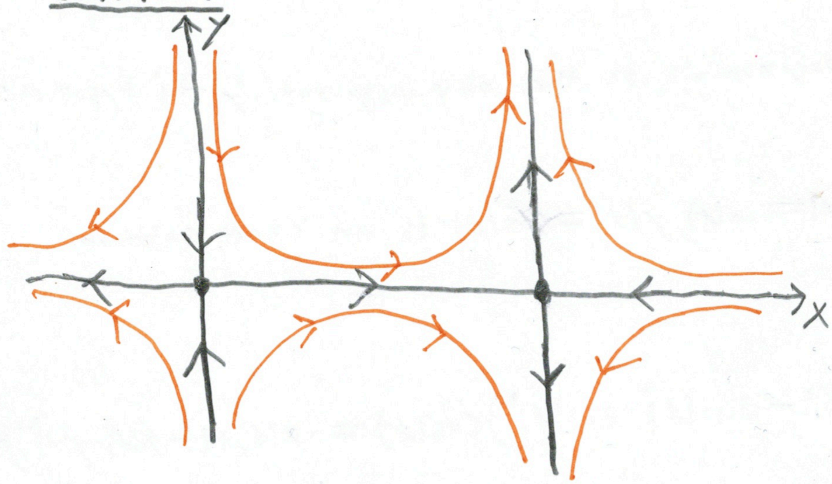
#6.1.12

A system has exactly two fixed points both of which are saddles.

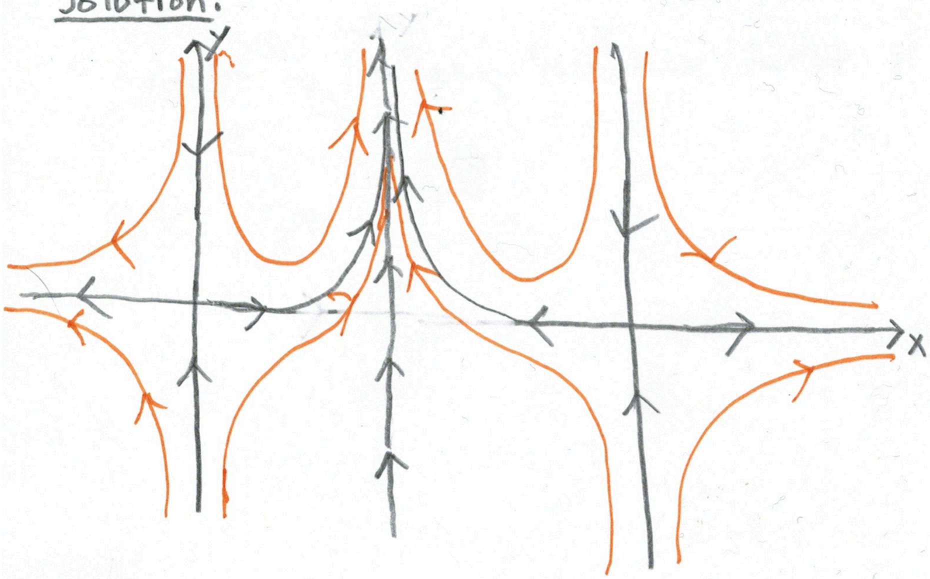
a.) Sketch a phase portrait in which there is a single trajectory connecting the saddles.

Solution:



b.) Sketch a phase portrait in which there is no trajectory that connects the saddle.

Solution:



#6.2.2.

Consider the system $\dot{x}=y,\ \dot{y}=-x+(1-x^2-y^2)y$.

a.) Let D be the open disk $x^2+y^2<4$. Verify that the system satisfies the hypothesis of the existence and uniqueness theorem.

Solution:

The governing equations are polynomials so they obviously satisfy the hypothesis of the theorem.

b.) By substitution, show that $x(t)=\sin(t),\ y(t)=\cos(t)$ is an exact solution.

Solution:

$$\dot{x}(t)=\cos(t)=y(t)$$

$$\dot{y}(t)=-\sin(t)=-\sin(t)+(1-\sin(t)^2-\cos(t)^2)\cos(t)=-x(t)+(1-x(t)^2-y(t)^2)y(t)$$

c.) Consider a solution with intial condition $x(0)=\frac{1}{2},\ y(0)=0$. Explain why this solution must satisfy $x(t)^2+y(t)^2<1$.

Solution:

From parts a) and b) it follows that $x(t)^2+y(t)^2<1$ or solutions will fail to be unique.