#3.6.3.

Homework #3

Consider the system $\dot{x} = rx + ax^2 - x^3$ where $a \in \mathbb{R}$.

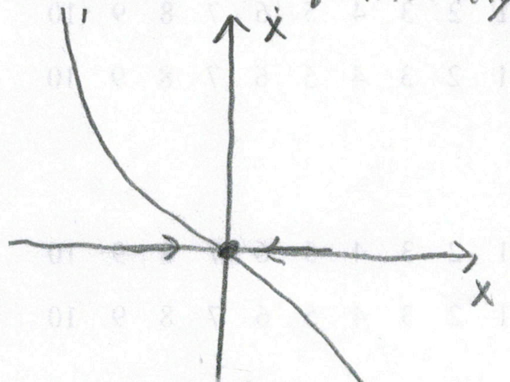
a.) Sketch all qualitatatively different bitorcation diagrams that can be obtained by varying a.

Solution:

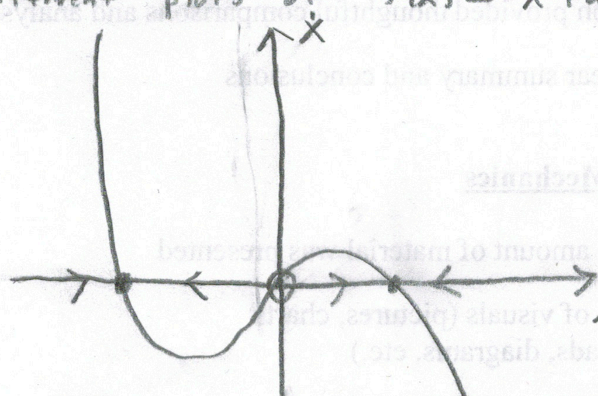
Lets first determine the fixed points:

$$x = 0 \quad \text{and} \quad x = -\frac{a}{2} \pm \frac{\sqrt{a^2+4r}}{2}.$$

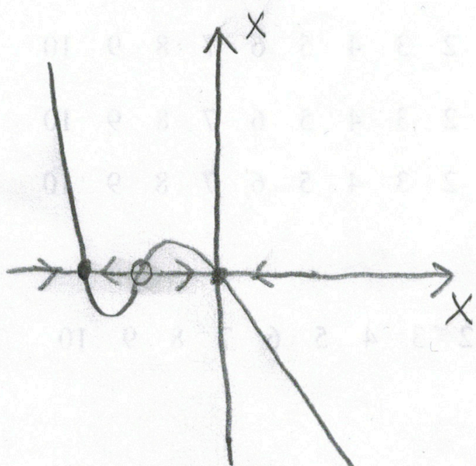
To determine existence and stability of these fixed points lets draw the four qualitatively different plots of $f(x) = -x^3 + ax^2 + rx$!



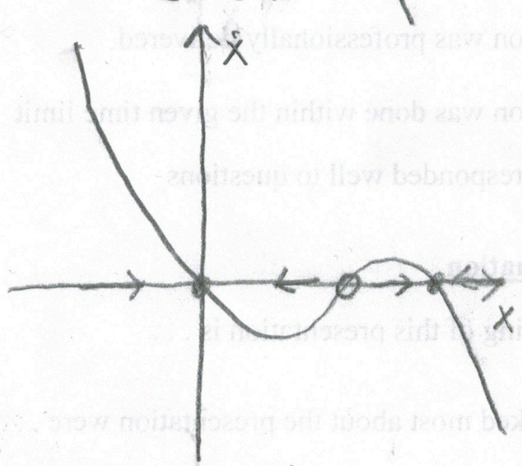
Case 1



Case 2



Case 3



Case 4

The stability of the origin and the sign of the two other roots determines which case we are in. Clearly if $r < -4a^2$ we are in case 1. Other cases depend on the sign of $-\frac{a}{2}$ and stability of the origin. Differentiating we have that:

$$f'(0) = r.$$

Consequently the origin is stable if and only if $r < 0$.

Therefore, we can deduce that the system is in the following cases for the following range of parameters:

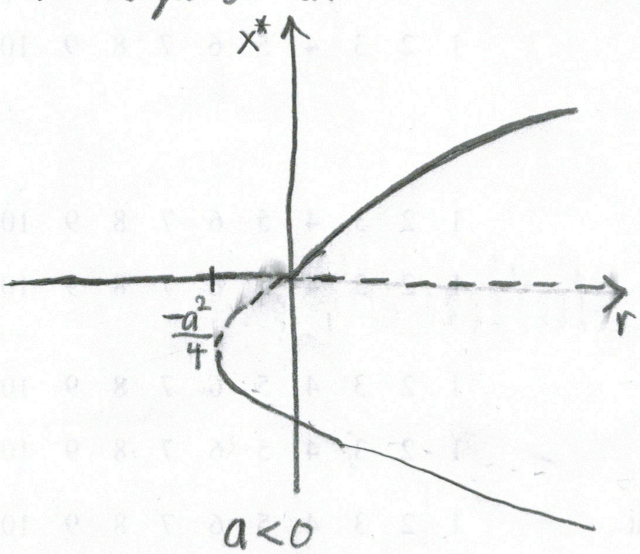
a.) Case 1 if $r < -\frac{a^2}{4}$.

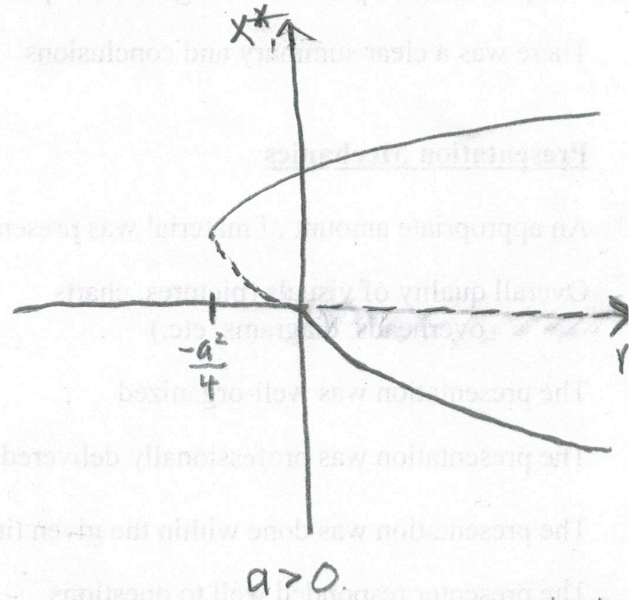
b.) Case 2 if $r > 0$.

c.) Case 3 if $-\frac{a^2}{4} < r < 0$ and $a > 0$

d.) Case 4 if $-\frac{a^2}{4} < r < 0$ and $a < 0$.

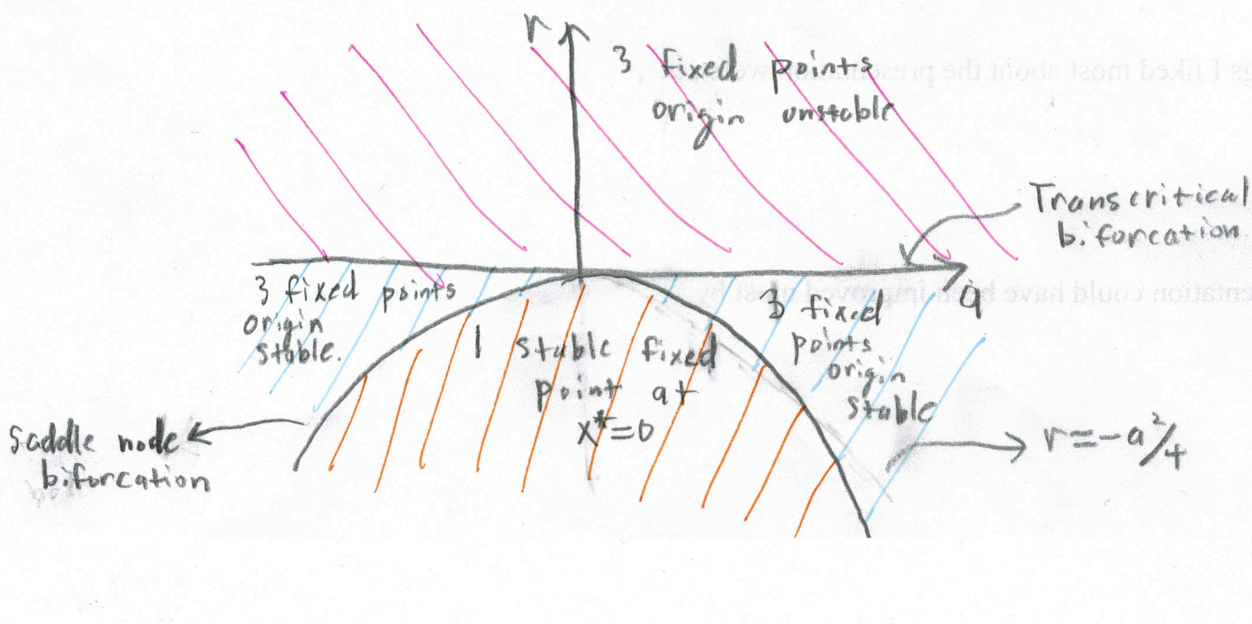
Consequently, we get two different bifurcation curves depending on the sign of a:





b.) Plot the regions in the (r, a) plane that corresponds to qualitatively different vector fields.

Solution:



#3.7.6

Consider the model

$$\dot{x} = -kxy$$
$$\dot{y} = kxy - ly$$
$$\dot{z} = ly$$

a.) Show that $x+y+z=N$, a constant.

<u>Solution:</u>

Differentiating:

$$\frac{d}{dt}(x+y+z) = \dot{x}+\dot{y}+\dot{z} = 0$$

Therefore, $x+y+z$ is a constant.

b.) Use the $\dot{x}$ and $\dot{z}$ equation to show that $\dot{z} = l[N - z - x_0 \exp(-kz/l)$

<u>Solution:</u>

We can solve for x in terms of z.

$$\frac{dx}{dz} = -\frac{k}{l}x$$

$$\Rightarrow \int_{x_0}^{x} \frac{1}{x}\,dx = -\frac{k}{l}z(t)$$

$$\Rightarrow \quad x(t) = x_0 \exp\left(-\frac{k}{l}z(t)\right).$$

c.) Show that z satisfies the first order equation $\dot{z} = l[N - z - x_0 \exp(-\frac{kz}{l}$

<u>Solution:</u>

Calculating:

$$\dot{z} = ly = l[N - z - x] = l\left[N - z - x_0 \exp\left(-\frac{kz}{l}\right)\right]$$

d.) Show that this equation can be nondimensionalized to

$$\frac{du}{d\tau} = a - bu - e^{-u}$$

<u>Solution:</u>

Let $u = kz/l$. Then,

$$\frac{du}{dt} = \frac{k}{l}\frac{dz}{dt} \Rightarrow \frac{dz}{dt} = \frac{l}{k}\frac{du}{dt}.$$

Therefore,

$$\frac{du}{dt} = k\left[N - \frac{\ell}{k}u - x_0 e^{-u}\right].$$

Setting $\tau = (x_0 k)^{-1} t$. Therefore,

$$\frac{du}{d\tau} = \frac{N}{x_0} - \frac{\ell}{k x_0} u - e^{-u}.$$

Defining $a = N x_0^{-1}$, $b = \ell k^{-1} x_0^{-1}$ we have that

$$\frac{du}{d\tau} = a - bu + e^{-u}.$$

e.) Show that $a \geq 1$ and $b > 0$.

Solution:

Calculating,

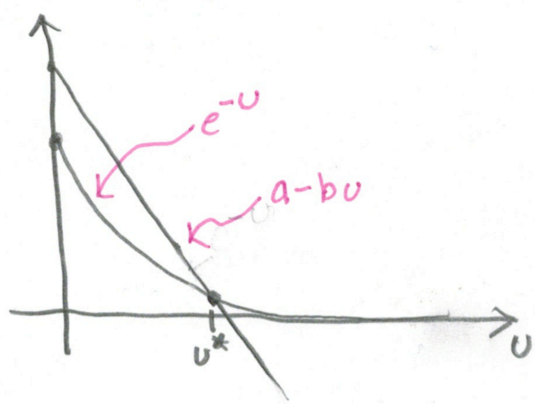
$$a = \frac{x_0 + y_0 + z_0}{x_0} \geq 1.$$

also $b = \ell k^{-1} x_0^{-1} > 0$ by definition.

f.) Determine the number of fixed points u^* and classify their stability.

Solution:

To determine the number of fixed points we plot $a-bu$ and e^{-u} and look for intersections.



Clearly we have one fixed point at $u = u^*$. To analyze stability we note that for all values of u:

$$\frac{d}{du}\left[a - bu + e^{-u}\right] = -b - e^{-u} < 0.$$

There fore, u^* is stable.

g.) Show that the maximum of $\dot{u}(t)$ occurs at the same time as the maximum of both $\dot{z}(t)$ and $y(t)$.

Solution:

Calculating we have that

$$\frac{du}{dt} = \frac{k}{l}\frac{dz}{dt}$$

$$\Rightarrow \frac{d^2u}{dt^2} = \frac{k}{l}\frac{d^2z}{dt^2}.$$

Therefore, the maximumum for $\dot{u}$ and $\dot{z}$ occur at the same time. Now,

$$\frac{d^2z}{dt^2} = l\frac{dy}{dt}.$$

So, if $\frac{d^2z}{dt^2} = 0$ then $\frac{dy}{dt} = 0$.

h.) Show that if $b<1$ then $\dot{u}$ is increasing at $t=0$.

Solution:

Differentiating we have that:

$$\frac{d\dot{u}}{dt} = \frac{1}{x_0^2K^2}\frac{d^2u}{d\tau^2} = \frac{1}{x_0^2K^2}\left(-b\frac{du}{d\tau} - \frac{du}{d\tau}e^{-u}\right)$$

$$\rightarrow \frac{d\dot{u}}{dt} = \frac{1}{x_0^2K^2}\left(-b-e^{-u}\right)\left(a-bu-e^{-u}\right)$$

Therefore,

$$\left.\frac{d\dot{u}}{dt}\right|_{t=0} = \frac{1}{x_0^2K^2}\left(-b+e^{-u_0}\right)\left(a-bu_0-e^{-u_0}\right) = \frac{1}{x_0^2K^2}\left(-b+e^{-u_0}\right)\left.\frac{du}{d\tau}\right|_{\tau=0}$$

Now, $\frac{du}{d\tau} > 0$ since u^* is a stable fixed point and since $e^{-u_0} \leq 1$ it follows that if $b<1$ that $\left.\frac{d\dot{u}}{dt}\right|_{t=0} > 0$.

i). Show that $t_{peak} = 0$ if $b>1$.

Solution:

In this case $\left.\frac{d\dot{u}}{dt}\right|_{t=0} < 0$ so the maximum value occurs at $t=0$.

In the limit $t\to\infty$ we have $\frac{d\dot{u}}{dt} = 0$ since the system goes to the equilibrium point.

$\blacksquare$

#4.3.8

Classify and identify all bifurcations that occur for the system

$$\dot{\theta} = \frac{\sin(2\theta)}{1+\nu\sin(2\theta)}$$

on S^1.

Solution:

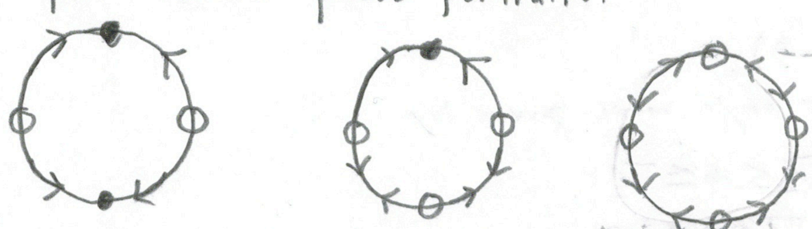
The fixed points are given by $\theta = 0, \pi/2, \pi$, and $\frac{3\pi}{2}$. Differentinny we have that

$$\frac{d\dot{\theta}}{d\theta} = \frac{2\cos(2\theta)}{1+\nu\sin(\theta)} - \frac{\nu\sin(2\theta)\cos(2\theta)}{(1+\nu\sin(\theta))^2}$$

$$\Rightarrow \left.\frac{d\dot{\theta}}{d\theta}\right|_0 = 2, \qquad \left.\frac{d\dot{\theta}}{d\theta}\right|_\pi = 2,$$

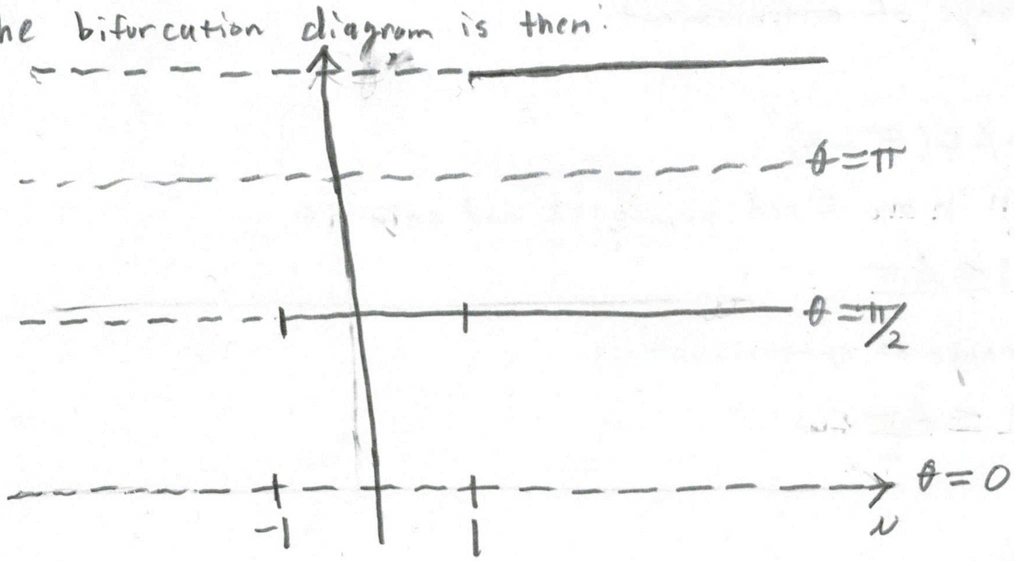
$$\left.\frac{d\dot{\theta}}{d\theta}\right|_{\pi/2} = \frac{-2}{1+\nu}, \qquad \left.\frac{d\dot{\theta}}{d\theta}\right|_{\frac{3\pi}{2}} = \frac{-2}{1-\nu}$$

Lets plot various phase portraits:



$\nu > 1$ $\qquad$ $-1 < \nu < 1$ $\qquad$ $\nu < -1$

The bifurcation diagram is then:



#4.4.1

Find conditions under which it is valid to approximate the equation $mL^2\ddot{\theta}+b\dot{\theta}+mgL\sin(\theta)=\Gamma$ by its overdamped limit $b\dot{\theta}+mgL\sin(\theta)=\Gamma$.

Solution:

Let T_{sc} be an arbitrary time scale. Setting $\tau = T_{sc}^{-1}t$ we have that:

$$\frac{L}{gT_{sc}^2}\frac{d^2\theta}{d\tau^2}+\frac{b}{mgLT_{sc}}\frac{d\theta}{d\tau}+\sin(\theta)=\frac{\Gamma}{mgL}$$

Therefore, to ensure the frictional effects are $O(1)$ we set $T_{sc}=\frac{b}{mgL}$. Consequently, to ensure inertial effects are negligible we must assume that

$$\frac{L}{g}\cdot\left(\frac{mgL}{b}\right)^2=\frac{L^3m^2g}{b}<<1.$$

#4.5.1

Consider the model

$$\dot{\Phi}=\Omega$$
$$\dot{\theta}=\omega+Af(\Phi-\theta).$$

where $f(\phi)$ is the triangle wave given by

$$f(\phi)=\begin{cases}\phi, & -\pi/2\leq\phi\leq\pi/2\\ \pi-\phi, & \pi/2\leq\phi\leq 3\pi/2\end{cases}$$

b.) Find the range of entrainment.

Solution:

$$\dot{\phi}=\Omega-\omega-Af(\Phi-\theta)$$

This system will have fixed points if and only if

$$|\Omega-\omega|\leq\frac{A\pi}{2}$$

Therefore, the range of entrainment is

$$-\frac{A\pi}{2}+\omega\leq\Omega\leq\frac{A\pi}{2}+\omega.$$

C. Assuming that the firefly is phase-locked to the stimulus, find a formula for the phase difference ϕ.

Solution:

We have that

$$\frac{d\phi}{dt} = \Omega - \omega - A f(\phi)$$

$$\Rightarrow \int_0^{\phi} \frac{1}{\Omega - \omega - A f(\phi)}\, d\phi = t$$

If $\phi(t) < \pi/2$ then:

$$t = \int_0^{\phi} \frac{1}{\Omega - \omega - A\phi}\, d\phi$$

$$\Rightarrow t = -\frac{1}{A} \ln(\Omega - \omega - A\phi)\Big|_0^{\phi}$$

$$= \frac{1}{A} \ln\left(\frac{\Omega - \omega}{\Omega - \omega - A\phi}\right)$$

$$\Rightarrow e^{At} = \frac{\Omega - \omega}{\Omega - \omega - A\phi}$$

$$\Rightarrow (\Omega - \omega)e^{-At} = \Omega - \omega - A\phi$$

$$\Rightarrow \phi = \frac{\Omega - \omega}{A} - \frac{(\Omega - \omega)e^{-At}}{A}$$

$$\phi(t) = \frac{\Omega - \omega}{A}\left(1 - e^{-At}\right)$$

We want to find when $\phi(t) = \pi/2$:

$$\frac{\pi}{2} = \frac{\Omega - \omega}{A}\left(1 - e^{-At}\right)$$

Solving we have that

$$t = \ln\left(\left|\frac{2(\omega - \Omega)}{A\pi + 2\omega - 2\Omega}\right|\right)$$

Therefore, $\phi(t) = \frac{\Omega - \omega}{A}\left(1 - e^{-At}\right)$ for $0 < t < \ln\left(\left|\frac{2(\omega - \Omega)}{A\pi + 2\omega - \Omega}\right|\right)$

Extending this function periodically we obtain a solution for all t.

d.) Find a formula for T_{drift}.

Solution:

Integrating and using periodicity we have that

$$T_{drift} = 4\ln\left(\left|\frac{2(\omega-\Omega)}{A\pi+2(\omega-\Omega)}\right|\right)$$