Homework #2.

#3.1.4

For the system

$$\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}.$$

Sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs. Finally sketch the bifurcation diagram of fixed points x^* versus r.

Solution:

The equilibrium points satisfy the equation

$$2(1+x^*)r + x^*(1+x^*) - 2x^* = 0$$

$$\Rightarrow x^{*2} + (2r-1)x^* + 2r = 0.$$

Therefore when the equilibrium points exist they are given by

$$x^* = \frac{1}{2} - r \pm \sqrt{r^2 - 3r + \frac{1}{4}}.$$

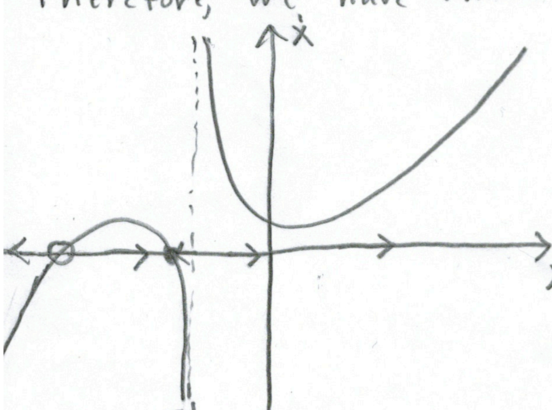
Existence of the equilibrium points is satisfied if and only if $r^2 - 3r + \frac{1}{4} \geq 0$. Solving this inequality yields there are no fixed points if

$$-\sqrt{2} + \frac{3}{2} < r < \sqrt{2} + \frac{3}{2}.$$

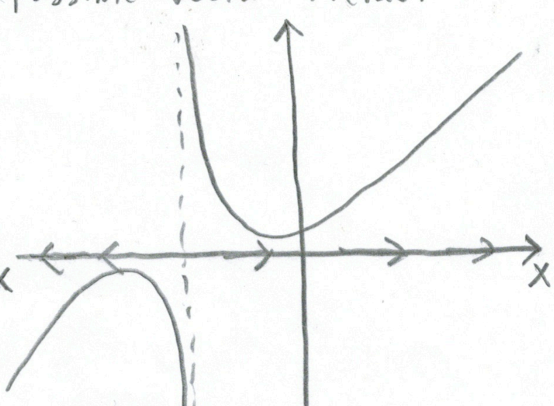
To understand the stability of the fixed points we determine some analytical properties of the function $f(x) = r + \frac{1}{2}x - \frac{x}{1+x}$.

1. $\lim_{x \to -1^+} f(x) = \infty$
2. $\lim_{x \to \infty} f(x) = \infty$
3. $\lim_{x \to -\infty} f(x) = -\infty$
4. $\lim_{x \to -1^-} f(x) = -\infty$

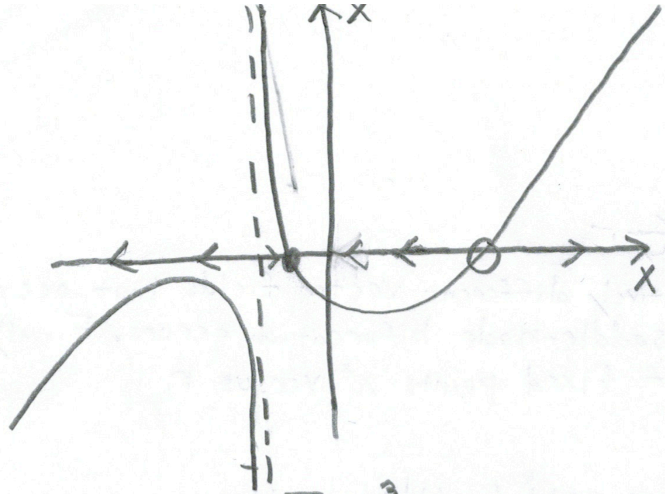
Therefore, we have three possible vector fields:



$r > \sqrt{2} + \frac{3}{2}$

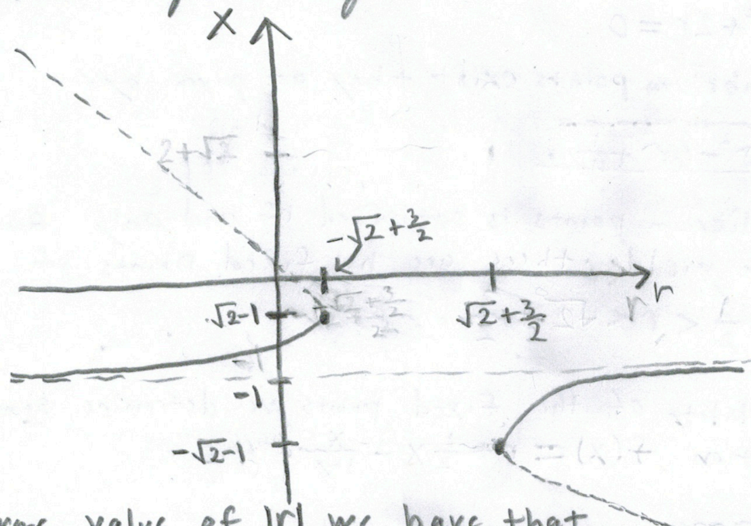


$-\sqrt{2} + \frac{3}{2} < r < \sqrt{2} + \frac{3}{2}$



$r \leq -\sqrt{2} + \frac{3}{2}$

The bifurcation diagram is given below:



Note, for large value of $|r|$ we have that

$$\frac{1}{2} - r + \sqrt{r^2 - 3r + \frac{1}{4}} = \frac{1}{2} - r + r\sqrt{1 - \frac{3}{r} + \frac{1}{4r^2}} \approx \frac{1}{2} - \frac{3}{2} = -1$$

$$\frac{1}{2} - r - \sqrt{r^2 - 3r + \frac{1}{4}} = \frac{1}{2} - r - r\sqrt{1 - \frac{3}{r} + \frac{1}{4r^2}} \approx 2 - 2r$$

$\blacksquare$

#3.2.4

Sketch all the qualitatively different vector fields for the system

$$\dot{x} = x(r - e^x).$$

Sketch the the bifurcation diagram for this system.

Solution:

The fixed points are given by

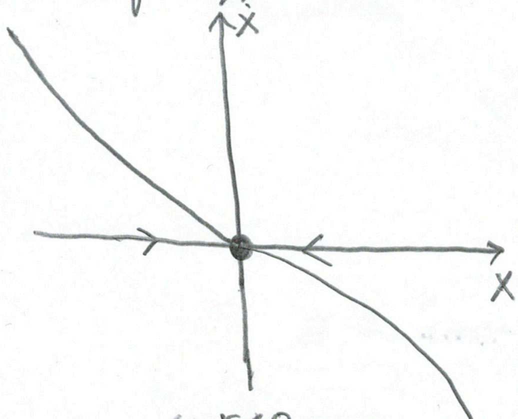
$$x^* = 0 \quad \text{and} \quad x^* = \ln(r)$$

if $r > 0$. Taking limits we have that

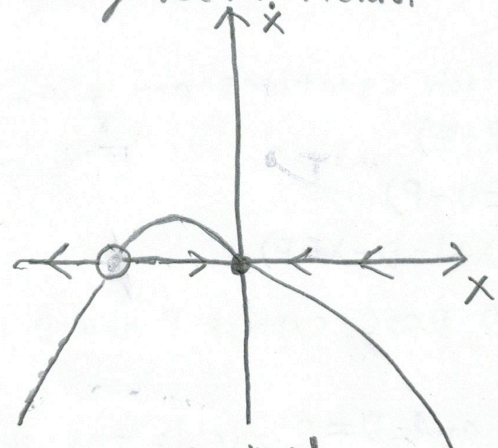
1. $\lim\limits_{x \to \infty} x(r - e^x) = -\infty$

2. $\lim\limits_{x \to -\infty} x(r - e^x) = -\infty$ if $r > 0$ else $\lim\limits_{x \to -\infty} x(r - e^x) = \infty$ if $r < 0$.

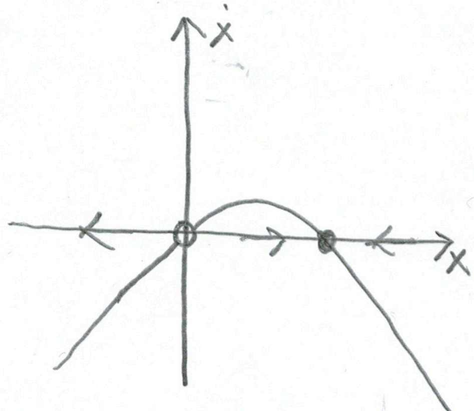
Consequently, we have the following vector fields:



$r < 0$

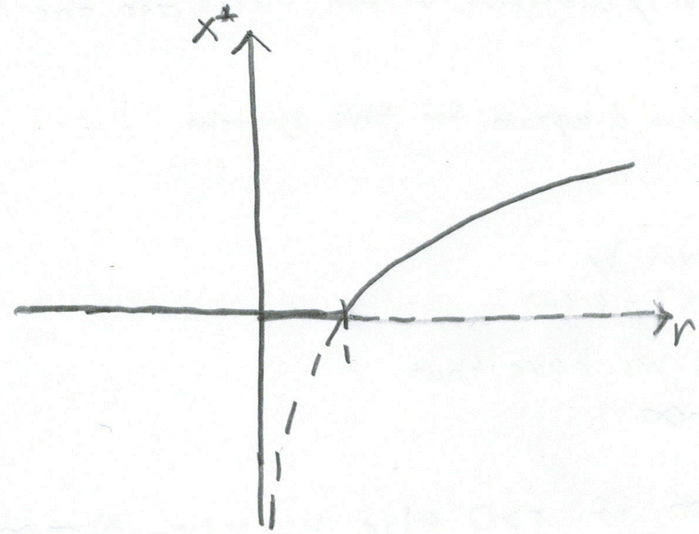


$0 < r < 1$



$r > 1$

From these vector fields we obtain the following bifurcation diagram:



#3.3.2

The Maxwell Bloch equations are given by:

$$\dot{E} = \kappa(P-E)$$
$$\dot{P} = \gamma_1(ED-P)$$
$$\dot{D} = \gamma_2(\lambda+1-D-\lambda EP)$$

a.) Assuming $\dot{P} \approx 0$, $\dot{D} \approx 0$ express P and D in terms of E.

Solution:

Setting $P=0$ and $D=0$ yields the system of equations

$$0 = ED - P$$
$$0 = \lambda + 1 - D - \lambda EP$$

Solving:

$$P = ED$$
$$\Rightarrow 0 = \lambda + 1 - D - \lambda E^2 D$$
$$\Rightarrow \frac{\lambda+1}{1+\lambda E^2} = D.$$

Therefore, we have that

$$P = \frac{(\lambda+1)E}{1+\lambda E^2} \quad \text{and} \quad D = \frac{\lambda+1}{1+\lambda E^2}.$$

Consequently, we have that

$$\dot{E} = \kappa\left(\frac{1-E^2}{1+\lambda E^2}\right)E\lambda$$

b. Find all the fixed points of the equation for E.

Solution:

Solving the equation $\dot{E}=0$ yields the fixed points

$$E^*=-1,0,1.$$

C. Draw the bifurcation diagram of E^* vs. λ.

Solution:

Calculating we have that

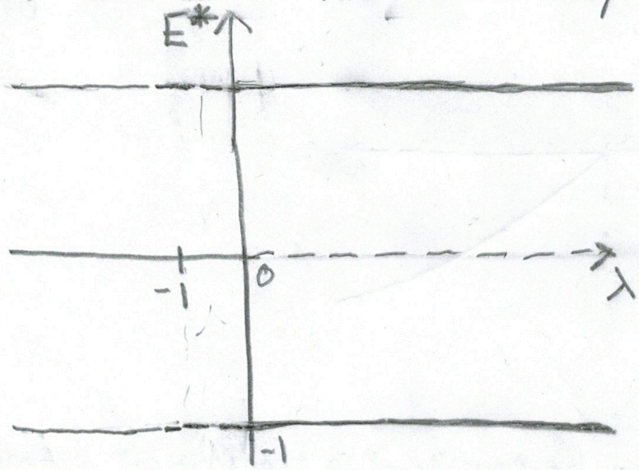
$$\frac{d}{dE}\left(K\left(\frac{1-E^2}{1+\lambda E^2}\right)E\lambda\right)=\lambda K\left(\frac{1-E^2}{1+\lambda E^2}\right)-\frac{2E^2K\lambda}{1+\lambda E^2}-K\lambda\frac{1-E^2}{(1+\lambda E^2)^2}\cdot 2\lambda E^2.$$

Evaluating at the fixed points we have that

$$\frac{d}{dE}\left(K\left(\frac{1-E^2}{1+\lambda E^2}\right)E\lambda\right)\Big|_0=\lambda K$$

$$\frac{d}{dE}\left(K\left(\frac{1-E^2}{1+\lambda E^2}\right)E\lambda\right)\Big|_{\pm 1}=-\frac{2K\lambda}{1+\lambda}$$

Therefore, $E^*=0$ is stable for $\lambda<0$ while $E^*=\pm 1$ are stable for $\lambda>0$ and $\lambda<-1$. The bifurcation diagram as below.



This bifurcation diagram may seem impossible but there are vertical asymptote. at $E^*=\sqrt{-\lambda}$ which act similar to fixed points.

■

#3.4.6

Sketch the bifurcation diagram for the system

$$\dot{x} = rx - \frac{x}{1+x}.$$

Solution:

Solving the equation $\dot{x}=0$ yields the equilibrium points

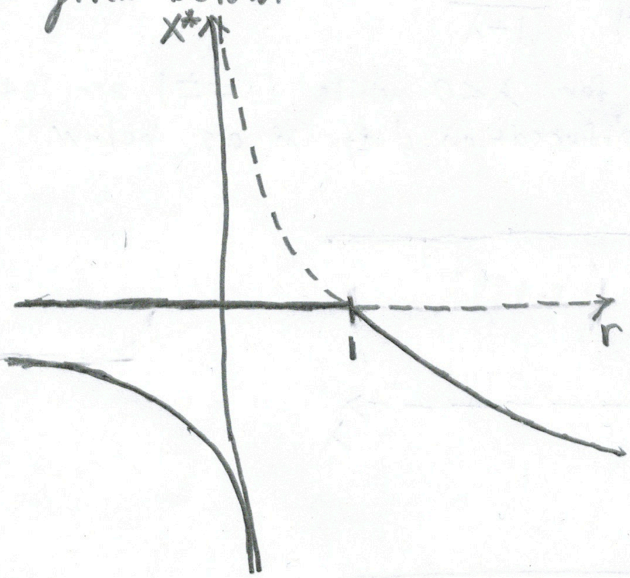
$$x^* = 0, \ \frac{1-r}{r}.$$

We also have that

$$\frac{d}{dx}\frac{dx}{dt} = r - \frac{1}{1+x} + \frac{x}{(1+x)^2}$$

$$\Rightarrow \left(\frac{d}{dx}\frac{dx}{dt}\right)\Bigg|_0 = r-1 \quad \text{and} \quad \left(\frac{d}{dx}\frac{dx}{dt}\right)_{\frac{1}{r}-1} = \frac{(1-r)}{r} r^2 = r(1-r)$$

Consequently, $x=0$ is stable for $r<1$ and unstable for $r>1$ while $x=\frac{1-r}{r}$ is stable for $r>1$ and $r<0$ and stable for $0<r<1$. The bifurcation diagram is given below.



The bifurcation point $r=1$ can be considered a transcritical bifurcation. ■

#3.4.14

Consider the system $\dot{x} = rx + x^3 - x^5$.

a.) Find algebraic expressions for all fixed points as r varies.

Solution:

Solving the equation $\dot{x} = 0$ we have that:

$$0 = -x(x^4 - x^2 - r)$$

Therefore, we obtain the fixed points:

$$x = \pm\sqrt{\frac{1}{2} \pm \frac{\sqrt{1+4r}}{2}} \quad \text{and} \quad x = 0.$$

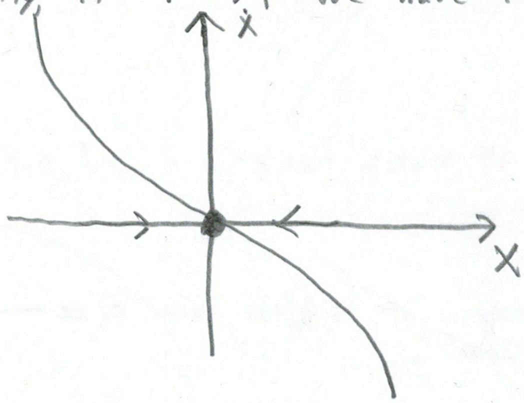
b.) Sketch the vector fields as r varies.

Solution:

For $r < -\frac{1}{4}$ there is only one fixed point at $x = 0$. Differentiating we have that

$$\frac{d}{dx}\left(\frac{dx}{dt}\right)\Big|_0 = r.$$

Consequently, if $r < -1/4$ we have the following vector field



We next check when

$$\frac{1}{2} - \frac{\sqrt{1+4r}}{2} > 0$$

$$\Rightarrow 4r < 0$$
$$\Rightarrow r < 0.$$

Therefore, if $-\frac{1}{4} < r < 0$ there will be five fixed points. The corresponding vector fields are plotted below: