Homework #1.

#2.1

In the next three exercises, interpret $\dot{x}=\sin(x)$ as a flow on the line.

#2.1.1

Find all fixed points of the flow.

Solution:

The fixed points occur when $\dot{x}=\frac{dx}{dt}=0$. Therefore, for this system the fixed points occur when $\sin(x)=0$, i.e. when $x=n\pi$ where $n\in\mathbb{Z}$.

$\blacksquare$

#2.1.2

At which points x does the the flow have the greatest velocity to the right?

Solution:

The flow has the greatest velocity to the right when $x=\sin(x)$ is a maximum, i.e. $x=\frac{\pi}{2}+2n\pi$ where $n\in\mathbb{Z}$.

$\blacksquare$

#2.1.3.

a.) Find the flow's accleration as a function of x.

Solution:

Calculating we have that:

$$\frac{d\dot{x}}{dt}=\cos(x)\frac{dx}{dt}=\cos(x)\sin(x)=\frac{\sin(2x)}{2}.$$

Therefore, the acceleration is simply

$$\ddot{x}=\frac{\sin(2x)}{2}.$$

$\blacksquare$

b.) Find the points where the flow has maximum accleration.

Solution:

The flow has maximum accleration when $\sin(2x)$ is maximum, i.e when

$$x=\frac{\pi}{4}+n\pi$$

where $n\in\mathbb{Z}$.

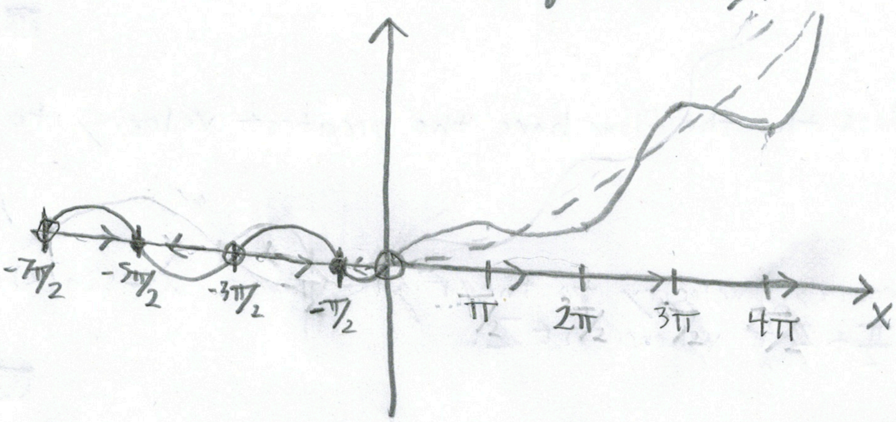
$\blacksquare$

#2.2.7

Andlye the equation $\dot{x}=e^{x}-\cos(x)$ graphically. Sketch the vector field on the real line, find all fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions.

Solution:

For positive values of x we know that $e^{x}>1$ so there are no fixed points in this regime and $e^{x}-\cos(x)$ will "look" like an oscillating exponential. If instead x is negative then $e^{x}\approx 0$ for $|x|>>1$. We can use this information to sketch $e^{x}-\cos(x)$ qualitatively:



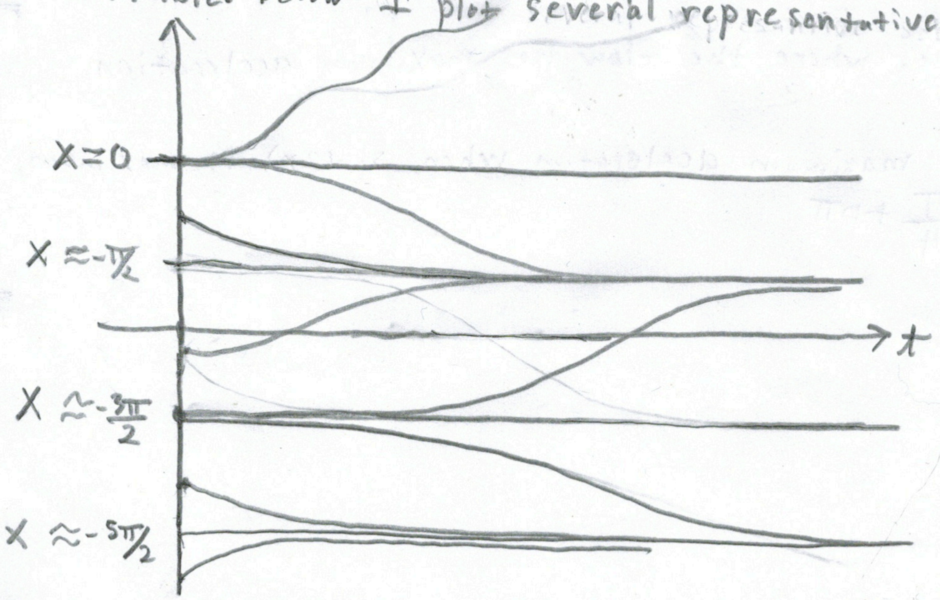
The vector field can now be drawn in as above and we can approximate the location of the fixed points and their stability. We get an alternating sequence of fixed points approximately described by:

$$x \approx -\pi/2, -5\pi/2, \ldots$$

which are stable and

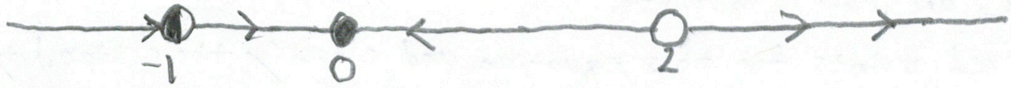
$$x \approx -\frac{3\pi}{2}, -\frac{7\pi}{2}, \ldots$$

which are unstable. Below I plot several representative solution curves.



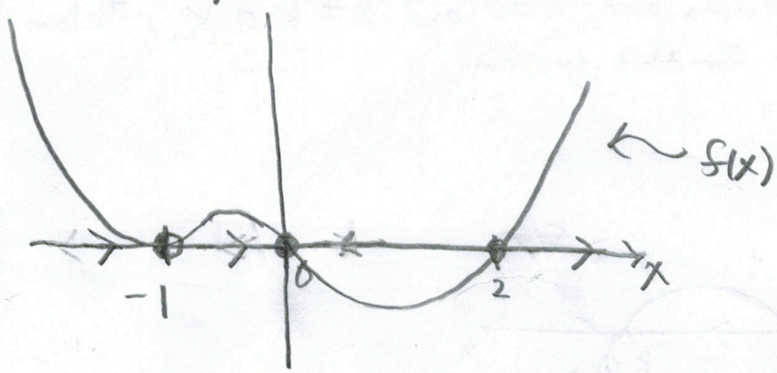
#2.2.8

For the phase portrait below find an equation that is consistent with it.



Solution:

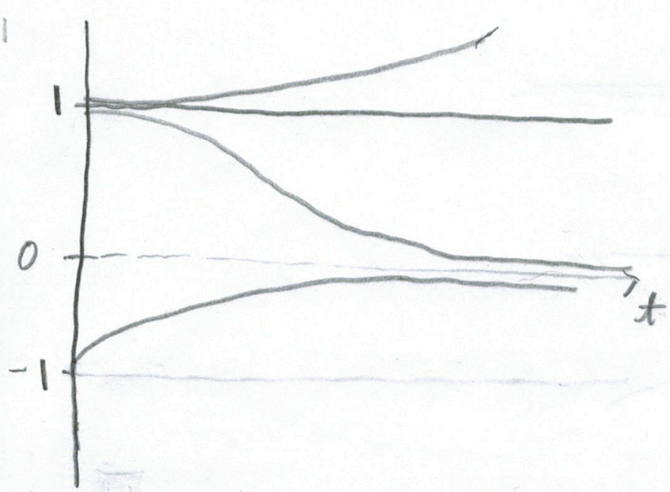
First I will sketch the graph of a polynomial $f(x)$ for which $\dot{x} = f(x)$ matches this phase portrait.



If we let $f(x) = (x+1)^2 x \cdot (x-2)$ we will have the desired phase portrait.

2.2.9

ind an equation $\dot{x} = f(x)$ whose solutions are consistent with the figure elow.



Solution:

Here we have a stable fixed point at $x=0$ and an unstable fixed point at $x=1$. From the figure it is clear that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$. Therefore, he polynomial $f(x) = x(x-1)$ works.

#2.3.2.

The equation for the kinetics of a molecule X is given by

$$\dot{x} = k_1 a x - k_{-1} x^2$$

where k_1 and k_{-1} are rate constants.

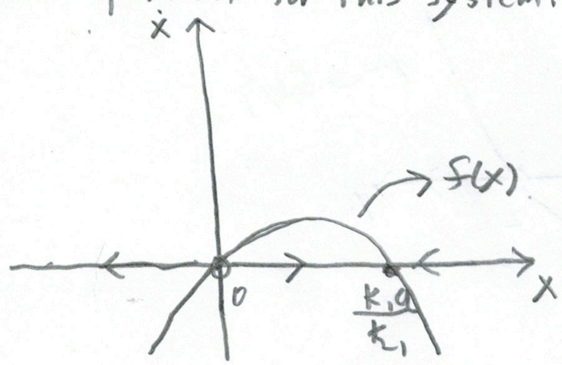
a.) Find all fixed points for this equation and classify their stability.

Solution:

The fixed points satisfy the equation

$$k_1 a x - k_{-1} x^2 = 0.$$

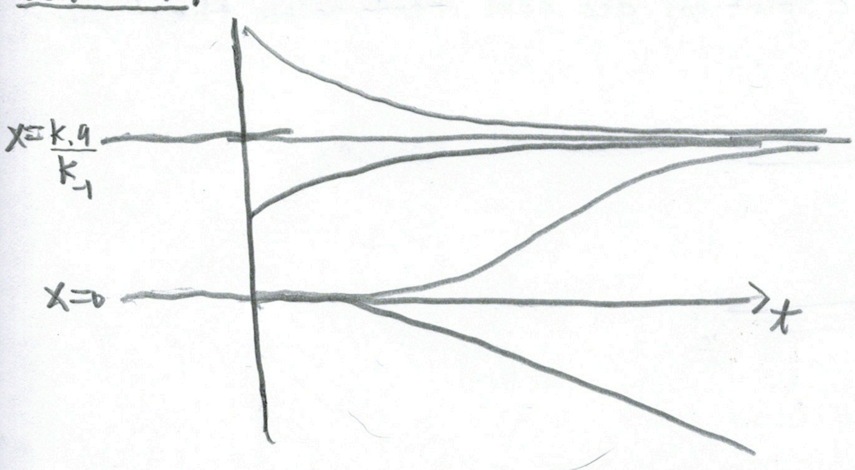
Therefore, the fixed points are $x=0$ and $x = k_1 a \cdot k_{-1}^{-1}$. Below I plot a typical phase portrait for this system.



Here it is clear that 0 is unstable and $\frac{k_1 a}{k_{-1}}$ is stable. ∎

b.) Sketch the graph of $x(t)$ for various initial values x_0.

Solution:



∎

#2.7.6

For the following vector field plot the potential function $V(x)$ and identify all the equilibrium points and their stability:

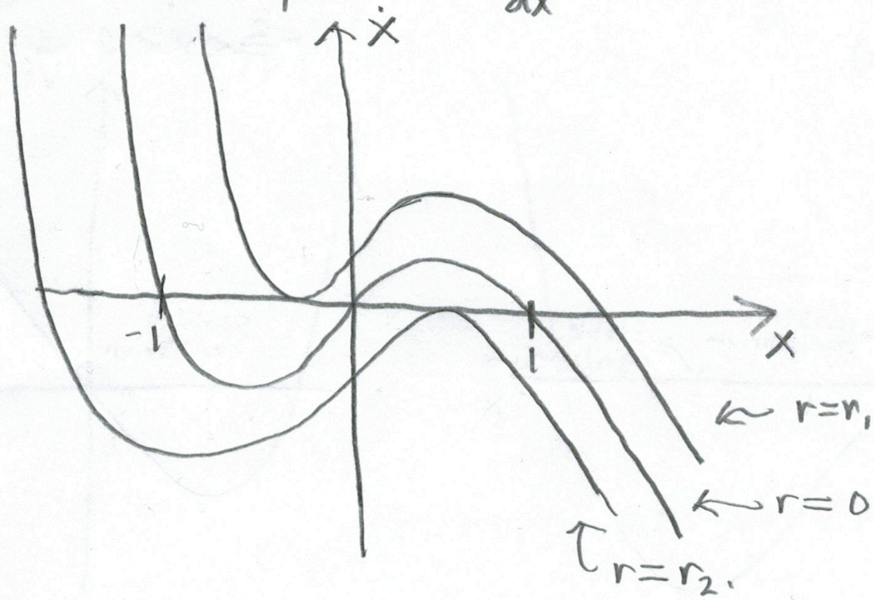
~~Solut~~ $\dot{x} = r + x - x^3$.

Solution:

The potential can be found by integrating

$$V = \int_0^x -(r+s-s^3)\,ds = -rx + \frac{r^2}{2} - \frac{r^4}{4}.$$

V has 1, 2 or 3 critical points depending on the value of r. Below are several plots of $-\frac{dV}{dx}$ for various values of r.



We want to find the value of r_1 and r_2. These values can be found by looking at the equation $\frac{d^2V}{dx^2} = 0$:

$$-\frac{d^2V}{dx} = 1 - 3x^2$$

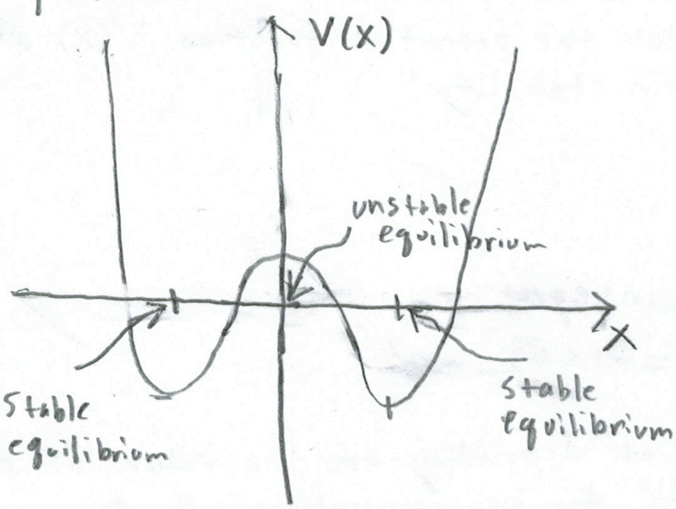
The critical points are at $x = \pm\frac{1}{\sqrt{3}}$. Now,

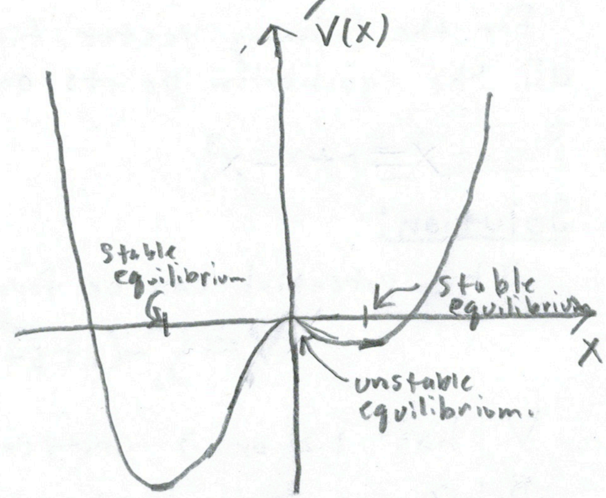
$$-\frac{dV}{dx}\bigg|_{\pm 1/\sqrt{3}} = r \pm \frac{1}{\sqrt{3}} - \frac{1}{3^{3/2}}$$

Consequently if $r = \mp\frac{2}{\sqrt{3}}$ the graph of $-\frac{dV}{dx}$ will have a repeated root at $x = \pm\frac{1}{\sqrt{3}}$. Therefore, we have the following cases:

1. 3 roots if $-\frac{2}{\sqrt{3}} < r < \frac{2}{\sqrt{3}}$
2. 1 root if $r > 2/\sqrt{3}$ or $r < -\frac{2}{3}$.

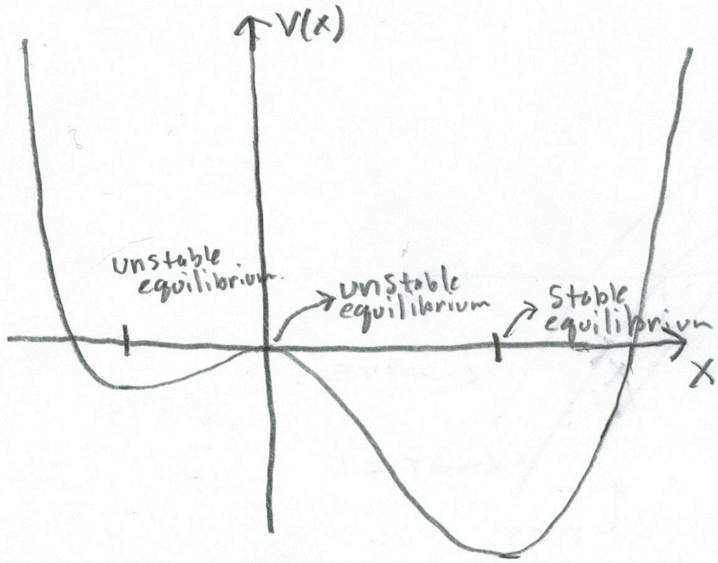
The potential in these cases looks like the following:

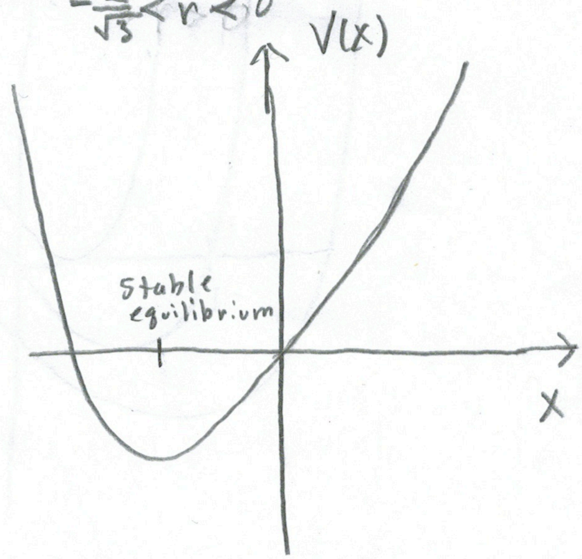




$r=0$

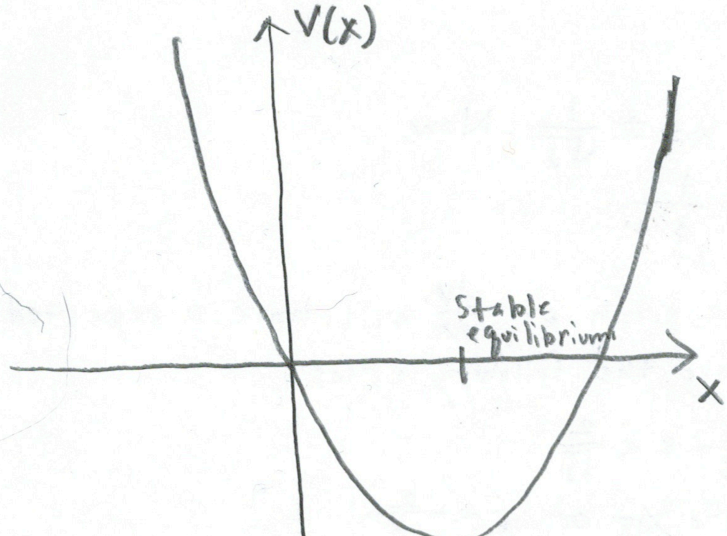
$-\frac{2}{\sqrt{3}}<r<0$





$0<r<2/\sqrt{3}$

$r<-2/\sqrt{3}$



∎