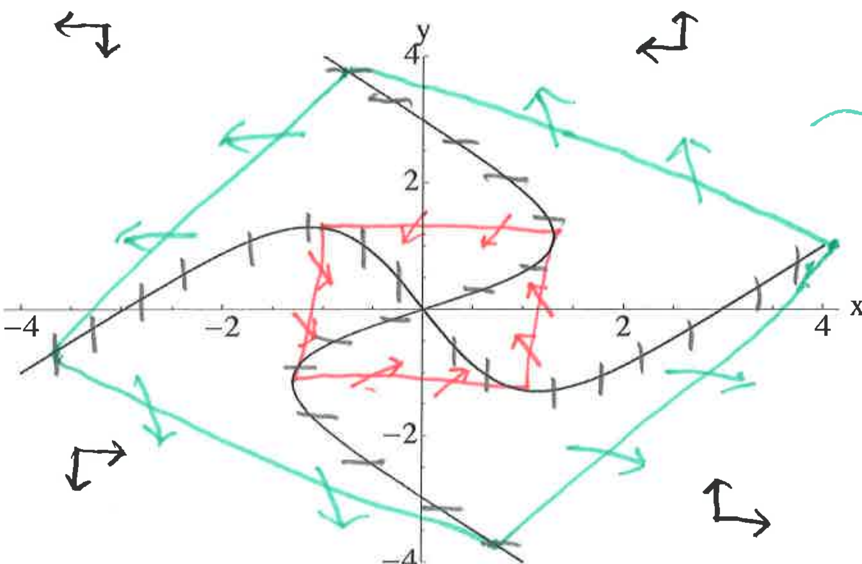
APMA: 1360
Classwork #2

March 21, 2014

1. Tell me everything you know about the system

$$\begin{cases} \dot{x} = x - y - 3\tanh(x) \\ \dot{y} = x + y - 3\tanh(y) \end{cases}$$

As a hint, the null-clines are plotted below.



This diamond has to actually be very big.

For large $|x|$, fixed y:

$$\frac{dy}{dx} \approx 1$$

For large $|y|$, fixed x:

$$\frac{dy}{dx} \approx -1$$

This tells us that by constructing an appropriate polygonal domain a repelling region can be constructed. In fact, diagonals work for large enough x, y:

$$\frac{dy}{dx} > 1 \Rightarrow y + x > x - y - 3\tanh(x) + 3\tanh(y)$$

$$\frac{dy}{dx} > -1 \Rightarrow x - y > -x - y + 3\tanh(x) - 3\tanh(y)$$

We can also look at polar coordinates:

$$r\dot{r} = x\cdot\dot{x} + y\cdot\dot{y}$$
$$= x^2 + y^2 - 3x\tanh(x) - 3y\tanh(y)$$

$$\Rightarrow \dot{r} = r - 3\cos\theta\tanh(r\cos\theta) - 3\sin\theta\sinh(r\sin\theta).$$

Therefore if $r > 6$ we have that $\dot{r} > 0$. Consequently, this system has an unstable limit cycle.

2. Let $a \in \mathbb{R}$. What type, if any, bifurcation occurs for the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = ax + (1-x^2)y \end{cases}.$$

Sketch representative phase portraits of the system.

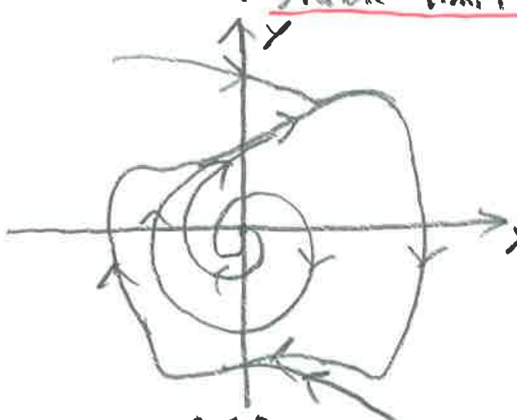
We can rewrite this system as:

$$\ddot{x} - (1-x^2)\dot{x} = ax$$

If $a<0$ we can let $\tau = \sqrt{|a|}t$ and obtain

$$\frac{d^2x}{d\tau^2} + \frac{1}{\sqrt{|a|}}(x^2-1)\frac{dx}{d\tau} = -x.$$

Therefore, if $a<0$ this system is the same as the van der Pol oscillator and therefore has a stable limit cycle.

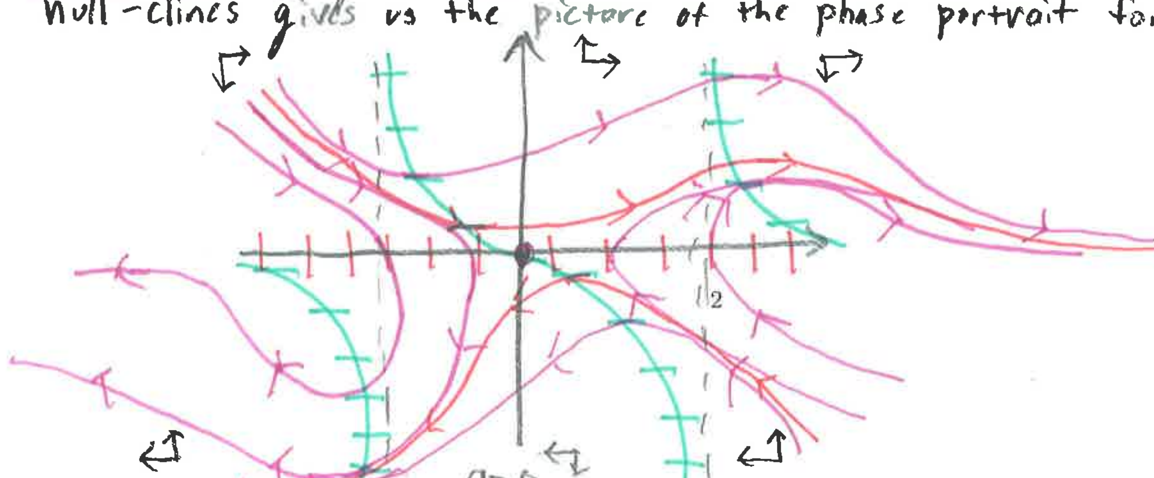


$a<0$

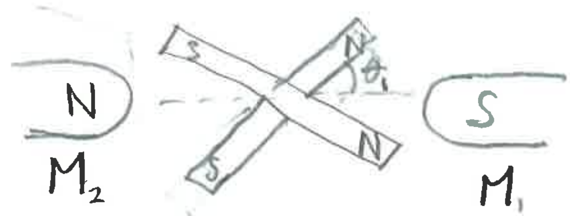
The fixed point for the system is $(0,0)$ and $J(0,0) = \begin{pmatrix} 0 & 1 \\ a & 1 \end{pmatrix}$. So the eigenvalues are given by:

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{1-4a^2}.$$

If $0<a<1/4$ we have an unstable spiral while if $a \geq 1/4$ it is a saddle. Index theory guarantees there is no closed orbit if $a>1/4$. Plotting the null-clines gives us the picture of the phase portrait for $a>0$.



$a>0$



3. Consider the system

$$\begin{cases} \dot\theta_1 = K\sin(\theta_1 - \theta_2) - \sin(\theta_1) \\ \dot\theta_2 = K\sin(\theta_2 - \theta_1) - \sin(\theta_2) \end{cases},$$

where $K \geq 0$. Determine if the system undergoes a bifurcation and of what type. Determine whether periodic orbits can exist or not.

Let $\phi = \theta_1 - \theta_2$ and $\varphi = \theta_1 + \theta_2$.

$$\Rightarrow \dot\phi = 2K\sin(\phi) - \sin\left(\frac{\phi+\varphi}{2}\right) + \sin\left(\frac{\phi-\varphi}{2}\right)$$

$$= 2K\sin(\phi) - \sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\varphi}{2}\right) - \sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\varphi}{2}\right) - \sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\phi}{2}\right)$$

$$= 4K\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) - 2\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\phi}{2}\right)$$

$$\Rightarrow \boxed{\dot\phi = 2\cos\left(\frac{\phi}{2}\right)\left(2K\sin\left(\frac{\phi}{2}\right) - \sin\left(\frac{\varphi}{2}\right)\right)}$$

$$\dot\varphi = -\sin\left(\frac{\phi+\varphi}{2}\right) - \sin\left(\frac{\phi-\varphi}{2}\right)$$

$$\Rightarrow \boxed{\dot\varphi = -2\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\varphi}{2}\right)}$$

Lets determine fixed points:

Cases:

1. $\phi = 0, \varphi = 0 \Rightarrow \theta_1 = \theta_2 = 0$ (Both aligned at M_1) $\Rightarrow$ (Stable) (Small K)
2. $\varphi = 0, \phi = 0 \Rightarrow \theta_1 = \theta_2 = \pi/2$ (Both pointing up) ↑↑ (Stable) (Small K).
3. $\varphi = -\pi, \phi = 0 \Rightarrow \theta_1 = \theta_2 = -\pi/2$ (Both pointing down) ↓↓ (Stable)
4. $\phi = 0, \varphi = 2\pi \Rightarrow \theta_1 = \theta_2 = \pi$ (Both point left) ⇇ (unstable)
5. $\phi = \pi, \varphi = \pi \Rightarrow \theta_1 = \pi, \theta_2 = 0$ (one left, one right) ⇆ (likely unstable).
6. $\phi = \pi, \varphi = -\pi \Rightarrow \theta_1 = 0, \theta_2 = \pi$ ⇄ (~~By~~ likely unstable)

If $K > 1/2$:

1. $\varphi = \pi$, $\sin\left(\frac{\phi}{2}\right) = \frac{1}{2K} \Rightarrow \theta_1 = \pi/2 + \sin^{-1}(1/2K)$, $\theta_2 = \pi/2 - \sin^{-1}(1/2K)$ (stable)
2. $\varphi = -\pi$, $\sin\left(\frac{\phi}{2}\right) = -1/2K \Rightarrow \theta_1 = \theta_1 = -\pi/2 + \sin^{-1}\left(\frac{-1}{2K}\right)$, $\theta_2 = -\pi/2 - \sin^{-1}\left(\frac{-1}{2K}\right)$ (stable).

This represent a pitchfork bifurcation where for large K —two strong central magnets— they end up forming an X shape.