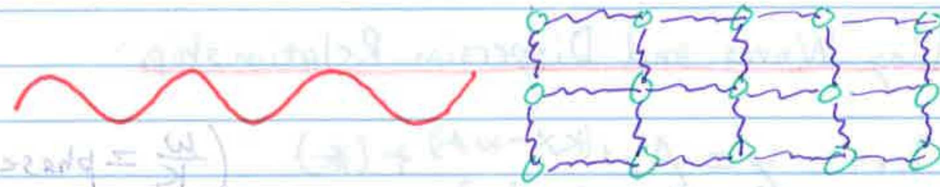
Chapter 4: Dispersive Waves

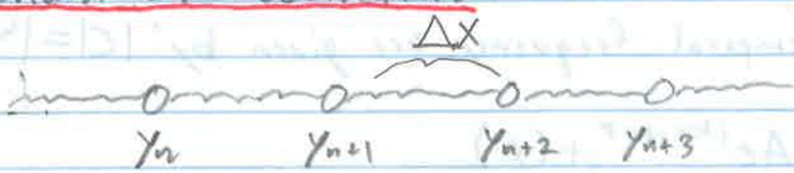
Motivation

Optics is the study of light's interaction with matter



Light carries an electric field with it that interacts with atoms causing them to vibrate and emit light.
Emitted light is caused by resonant interactions.

Chain of Oscillators



$$\ddot{y}_n = -\frac{d}{dy} U(y_{n-1}, y_n, y_{n+1})$$ (Nearest Neighbor Model)

1. $U = \alpha(y_{n+1} - y_n)^2 + (y_n - y_{n-1})^2$

$$\Rightarrow \ddot{y}_n = \alpha(y_{n+1} - 2y_n + y_{n-1})$$

Letting $\alpha = c^2/\Delta x$ and taking $\Delta x \to 0$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$ (Wave equation)

2. $U = -\alpha(y_{n+1} - y_n)^2 + \alpha(y_n - y_{n-1})^2 + \beta y_n^2$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \beta y^2}$$

3. $U = \alpha(y_{n+1}-y_n)^2 + \alpha(y_n - y_{n-1})^2 + \beta_1 y_n^2 + \beta_2 y_n^4$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - y - \varepsilon y^3}$$ (Klein-Gordon Equation)

Travelling Waves and Dispersion Relationship.

1. Let $y = Ae^{i(kx-\omega t)} + (*)$ ($\frac{\omega}{k}$ = phase velocity)

$\Rightarrow A\omega^2 = Ak^2c^2$

$\Rightarrow \omega = \pm kc.$

The relationship $\omega(k)$ is called the dispersion relationship.

(The speed of the wave is $\pm c$ and the wavelength and temporal frequency are given by: $|c| = |\omega/k|$)

2. Let $y = Ae^{ikx - i\omega t} + (*)$.

$\Rightarrow -\omega^2 A = (-c^2k^2 - \beta)A$

$\Rightarrow \omega = \pm\sqrt{c^2k^2+\beta}$

The phase velocity depends on k!

$$\frac{\omega}{k} = \pm\sqrt{c^2 + \beta/k^2}.$$

Waves of different wavelengths move at different speeds.

3. Let $y = Ae^{ikx - i\omega t} + (*)$

At $O(1)$:

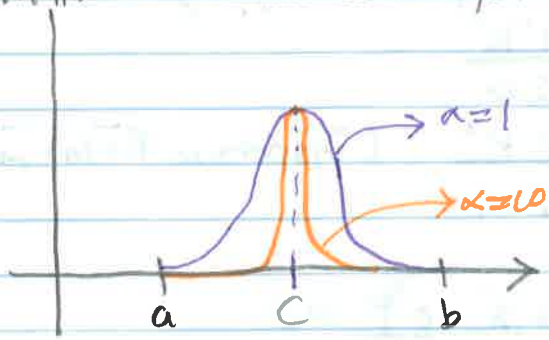
$$\frac{\omega}{k} = \pm\sqrt{1 + 1/k^2}$$

Laplace's Method

Suppose we want to approximate something like

$$I[\alpha] = \int_a^b f(x)\, e^{-\alpha h(x)}\, dx \quad , \text{ as } \alpha \to \infty$$

where $h: \mathbb{R} \to \mathbb{R}^+$ and $\exists c \in (a,b)$ where $h'(c)=0$, $h''(c) > 0$



$$I[\alpha] \approx \int_a^b f(c)\, e^{-\alpha h(c) + \alpha h''(c)(x-c)^2} dx$$

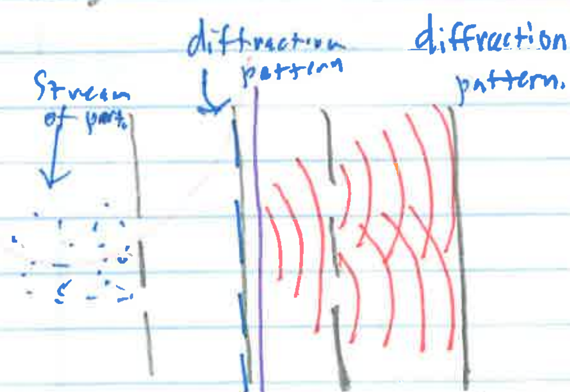
$$\approx f(c) e^{-\alpha h(c)} \int_a^b e^{-h''(c)(x-c)^2} dx$$

$$\approx f(c) e^{-\alpha h(c)} \int_{-\infty}^{\infty} e^{-\alpha h''(c)(x-c)^2} dx$$

$$= f(c)\, e^{-\alpha h(c)} \sqrt{\frac{2\pi}{h''(c)\alpha}}$$

$$I[\alpha] e^{\alpha h(c)} \sim \sqrt{\frac{2\pi}{h''(c)\alpha}} + O(1/\alpha)$$

Schrödinger's equation



• → particle in space

$$E = \frac{p^2}{2m}$$

A stream of particles diffracts like a wave with:

$p = \hbar k$ (de Broglie relationship)

The energy of a free particle is:

$$E = \hbar \omega \quad \text{(Planck / Einstein Photoelectric effect).}$$

- → particle absorbs one quantum of electromagnetic frequency ω.

$$\Rightarrow \hbar \omega = \frac{\hbar^2 K^2}{2m}$$

$$\Rightarrow \omega = \frac{\hbar K^2}{2m} \quad \text{(Dispersion Relationship)}$$

$$\Downarrow$$

$$i\hbar \frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = 0$$

Here Ψ is the wavefunction of the particle.

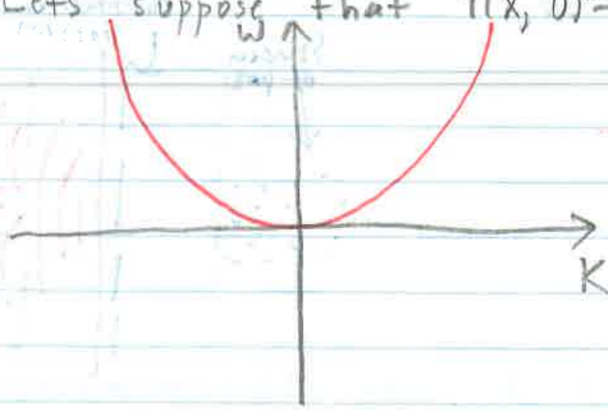
* Dispersion relationship ⇒ P.D.E.

The general solution to schrödingers equation is

$$\Psi = \int_{-\infty}^{\infty} A(k)\, e^{(ikx - i\hbar k^2/2m\, t)}\, dk$$

$$A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx}\, dx.$$

Lets suppose that $\Psi(x,0) = e^{-x^2/w^2} e^{ik_0 x}$ (pulse centered at wave number k_0)



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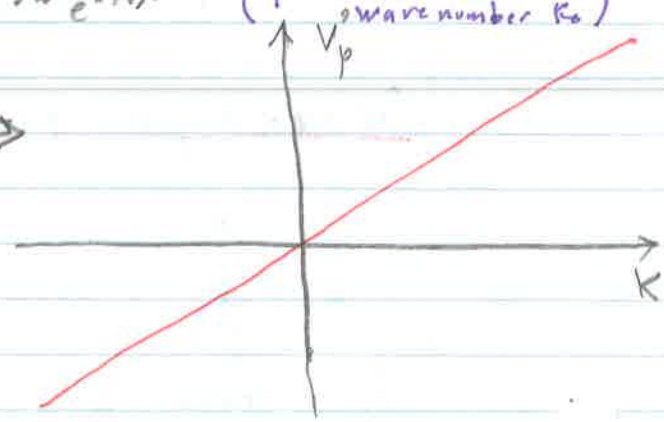
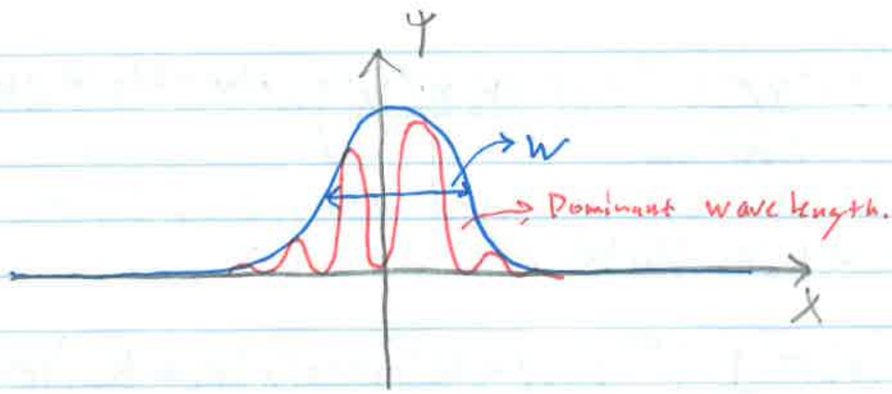




We can calculate:

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{w^2}\right) e^{i(k_0-k)x}\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{w^2}\left(x^2 - i w^2(k_0-k)x\right)\right) dx$$

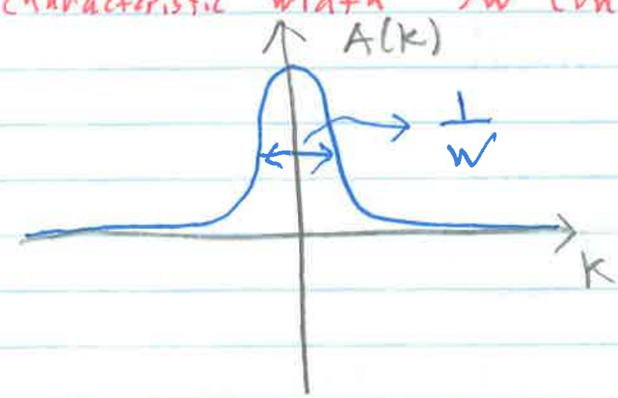
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{w^2}\left(x^2 - i w^2(k_0-k)x + \left(\frac{i(k_0-k)w^2}{2}\right)^2\right)\right) dx$$

$$\times \exp\left(-\frac{(k_0-k)^2 w^2}{4}\right)$$

$$= \frac{1}{2\pi}\exp\left(-\frac{(k_0-k)^2 w^2}{4}\right)\int_{-\infty}^{\infty} \exp\left(-\frac{1}{w^2}\left(x - \frac{i(k_0-k)w^2}{2}\right)^2\right) dx$$

$$\Rightarrow \boxed{A(k) = \frac{w}{2\sqrt{\pi}} \exp\left(-\frac{(k_0-k)^2 w^2}{4}\right)}$$

The pulse is localized in spectrum near $k_0 = k$ with characteristic width $1/w$ (Uncertanty Principle).



Therefore,

$$\Psi(x,t) = \frac{w}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-(k_0-k)^2 w^2}{4}\right) e^{ikx - \hbar k^2/2m\, t}\, dk$$

This can also be calculated explicitely. Let

$$k = k_0 + 1/w\, K$$

$$\Rightarrow \Psi(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{K^2}{4}\right) \exp\left(i\left(k_0 + \frac{K}{w}\right)\right) \exp\left(\frac{-\hbar}{2m}\left(k_0 + \frac{K}{w}\right)^2 t\right) dK$$

$$\Rightarrow \Psi(x,t) = \underbrace{\frac{1}{2\sqrt{\pi}} \exp\left(i\left(k_0 x - \frac{\hbar k_0^2}{2m} t\right)\right)}_{\text{Dominant wave}} \sqrt{\frac{1}{1 + 2i \frac{1}{w^2} \cdot 2\hbar/2m}} \times \underbrace{\exp\left(\frac{-1/w^2 \left(x - \hbar k/m\, t\right)^2}{1 + 2i \cdot \frac{1}{w^2} t \cdot \hbar/m\, t}\right)}_{\text{Pulse Shape.}}$$

$$\Rightarrow \Psi(x,t) = \frac{1}{2\sqrt{\pi}} \exp\left(i\left(k_0 x - \frac{\hbar k_0^2}{2m} t\right)\right) \sqrt{\frac{1}{1 + 2i \frac{1}{w^2} \hbar/m}}\, \exp\left(\frac{-\frac{1}{w^2}\left(x - \hbar k/m\, t\right)^2}{1 + 2i \frac{1}{w^2} \hbar/m\, t}\right)$$

For $t\,\hbar/mw^2 \ll 1$ the pulse moves at a group velocity:

$$\boxed{v_g = \frac{\hbar k}{m} \qquad v_p = \frac{2\hbar k}{m}}$$

(lifetime of electron):

$w = 10^{-15}$ m

$\hbar = 10^{-34}$ $m^2 kg/s$

$m = 10^{-31}$ kg

$$\Rightarrow t \ll \frac{10^{30} \cdot 10^{31}}{10^{34}}\ s$$

$t \ll 10^{27}$ seconds $= 3 \times 10^{19}$ years

age of universe 10^{10} years.

Example - (Linearized KdV)

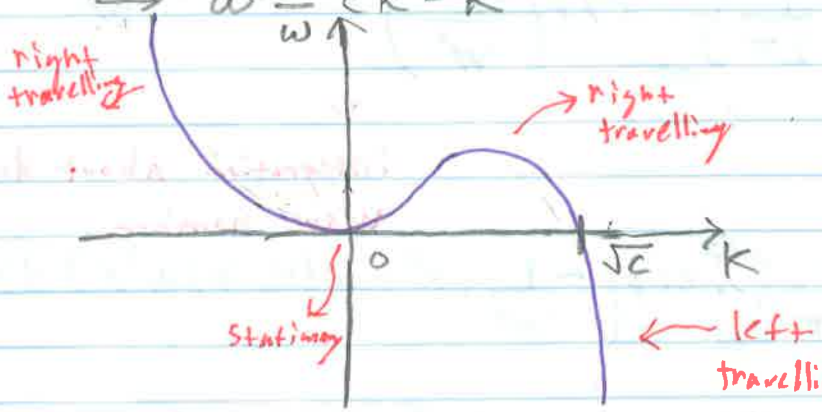
$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

To determine the dispersion relationship look for solutions of the form:

$$u(x,t) = Ae^{ikx - i\omega t}$$

$$\Rightarrow -i\omega + cik - ik^3 = 0$$

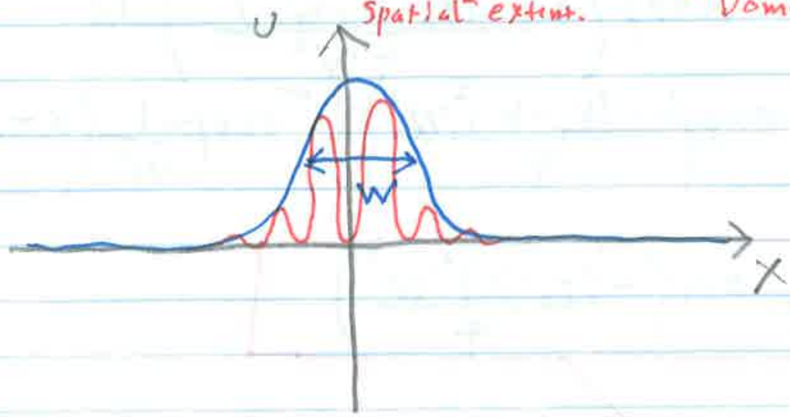
$$\Rightarrow \omega = ck - k^3$$



Lets investigate the behavior of pulses:

$$u(x,0) = e^{-x^2/w^2} e^{ik_0 x}$$

Spatial extent.
Dominate wavenumber



Lets try to solve the P.D.E.

$$u_t + cu_x + u_{xxx} = 0$$

The general solution is given by:

$$u(x,t) = \int_{\infty}^{\infty} A(k) e^{ikx - i\omega(k)} dk.$$

We also have at $t=0$ that

$$u(x,0) = \exp\left(\frac{-x^2}{W^2}\right)\exp(ik_0 x) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk$$

Take inverse Fourier transform:

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(\frac{-x^2}{W^2}\right)\exp(ik_0x)\,e^{-ikx}\,dx.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(\frac{-x^2}{W^2}\right)e^{i(k_0-k)x}\,dx$$

integration about dominant wave number.

$$\Rightarrow A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{W^2}\left(x^2 - i(k_0-k)W^2x\right)\right)dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(\frac{-1}{W^2}\left(x^2 - i(k_0-k)W^2x + \frac{(i(k_0-k)W^2x)^2}{4}\right)\right)$$

$$\times\exp\left(+\frac{(i(k_0-k)W^2)^2}{4W^2}\right)dx$$

$$= \frac{1}{2\pi}\exp\left(-\frac{(k_0-k)^2W^2}{4}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{1}{W^2}\left(x-\frac{i(k_0-k)W}{2}\right)^2\right)dx$$

$$\boxed{A(k) = \frac{W}{2\sqrt{\pi}}\exp\left(\frac{-(k_0-k)^2W^2}{4}\right)}$$

localized near $k_0=k$ if W is small.
(Uncertainty Principle).

Therefore,

$$\boxed{u(x,t) = \frac{W}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left(\frac{-(k_0-k)^2W^2}{4}\right)e^{ikx-i\omega(k)t}\,dk.}$$

We can try to recover a group velocity let $k = k_0 + \frac{1}{w^2}K$

Then,

$$\omega(k) = \omega(k_0 + \tfrac{1}{w^2}K)$$
$$\approx \omega(k_0) + \omega'(k_0)\frac{K}{w^2} + \frac{\omega''(k_0)}{2w^4}K^2 + \ldots$$

$$\Rightarrow u(x,t) = e^{ik_0x - i\omega(k_0)t}\sqrt{\frac{1}{1+2i\frac{1}{w^2}\omega''(k_0)t}}\exp\left(\frac{-\frac{1}{w^2}(x-\omega'(k_0)t)^2}{1+2i\frac{1}{w^2}t\,\omega''(k_0)}\right)$$

The group velocity is

$$\boxed{\begin{aligned} V_g &= \omega'(k_0) \\ V_p &= \frac{\omega}{K} \end{aligned}}$$

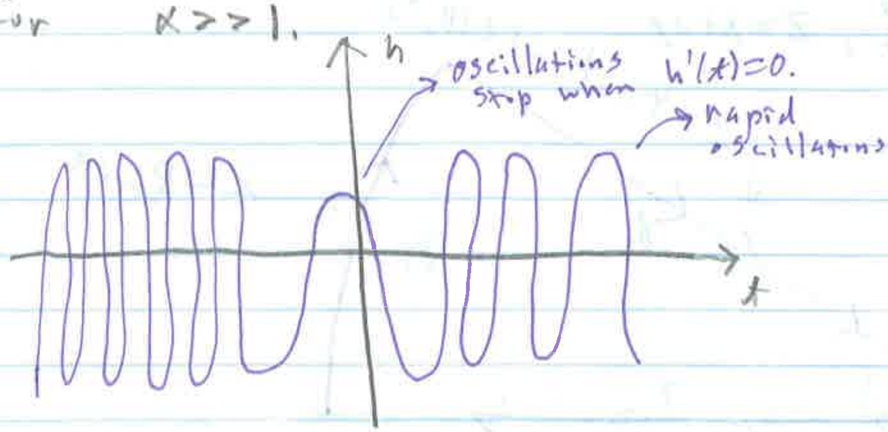
What happens for large t?

Method of Stationary Phase

We want to approximate an integral of the form:

$$I[\alpha] = \int_{-\infty}^{\infty} f(t)\, e^{i\alpha h(t)}\, dt$$

for $\alpha \gg 1$.



Lets first consider

$$I[\alpha] = \int_a^b f(t) e^{i\alpha h(t)} dt \quad , \quad \alpha >> 1.$$

1. Suppose $h'(a) = 0$ and $h' \neq 0$ for all $t \in (a, b]$.

The idea is to again expand

$$I[\alpha] \approx \int_a^b f(a) \exp\left(i\alpha\, h(a) + i\frac{(t-a)^2}{2} h''(a)\right) dt$$

Let $z = t - a$

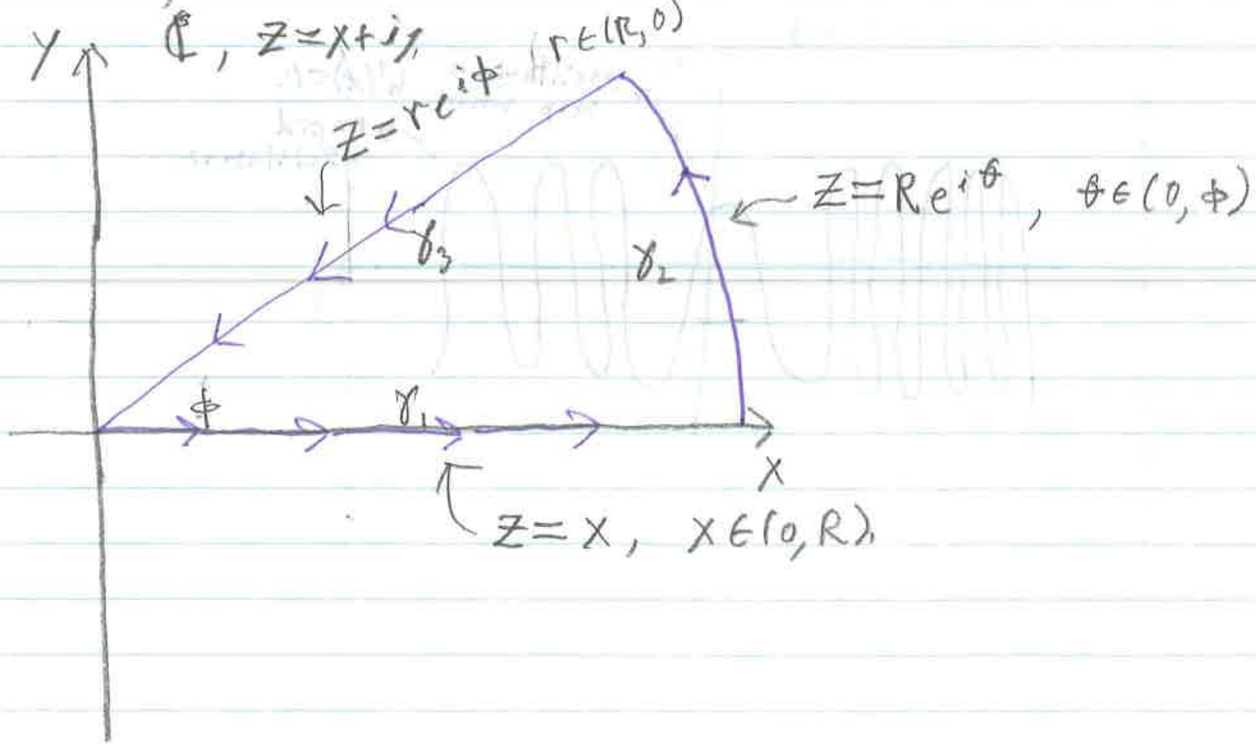
$$\Rightarrow I[\alpha] \approx \int_0^{b-a} f(a) \exp\left(i\alpha h(a)\right) \exp\left(\frac{i z^2}{2} h''(a)\right) dz$$

$$\approx \int_0^{\infty} f(a) \exp\left(i\alpha h(a)\right) \exp\left(\frac{i z^2}{2} h''(a)\right) dz$$

To evaluate this integral we will use Cauchy's theorem:

$$\oint_{\gamma} F(z)\, dz = 0 \quad , \quad \text{let } F(z) = \exp\left(\frac{i z^2}{2} h''(a)\right)$$

If F, F' are defined:



$\underline{\gamma_1:}$

$$dz = dx$$

$$\int_{\gamma_1} F(z)\,dz = \int_0^R \exp\left(\frac{i x^2}{2} h''(a)\right) dx$$

$\underline{\gamma_2:}$

$$dz = R i e^{i\theta} d\theta$$

$$\int_{\gamma_2} F(z)\,dz = \int_0^{\phi} \exp\left(i R^2 e^{i2\theta} h''(a)\right) d\theta$$

If $h''(a) < 0$ we have

$$\lim_{R\to\infty} \int_{\gamma_2} F(z)\,dz = 0$$

$\underline{\gamma_3:}$

$$I[\alpha] \sim \frac{\sqrt{\pi}\, f(a)\, e^{i\alpha h(a) + (1/4) i\pi}}{\sqrt{2\alpha h''(a)}}$$

This integral is difficult to evaluate. Instead we will use a technique called the method of stationary phase to evaluate these integrals.

Method of Stationary Phase:

We want to approximate an integral of the form:

$$I[\alpha] = \int_a^b f(t)\, e^{i\alpha h(t)}\, dt \quad \text{as} \quad \alpha \to \infty.$$

example:

Suppose $h'(t) \neq 0$ on (a,b). Integrating by parts we have

$$I[\alpha] = \frac{1}{i\alpha} \int_a^b \frac{f(t)}{h'(t)} \left(\frac{d}{dt} e^{i\alpha h(t)} \right) dt$$

$$= \frac{1}{i\alpha} \frac{f(t)}{h'(t)} e^{i\alpha h(t)} \Big|_a^b - \frac{1}{i\alpha} \int_a^b \frac{d}{dt}\left(\frac{f(t)}{h'(t)} \right) e^{i\alpha h(t)} dt$$

$$\Rightarrow |I[\alpha]| \sim \frac{1}{\alpha} \left(\frac{f(b)}{h'(b)} e^{i\alpha h(a)} - \frac{f(a)}{h'(a)} e^{i\alpha h(b)} \right)$$

The integral goes to zero at rate $O(1/\alpha)$ as $\alpha \to \infty$.

What happens if $h'(t) = 0$ at some point? We need to use Cauchy's theorem and Laplace's method.

Theorem - If $f: \mathbb{C} \to \mathbb{C}$ satisfies $f'(z)$ exists then for any closed contour:

$$\int_C f(z)\, dz = 0.$$

Universal Equations – Nonlinear Schrödinger Equation

We will analyze the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + \omega_0^2 u \overset{?}{=} \varepsilon^2 \omega_0^2 u^3$$

We will look at the behavior of nearly monochromatic pulses:

First Guess:

$$u = A(X=\varepsilon^2 x, T=\varepsilon^2 t)\, e^{ik_0 x - i\omega(k_0)t} + (*).$$

$$\omega(k_0) = \sqrt{\omega_0^2 + c^2 k_0^2}$$

$O(\varepsilon^2)$:

$$\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} + u_1 - 2ik_0\frac{\partial A}{\partial X} - 2i\omega(k_0)\frac{\partial A}{\partial T} = A^3 e^{3ik_0 x - 3i\omega(k_0)t} + 3A^2A^* e^{ik_0 x - i\omega(k_0)t} + (*)$$

To remove secular terms:

$$k_0 c^2 \frac{\partial A}{\partial X} + \omega(k_0)\frac{\partial A}{\partial T} = +\frac{3}{2} i\omega_0^2 A^2 A^*$$

$$\Rightarrow \frac{k_0 c^2}{\omega(k_0)}\frac{\partial A}{\partial X} + \frac{\partial A}{\partial T} - \frac{3}{2}\frac{i\omega_0^2}{\omega(k_0)} A^2 A^* = 0$$

$$\Rightarrow \omega'(k_0)\frac{\partial A}{\partial X} + \frac{\partial A}{\partial T} - \frac{3}{2}\frac{i\omega_0^2}{\omega(k_0)} A^2 A^* = 0$$

Method of characteristics:

$$\frac{dX}{dT} = \omega'(k_0) \Rightarrow X = \omega'(k_0)t + X_0$$

$$\frac{dA}{dT} = \frac{3}{2}\frac{i\omega_0^2}{\omega(k_0)} A^2 A^*.$$

$$\boxed{\omega'(k_0) = \text{The group velocity.}}$$

$$\Rightarrow A(X,T) = A_0(X - \omega'(k_0)T)\exp\left(\frac{3}{2}\frac{i\omega_0^2}{\omega(k_0)} |A_0(X-\omega'(k_0)T)|^2\right)$$

This says the pulse simply propagates to the right at the group velocity $V_g = \omega'(k_0)$. This says dispersion has no effect!

Second Guess:

$$u = A(X_1 = \varepsilon x, t_1 = \varepsilon t, t_2 = \varepsilon^2 t)\, e^{ik_0 x - i\omega(k_0)t}$$

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$$

$\mathcal{O}(\varepsilon)$:

$$\frac{\partial^2 u_1}{\partial t^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} + \omega_0^2 u_1 - 2i\left(\omega(k_0)\frac{\partial A}{\partial t_1} + k_0^2 c^2 \frac{\partial A}{\partial X_1}\right) e^{ik_0 x - i\omega(k_0)t} = 0$$

$$\Rightarrow \frac{\partial A}{\partial t_1} + \omega'(k_0)\frac{\partial A}{\partial X_1} = 0$$

$\boxed{\text{On time } 0 < t < 1/\varepsilon \text{ travels at group velocity } V_g = \omega'(k_0)}$

$\mathcal{O}(\varepsilon^2)$:

$$\frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} + \omega_0^2 u_2 + \left(-2i\omega(k_0)\frac{\partial A}{\partial t_1} + \frac{\partial^2 A}{\partial t_2^2} - c^2 \frac{\partial^2 A}{\partial X_1^2}\right) e^{ik_0 x - i\omega(k_0)t}$$

$$= \omega_0^2 A^2 A^* e^{ik_0 x - i\omega(k_0)t} + \text{other terms}$$

$$\Rightarrow 2i\omega(k_0)\frac{\partial A}{\partial t_2} - \frac{\partial^2 A}{\partial t_1^2} + c^2 \frac{\partial^2 A}{\partial X_1^2} + \omega_0^2 A^2 A^* = 0$$

Change variables:

$$\xi = X_1 - \omega'(k_0)t_1, \quad \tau = t_1$$

$$X_1 = \xi + \omega'(k_0)\tau, \quad t_1 = \tau$$

$$\frac{\partial}{\partial t_1} = \frac{\partial \xi}{\partial t_1}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t_1}\frac{\partial}{\partial \tau} = -\omega'(k_0)\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau}$$

$$\frac{\partial}{\partial X_1} = \frac{\partial \xi}{\partial X_1}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial X_1}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \xi}$$

Also,

$$\frac{\partial A}{\partial \tau} = \frac{\partial A}{\partial t_1}\frac{\partial t_1}{\partial \tau} + \frac{\partial A}{\partial x_1}\frac{\partial x_1}{\partial \tau} = 0$$

$$\Rightarrow 2i\omega(k_0)\frac{\partial A}{\partial t_2} + (c^2 - \omega'(k_0)^2)\frac{\partial^2 A}{\partial \xi^2} + \frac{1}{2}A^2A^* = 0$$

NLS: $$i\frac{\partial A}{\partial t_2} + \gamma\frac{\partial^2 A}{\partial \xi^2} + \beta A^2 A^* = 0, \quad \beta = \frac{1}{4\omega(k_0)}$$

Group velocity dispersion

nonlinear coupling.

Benjamin-Feir Instability

Let make the guess:

$$A = A_0 \exp(i\alpha t_2)$$

$$\Rightarrow -\alpha A_0 + \beta A_0 |A_0|^2 = 0$$

$$\Rightarrow \alpha = \beta |A_0|^2$$

$A = A_0 \exp(i\beta|A_0|^2 t_2)$ is an exact solution that corresponds to the solution obtained by the first guess.

We now check stability. Take the following perturbation

$$A = (a e^{ik\xi + \sigma t_2} + A_0 e^{i\beta|A_0|^2 t_2}$$

perturb by Fourier modes

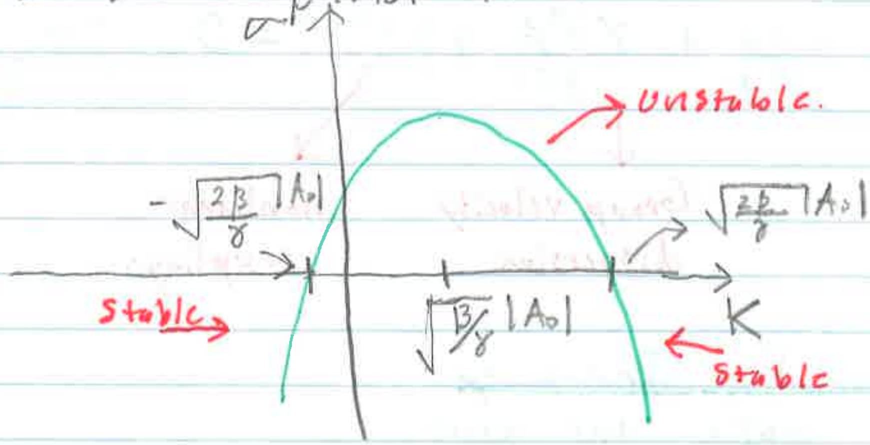
$$\Rightarrow -\beta A_0|A_0|^2 + a i\sigma e^{ik\xi+\sigma t_2} - k^2 a e^{ik\xi+\sigma t_2}\gamma$$

$$+\beta(A_0 + a e^{ik\xi+\sigma t_2})^2(A_0^* + a^* e^{-ik\xi+\sigma t_2}) = 0$$

$$\Rightarrow -\beta A_0|A_0|^2 + (-a i\sigma)e^{ik\xi+\sigma t_2} - k^2 a e^{ik\xi+\sigma t_2}\gamma$$
$$+ 2\beta|A_0|^2 a\, e^{ik\xi+\sigma t_2} + \beta A_0|A_0|^2 = 0$$

nonlinear terms in a have been dropped.

$$\Rightarrow i\sigma - k^2\gamma + 2\beta|A_0|^2 = 0$$
$$\Rightarrow \sigma^2 = 2\beta|A_0|^2 - k^2\gamma.$$



Therefore, this solution is unstable to small perturbations in Fourier space.

Nearly Monochromatic Waves

Recall for linear dispersive P.D.E. solutions can be expressed as:

$$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega(k)t)}dk.$$

A Slowly varying wave train is:

$$u(x,t) = A(x,t)e^{i\theta(x,t)}$$

- slowly varying wave number:

$$k = \frac{\partial\theta}{\partial x}$$

- slowly varying frequency:

$$\omega = -\frac{\partial\theta}{\partial t}$$

We can get a simple consvertation of waves law:

$$\boxed{\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0}$$

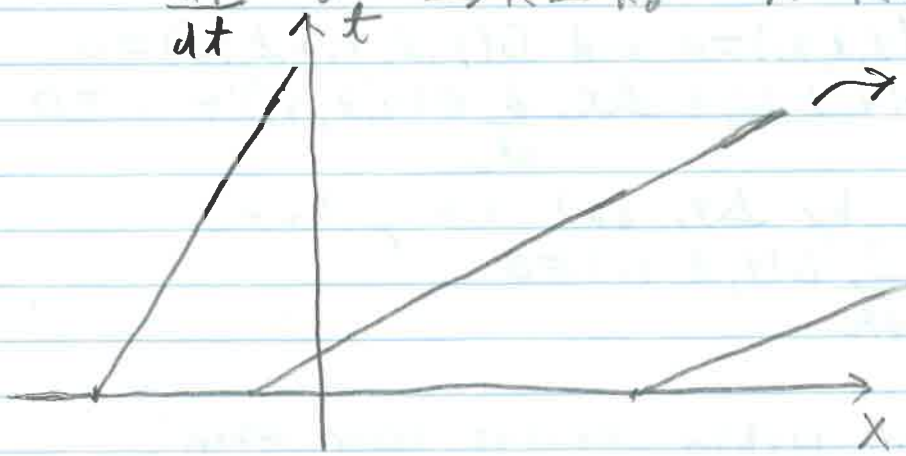
Now, assuming ω satisfies a dispersion relationship $\omega(k)$ we have that:

$$\frac{\partial k}{\partial t} + \frac{d\omega}{dk}\frac{\partial k}{\partial x} = 0$$

We can solve this using the method of characteristics.

$$\frac{dx}{dt} = \frac{d\omega}{dk} \Rightarrow x = \frac{d\omega}{dk} t + x_0$$

$$\frac{dk}{dt} = 0 \Rightarrow k = k_0 \Rightarrow k = k\left(x - \frac{d\omega}{dk} t\right)$$



k is constant along the characteristics.

k moves at the group velocity $V_g = \frac{d\omega}{dk}$.

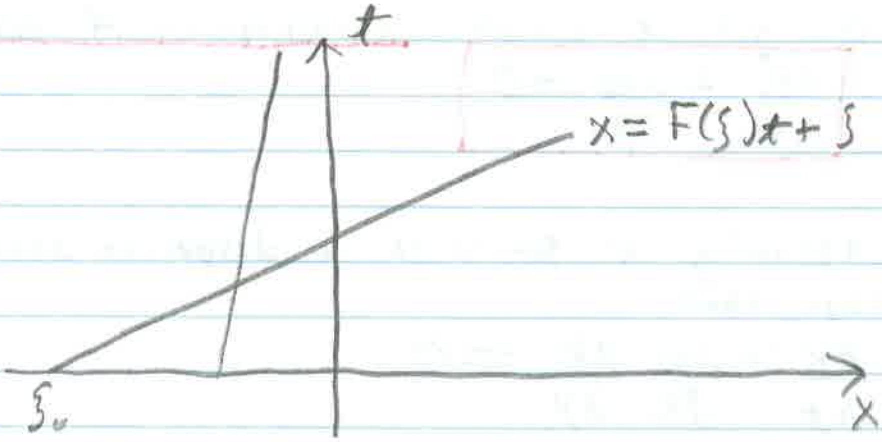
Caustics -

Characteristics all have slope $\frac{d\omega}{dk}$.

$$x = \omega'(k(\xi, 0))t + \xi \quad , \quad \xi = x - \frac{d\omega}{dk} t$$

$$x = F(\xi)t + \xi$$

Characteristics intersect if characteristics to the right move more slowly: $F'(\xi) < 0$.



To locate the point of intersection, let

$$G(x, t, x_0) = 0$$

describe family of curves parametrized by x_0. Now suppose

$$G(x, t, x_0) = 0 \quad \text{and} \quad G(x, t, x_0 + \Delta x_0) = 0$$

$$\Rightarrow G(x, t, x_0) + \Delta x_0 \frac{\partial}{\partial x_0} G(x, t, x_0) + \dots = 0$$

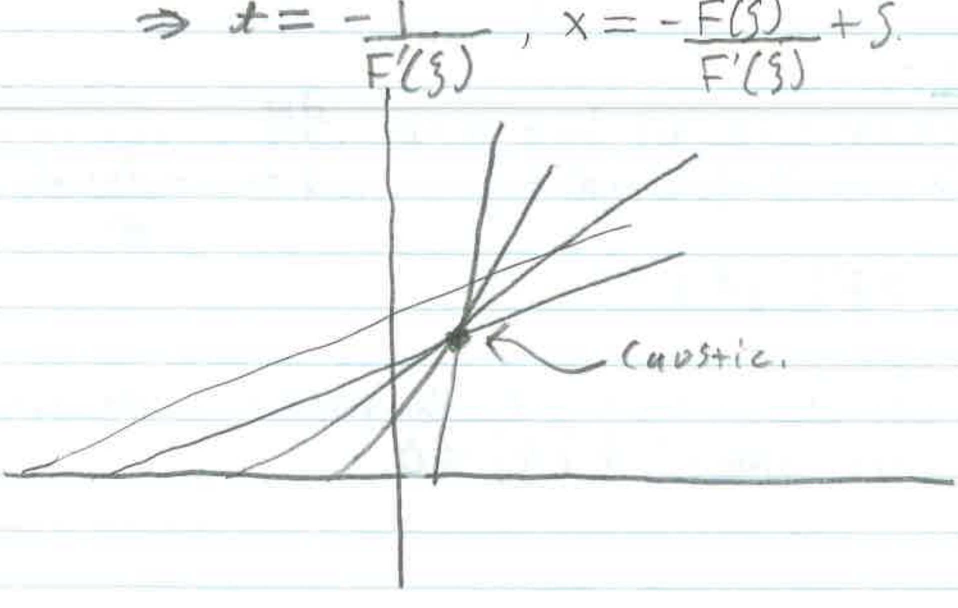
Dividing by Δx_0 and taking limit

$$\frac{\partial}{\partial x_0} G(x, t, x_0) = 0$$

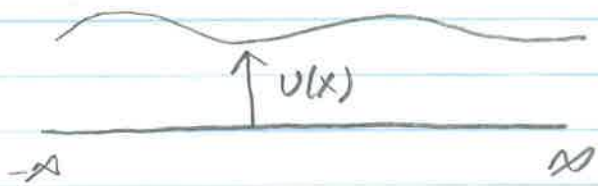
For our problem caustics occur when:

$$0 = F'(\xi) t + 1 = 0$$

$$\Rightarrow t = -\frac{1}{F'(\xi)}, \quad x = -\frac{F(\xi)}{F'(\xi)} + \xi$$



Korteweg-de Vries Equation



height of shallow water waves in a channel.

$$\underbrace{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}}_{\text{advection}} + \underbrace{\beta u \frac{\partial u}{\partial x}}_{\text{conservation law}} = \underbrace{\gamma \frac{\partial^3 u}{\partial x^3}}_{\text{dispersion}}$$

Conservation of Mass:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} - \beta \frac{\partial}{\partial x}\frac{u^2}{2} + \gamma \frac{\partial^3 u}{\partial x^3}$$

$$\Rightarrow \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x)\,dx = 0$$

Conservation of Momentum:

$$u\frac{\partial u}{\partial t} = -cu\frac{\partial u}{\partial x} - \beta u^2 \frac{\partial u}{\partial x} + \gamma u \frac{\partial^3 u}{\partial x^3}$$

$$\Rightarrow \int \frac{1}{2}\frac{\partial}{\partial t} u^2\,dx = -\frac{c}{2}\int \frac{\partial}{\partial x} u^2\,dx - \frac{\beta}{3}\int \frac{\partial}{\partial x} u^3\,dx$$

$$+ \gamma \int u \frac{\partial^3 u}{\partial x^3}\,dx$$

$$= -\gamma \int u_x u_{xx}\,dx$$

$$= -\frac{\gamma}{2}\int \frac{\partial}{\partial x} u_x^2\,dx$$

$$= 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int u^2\,dx = 0.$$

Travelling frame:

$$z = x - ct$$
$$\tau = t$$

$$\Rightarrow \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{c}{\partial z}$$
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial \tau} + \beta u \frac{\partial u}{\partial z} = \gamma \frac{\partial^3 u}{\partial z^3}.$$

We can rescale again to obtain KdV in standard form:

$$\frac{\partial u}{\partial \tau} + 6u\frac{\partial u}{\partial z} + \frac{\partial^3 u}{\partial z^3} = 0.$$

Conservation of Energy:

$$u^2\frac{\partial u}{\partial \tau} + 6u^3\frac{\partial u}{\partial z} + u^2\frac{\partial^3 u}{\partial z^3} = 0$$

$$\Rightarrow \frac{1}{3}\frac{\partial}{\partial \tau}u^3 + \frac{3}{2}\frac{\partial}{\partial z}u^4 + u^2\frac{\partial^3 u}{\partial z^3} = 0$$

$$\Rightarrow \frac{\partial}{\partial \tau}\int \frac{1}{3}u^3\,dx + \int u^2\frac{\partial^3 u}{\partial z^3} = 0$$

$$\Rightarrow \frac{\partial}{\partial \tau}\int \frac{1}{3}u^3\,dx - \int 2u\cdot\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} = 0$$

$$\Rightarrow \frac{\partial}{\partial \tau}\int \frac{1}{3}u^3\,dx + \frac{1}{3}\int\left(\frac{\partial u}{\partial \tau} + \frac{\partial^3 u}{\partial z^3}\right)\frac{\partial^2 u}{\partial z^2} = 0$$

$$\Rightarrow \frac{\partial}{\partial \tau}\int \frac{1}{3}u^3\,dx - \frac{1}{3}\int \frac{\partial^2 u}{\partial \tau\,\partial x}\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{\partial}{\partial \tau}\int\left(\frac{1}{3}u^3-\frac{1}{6}\cdot u_x'^2\right)dx=0$$

We are interested in solutions with finite energy.

Traveling Wave Solutions.

Let $\xi = z-\nu\tau$, $u(z,\tau)=f(\xi)$:

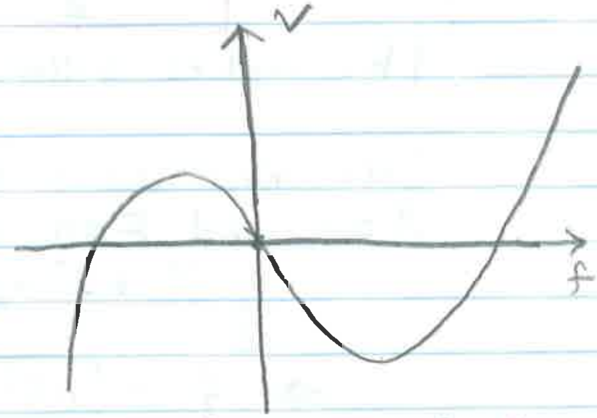
$$-\nu f' + 6ff' + f''' = 0$$

$$\Rightarrow -\nu f' + 3\frac{d}{d\xi}f^2 + f''' = 0$$

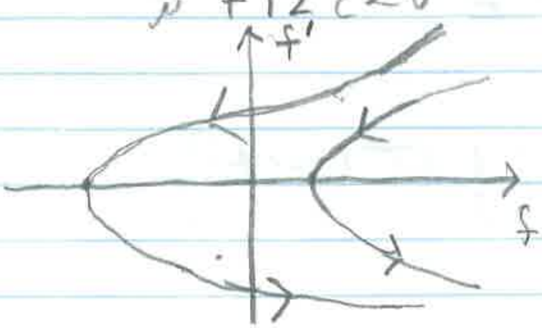
$$\Rightarrow -\nu f + 3f^2 + f'' = C.$$

$$\Rightarrow \frac{1}{2}f'^2 + f^3 - \nu\frac{f^2}{2} - Cf = E$$

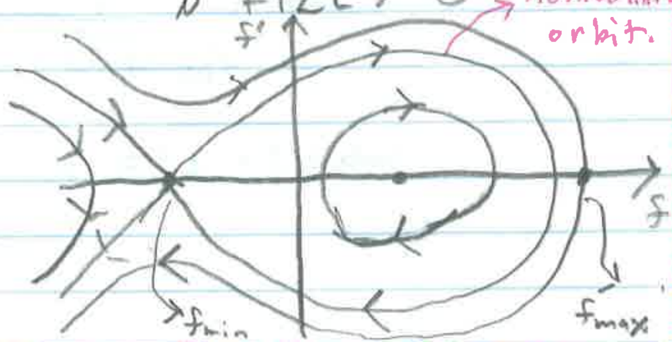
If $3f^2 - \nu f - C = 0$ the potential will have two fixed points.



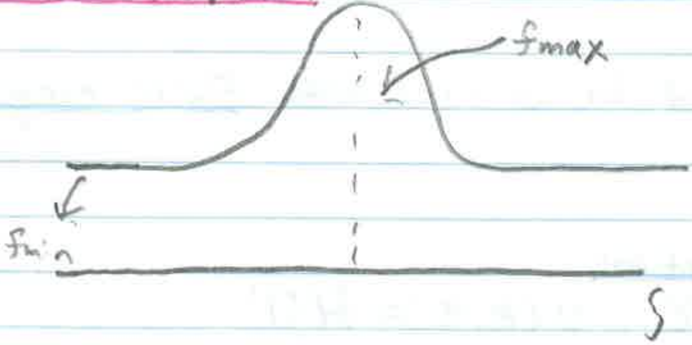
$\nu^2 + 12c < 0$



$\nu^2 + 12c > 0$



Only one solution has finite energy; the homoclinic orbit.



Can we calculate f_{max}? If we choose E correctly:

$$\frac{1}{2} f'^2 = -(f-f_{max})(f-f_{min})^2$$

$$= -(f-f_{max})(f^2-2ff_{min}+f_{min}^2)$$

$$= -f^3 - 2f^2 f_{min} + f f_{min}^2 - f_{max} f^2 + 2f f_{min} f_{max} - f_{max} f_{min}^2$$

When $C=0$ we get:

$$f_{min} = 0$$

$$\boxed{\nu = \frac{1}{2} f_{max}.}$$

This implies taller solitary waves move faster.

Lets now derive the form of the solitary wave. Setting $A=0$ and $E=0$ we have that:

$$\frac{1}{2} f'^2 + f^3 - \frac{\nu f^2}{2} = 0$$

$$\Rightarrow f' = \sqrt{\nu f^2 - 2f^3}$$

$$\Rightarrow \int_{-\infty}^{z} \frac{1}{\sqrt{\nu f^2 - 2f^3}} = z$$

$$\Rightarrow u(x,t) = \frac{1}{2} c \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c}(x-ct)\right)$$

Weakly nonlinear kdV

Suppose $U = \mathcal{O}(\varepsilon)$

$$\frac{\partial U}{\partial t} + \varepsilon U \frac{\partial U}{\partial X} + \frac{\partial^3 U}{\partial X^3} = 0$$

$$U = A(\mathcal{X}, T_1, T_2) e^{iK_0 X - iK_0^3 t} + \varepsilon U_1 + \varepsilon^2 U_2 + \dots$$

$\mathcal{X} = \varepsilon X$, $\dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial t} + \varepsilon \dfrac{\partial}{\partial T_1} + \varepsilon^2 \dfrac{\partial}{\partial T_2}$

$T_1 = \varepsilon t$,

$T_2 = \varepsilon^2 t$, $\dfrac{\partial}{\partial X} = \dfrac{\partial}{\partial X} + \varepsilon \dfrac{\partial}{\partial \mathcal{X}}$

$$\frac{\partial^2}{\partial X^2} = \frac{\partial^2}{\partial X^2} + 2\varepsilon \frac{\partial^2}{\partial X \partial \mathcal{X}} + \varepsilon^2 \frac{\partial^2}{\partial \mathcal{X}^2}$$

$$\frac{\partial^3}{\partial X^3} = \frac{\partial^3}{\partial X^3} + 3\varepsilon \frac{\partial^3}{\partial X^2 \partial \mathcal{X}} + 3\varepsilon^2 \frac{\partial^3}{\partial X \partial \mathcal{X}^2} + \varepsilon^3 \frac{\partial^3}{\partial \mathcal{X}^3}$$

$\underline{\mathcal{O}(\varepsilon)}$:

$$\frac{\partial U_1}{\partial t} - \frac{\partial^3 U_1}{\partial X^3} + \frac{\partial A}{\partial T_1} e^{ik_0 X + ik_0^3 t} + iK_0 A^2 e^{2i(K_0 X + K_0^3 t)}$$

$$= +3K \frac{\partial A}{\partial X} e^{iK_0 X - iK_0^3 t}$$

$$\Rightarrow \quad \frac{\partial A}{\partial T_1} - 3K^2 \frac{\partial A}{\partial X} = 0$$

$$\frac{\partial U_1}{\partial t} - \frac{\partial^3 U_1}{\partial X^3} + iK_0 A^2 e^{2i(K_0 X + K_0^3 t)}$$

$$\Rightarrow U_1 = \frac{A^2}{6K_0} e^{i(2K_0 X + 2K_0^3 t)}$$

$\underline{\mathcal{O}(\varepsilon^2)}$