Chapter 3: Multiple Scales

Example

$$\ddot{y} + \varepsilon\dot{y} + y = 0$$
$$y(0) = 0$$
$$\dot{y}(0) = 1$$

The exact solution is:

$$y(t) = \frac{1}{\sqrt{1-\varepsilon^2/4}}\, e^{-\varepsilon t/2} \sin\left(t\sqrt{1-\varepsilon^2/4}\right)$$

period $= 2\pi/\sqrt{1-\varepsilon^2/4} \sim 2\pi(1+\varepsilon^2/8+\dots$

Damping scale $= 1/\varepsilon$

Lets try a naive expansion:

$$y = y_0 + \varepsilon y_1 + \dots$$

$O(1)$:

$$\ddot{y}_0 + y_0 = 0$$
$$\Rightarrow y_0 = \sin(t)$$

$O(\varepsilon)$:

$$\ddot{y}_1 + y_1 = -\cos(t)$$

Guess:

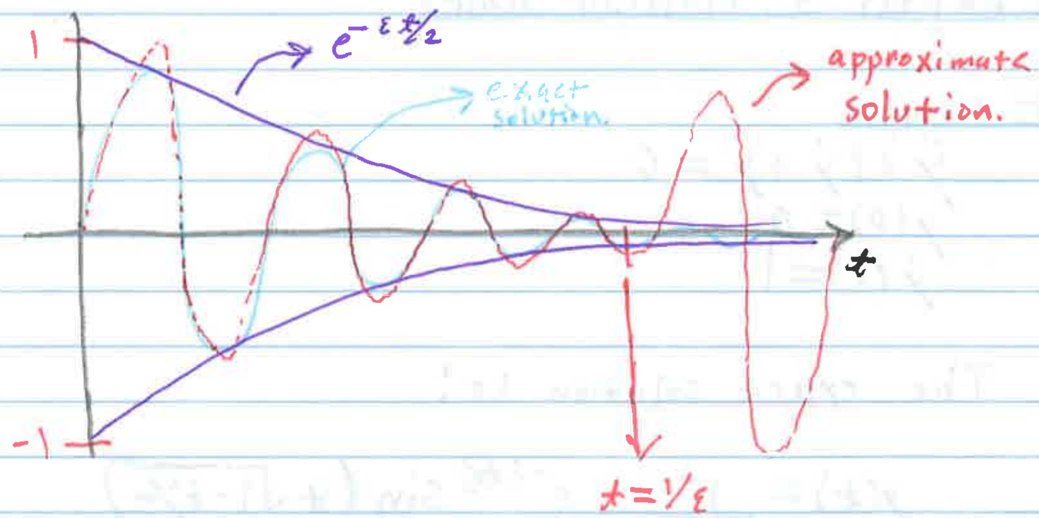
$$y_1 = At\sin(t) + Bt\cos(t)$$
$$\Rightarrow \dot{y}_1 = A\sin(t) + At\cos(t) + B\cos(t) - Bt\sin(t)$$
$$\Rightarrow \ddot{y}_1 = 2A\cos(t) - At\sin(t) - 2B\sin(t) - Bt\cos(t)$$

Set $A = 1/2$, $B = 0$

$$\Rightarrow y_1 = \frac{1}{2}t\sin(t).$$

Therefore, we get

$$y \sim \sin(t) + \frac{1}{2}\varepsilon t\sin(t).$$



The problem has two time scales:

$t_1 = t$ and $t_2 = \varepsilon t$.

A useful expansion has to account for both of these scales.

The idea is to assume the solution depends on the two scales,

$$y(t, \varepsilon) = y(t_1, t_2, \varepsilon).$$

Now,

$$\frac{d}{dt} = \frac{\partial t_1}{\partial t}\frac{\partial}{\partial t_1} + \frac{\partial t_2}{\partial t}\frac{\partial}{\partial t_2}$$

$$= \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2}$$

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}\right)\left(\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}\right)$$

$$= \frac{\partial^2}{\partial t_1^2} + \varepsilon\frac{\partial^2}{\partial t_2 \partial t_1} + \varepsilon\frac{\partial}{\partial t_1 \partial t_2} + \varepsilon^2\frac{\partial^2}{\partial t^2}$$

Can't always assume mixed partials commute.

The ODE now becomes:

$$\frac{\partial^2 y}{\partial t_1^2} + 2\varepsilon \frac{\partial^2 y}{\partial t_1 \partial t_2} + \varepsilon^2 \frac{\partial^2 y}{\partial t_2^2} + \varepsilon \frac{\partial y}{\partial t_1} + \varepsilon^2 \frac{\partial y}{\partial t_2} + y = 0$$

Again assume:

$$y = y_0 + \varepsilon y_1 + \ldots$$

$\mathcal{O}(1)$:

$$\frac{\partial^2 y_0}{\partial t_1^2} + y_0 = 0$$

$$\Rightarrow y_0(t_1, t_2) = A(t_2)\sin(t_1)$$

$\mathcal{O}(\varepsilon)$:

$$\frac{\partial^2 y_1}{\partial t_1^2} + 2\frac{\partial A}{\partial t_2}\cos(t_1) + A(t_2)\cos(t_1) + y_1 = 0$$

new term

$$\Rightarrow \frac{\partial^2 y_1}{\partial t_1^2} + y_1 = -2\frac{\partial A}{\partial t_2}\cos(t_1) - A(t_2)\cos(t_1) = 0$$

To remove the secular terms we must have:

$$\frac{\partial A}{\partial t_2} = -\frac{1}{2} A(t_2)$$

$$\Rightarrow A(t_2) = e^{-t_2/2}$$

Therefore,

$$y(t) \sim e^{-t/2\varepsilon}\sin(t_1)$$

Note, the exact solution was

$$y(t) = \frac{1}{\sqrt{1-\varepsilon^2/4}} e^{-\varepsilon t/2}\sin(t\sqrt{1-\varepsilon^2/4})$$

Remark:

We derived an amplitude equation:

$$\frac{\partial A}{\partial t_2} = -\frac{1}{2} A(t_2)$$

which governed the information transfer in the system.

Example:

$\ddot{y} + \varepsilon \dot{y}^3 + y = 0$

$y(0) = 1, \quad \dot{y}(0) = 0$

Naive Expansion:

$y = y_0 + \varepsilon y_1 + \ldots$

$O(1)$:

$y = \sin(t)$

$O(\varepsilon)$:

$\ddot{y}_1 + y_1 = -\cos^3(t) = \cos 3t + 3 \sin t \cos^2 t$

$= \cos(3t) + 3 \sin t (1 - \sin^2 t)$

$\downarrow$

resonant term.

To remove secular terms introduce two scales:

$t_1 = t, \quad t_2 = \varepsilon t.$

$$\frac{d}{dt} = \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_1^2} + 2\varepsilon \frac{\partial}{\partial t_1 \partial t_2} + \varepsilon^2 \frac{\partial^2}{\partial t_2^2}$$

$O(1)$.

$$\frac{\partial^2 y_0}{\partial t_1^2} + y_0 = 0$$

$\Rightarrow y_0 = A(t_2) \sin(t).$

$O(\varepsilon)$:

new term.

$$2 \frac{\partial A}{\partial t_2} \cos(t) + A^3(t_2) \cos(t)^3 + A(t_2) \cos(t)$$

$$+ \frac{\partial^2 y_1}{\partial t_1^2} + y_1 = 0$$

$$\Rightarrow 2\frac{\partial A}{\partial t_2}\cos(t) + \frac{A^3(t_2)}{4}\left(3\cos(t) + \cos(3t)\right) + A(t_2)\cos(t)$$

$$+\frac{\partial^2 y_1}{\partial t_1^2} + y_1 = 0$$

To remove secular terms set

$$2\frac{\partial A}{\partial t_2} = -\frac{3}{4}A^3(t_2) + A_2$$

$$A(t_2) = \frac{1}{\sqrt{3 - 2e^{-t_2/2}}}$$

This gives us

$$y \sim \frac{1}{\sqrt{3-2e^{-t/2\varepsilon}}}\sin(t).$$

Example.

$$\frac{d^2 u}{dt^2} + u = -\varepsilon\left(\frac{du}{dt}\right)^3$$

Lets solve the problem but use complex numbers to simplify the analysis.

Assume two scales

$$t_1 = t \text{ and } t_2 = \varepsilon t$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_1^2} + 2\varepsilon\frac{\partial^2}{\partial t_1\partial t_2} + \varepsilon^2\frac{\partial^2}{\partial t_2^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t_1^2} + 2\varepsilon\frac{\partial^2 u}{\partial t_1\partial t_2} + \varepsilon^2\frac{\partial^2 u}{\partial t_2^2} + u = -\varepsilon\left(\frac{\partial u}{\partial t_1} + \varepsilon\frac{\partial u}{\partial t_2}\right)^3$$

Assume

$$U = U_0 + \varepsilon U_1 + \dots$$

$\mathcal{O}(1)$:

$$\frac{\partial^2 U_0}{\partial t_1^2} + U_0 = 0$$

$$\Rightarrow U_0 = A(t_2) e^{i t_1} + (*)$$

this symbol means complex conjugate of all $+iw$ terms.

$\mathcal{O}(\varepsilon)$:

$$\varepsilon\left(\frac{\partial^2 U_1}{\partial t_1^2} + 2\frac{\partial^2 U_0}{\partial t_1 \partial t_2} + U_1\right) = -\varepsilon\left(\frac{\partial U_0}{\partial t_1} + \varepsilon\frac{\partial U_0}{\partial t_2}\right)^3$$

$$\frac{\partial^2 U_1}{\partial t_1^2} + 2\frac{\partial A}{\partial t_2} i e^{iwt_1} - 2\frac{\partial A^*}{\partial t_2} i e^{-iwt_1} + U_1$$

$$= -\left(A i e^{it_1} - A^* i e^{-it_1}\right)^3$$

$$\Rightarrow \frac{\partial^2 U_1}{\partial t_1^2} + 2\frac{\partial A}{\partial t_2} i e^{iwt} - 2\frac{\partial A^*}{\partial t_2} i e^{-iwt_1} + U_1$$

$$= + A^3 i e^{3t_1} - 3A^2A^* i e^{it_1} + 3AA^{*2} i e^{-it_1} - A^{*3} i e^{-3it_1}$$

$$\Rightarrow \frac{\partial^2 U_1}{\partial t_1^2} + U_1 = -2\frac{\partial A}{\partial t_2} i e^{iwt_1} + A^3 i e^{3t_1} - 3A^2A^* i e^{it_1} + (*)$$

To remove the secular terms we need to solve:

$$2\frac{\partial A}{\partial t_2} = -3 i A^2 A^*$$

amplitude

phase shift.

To solve set $A = r(t_2)e^{i\phi(t_2)}$.

$$\frac{\partial A}{\partial t_2} = \left(\frac{\partial r}{\partial t_2} + i r(t_2)\frac{\partial \phi}{\partial t_2}\right)e^{i\phi(t_2)}$$

Comparing real and imaginary parts:

$$\begin{cases} 2\dfrac{dr}{dt_2} = -3r^3 \\ r\dfrac{d\phi}{dt_2} = 0 \end{cases}$$

$$\Rightarrow r(t_2) = \frac{r(0)}{\sqrt{1+3r^2(0)t_2}}$$

$$\phi = \phi_0$$

The $O(1)$ solution is given by:

$$u_0(t) = \frac{r(0)}{\sqrt{1+3r(0)^2\varepsilon t}}e^{i(t+\phi_0)}$$

Example (Weak resonance)

$$\ddot{u} + \varepsilon\dot{u}(u^2-1) + (1+\varepsilon k)u = \varepsilon\cos(t)$$

Clearly a naive expansion will have resonances. We look for two time scales.

$$t_1 = t, \quad t_2 = \varepsilon t$$

$$u = u_0 + \varepsilon u_1 + \dots$$

$$\frac{\partial^2 u}{\partial t_1^2} + 2\varepsilon\frac{\partial^2 u}{\partial t_1 \partial t_2} + \varepsilon^2\frac{\partial^2 u}{\partial t_2^2} + \varepsilon\left(\frac{\partial u}{\partial t_1} + \varepsilon\frac{\partial u}{\partial t_2}\right)(u^2-1)$$

$$+ (1+\varepsilon k)u = \varepsilon\cos(t).$$

$\mathcal{O}(1)$:

$$\frac{\partial^2 u_0}{\partial t_1^2} + u_0 = 0$$

$$u_0 = A e^{it_1} + *$$

$\mathcal{O}(\varepsilon)$:

$$\frac{\partial^2 u_1}{\partial t_1^2} + u_1 + 2\frac{\partial A}{\partial t_2} i e^{it_1} - 2\frac{\partial A^*}{\partial t_2} i e^{-it_1} + \left(A i e^{it_1} - A^* i e^{-it_1}\right) \times$$

$$\left(A^2 e^{2it_1} + 2AA^* + A^{*2} e^{-2it_1} - 1\right) + KAe^{it_1} + KA^* e^{-it_1} = \cos(t_1)$$

We need to collect secular terms.

$$2\frac{\partial A}{\partial t_2} i + 2A^2A^* i - iA - iA^*A^2 + K = \frac{1}{2}$$

Let $A = r(t_2) e^{i\phi(t_2)}$

$$\frac{\partial A}{\partial t_2} = \left(\frac{dr}{dt_2} + i\frac{d\phi}{dt_2} r\right) e^{i\phi(t_2)}$$

$$\frac{dA^*}{dt_2} = \left(\frac{dr}{dt_2} - i\frac{d\phi}{dt_2} r\right) e^{-i\phi(t_2)}$$

$$\Rightarrow 2\left(\frac{dr}{dt_2} + i\frac{d\phi}{dt_2} r\right) i + 2r^3 i - ir - ir^3 + K$$

$$= \frac{1}{2}\left(\cos(\phi(t_2)) - i\sin(\phi(t_2))\right)$$

Coupled system:

$$\begin{cases} -2r\dfrac{d\phi}{dt_2} + K = \dfrac{1}{2}\cos(\phi) \\ 2\dfrac{dr}{dt_2} = -r^3 + r - \dfrac{1}{2}\sin(\phi). \end{cases}$$

To analyze this system we can look at the fixed points:

$$\frac{d\phi}{dt_2} = \frac{1}{2r}\left(K - \frac{1}{2}\cos(\phi)\right)$$

$$\frac{dr}{dt_2} = \frac{1}{2}\left(r - r^3 + \frac{1}{2}\sin(\phi)\right)$$

For a fixed point to exist we must have that:

$$-\tfrac{1}{2} \le K \le \tfrac{1}{2}$$

In this case:

$$\phi^* = \cos^{-1}(2K)$$

$$\Rightarrow r^* - r^{*3} + \sqrt{1-4K^2} = 0$$

There is only one positive root.

$\Rightarrow$ Amplitude is locked for $-\tfrac{1}{2} \le K \le \tfrac{1}{2}$

For K outside this range the radius oscillates.

Example:

A child on a swing knows that by standing and sitting on a swing one can go over the handle bars.

Model (Mathieu's Equation)

$$\ddot{x} + (1 + k\varepsilon^2 + \varepsilon\cos(t))x = 0$$

$$t_1 = t, \quad t_2 = \varepsilon^2 t \quad \text{and} \quad x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$$

$$\frac{\partial^2 x}{\partial t_1^2} + 2\varepsilon^2 \frac{\partial^2 x}{\partial t_1 \partial t_2} + \varepsilon^4 \frac{\partial^2 x}{\partial t_2^2} + (1 + k\varepsilon^2 + \varepsilon\cos(t))x = 0$$

$O(1)$:

$$x_0 = A(t_2)e^{it_1} + (*)$$

$\underline{O(\varepsilon)}$:

$$\frac{\partial^2 x_1}{\partial t_1^2} + x_1 + \frac{1}{2}(e^{it_1}+e^{-it_1})(Ae^{it_1}+A^*e^{-it_1}) = 0$$

$$\Rightarrow \frac{\partial^2 x_1}{\partial t_1^2} + x_1 + \frac{1}{2}(Ae^{2it_1}+A+A^*+A^*e^{-2it_1}) = 0$$

$$\Rightarrow x_1 = -\frac{1}{2}A + \frac{1}{6}Ae^{2it_1} + (*).$$

$\underline{O(\varepsilon^2)}$:

$$\frac{\partial^2 x_2}{\partial t_1^2} + x_2 + 2\frac{\partial A}{\partial t_2}ie^{it_1} - 2i\frac{\partial A^*}{\partial t_2}e^{-it_1} + \frac{1}{2}(e^{it_1}+e^{-it_1})\times\left(-\frac{1}{2}A + \frac{A}{6}e^{2it_1} - \frac{1}{2}A^* + \frac{1}{6}A^*e^{-2it_1}\right) + KAe^{it_1} + KA^*e^{-it_1}$$

$$2\frac{\partial A}{\partial t_2}i - \frac{1}{4}A - \frac{1}{4}A^* + \frac{A}{12} + KA = 0$$

$$2i\frac{\partial A}{\partial t_2} - \frac{1}{6}A - \frac{1}{4}A^* + KA = 0$$

Let $A = r(t_2)e^{i\phi(t_2)}$

$$\Rightarrow 2i\frac{dr}{dt_2}e^{i\phi(t_2)} - 2r\frac{d\phi}{dt_2}e^{i\phi(t_2)} - \frac{1}{6}re^{i\phi} - \frac{1}{4}re^{-i\phi} + Kre^{i\phi} = 0$$

$$\Rightarrow 2i\frac{dr}{dt_2} - 2r\frac{d\phi}{dt_2} - \frac{1}{6}r + Kr - \frac{1}{4}r(\cos(2\phi) - i\sin(2\phi)) = 0$$

$$\begin{cases} \dfrac{dr}{dt_2} = -\dfrac{1}{8}r\sin(2\phi). \\ \dfrac{d\phi}{dt_2} = -\dfrac{1}{2}\left(\dfrac{1}{6} - K + \dfrac{1}{4}\cos(2\phi)\right) \end{cases}$$

We can look at fixed points:

$$r = 0, \quad 1/6 - k + \frac{1}{4}\cos(2\phi) = 0$$

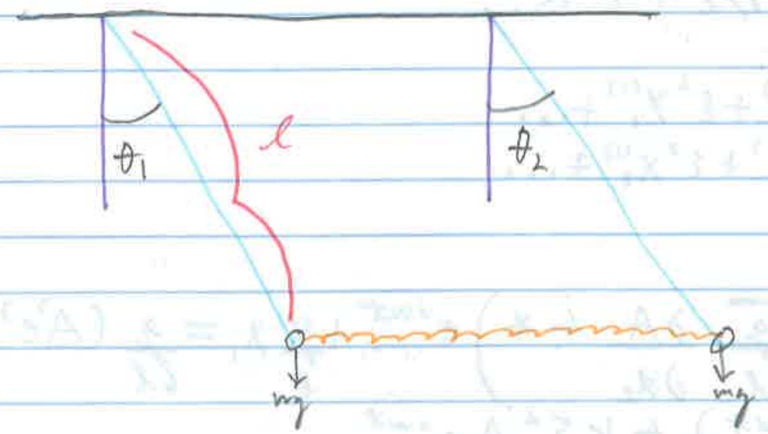
We must have that

$$\left|\frac{1}{6} - k\right| < 1/4$$

$$\Rightarrow -1/4 < 1/6 - k < 1/4$$

$$\Rightarrow -\frac{5}{12} < k < \frac{1}{12}$$

Coupled Oscillators



$$\frac{d^2\theta_1}{dt^2} + \frac{g}{\ell}\sin(\theta_1) = K'(\theta_2 - \theta_1)$$

$$\frac{d^2\theta_2}{dt^2} + \frac{g}{\ell}\sin(\theta_2) = -K'(\theta_2 - \theta_1)$$

small initial deflection

Let $x_i = \varepsilon\theta_i$ and expand:

$$\begin{cases} \dfrac{d^2x_1}{dt^2} + \dfrac{g}{\ell}x_1 = \dfrac{g}{6\ell}\varepsilon^2 x_1 + K'(x_2 - x_1) \\ \dfrac{d^2x_2}{dt^2} + \dfrac{g}{\ell}x_2 = \dfrac{g}{6\ell}\varepsilon^2 x_2 - K'(x_2 - x_1) \end{cases}$$

Assuming weak coupling of the form: $k' = \varepsilon^2 k$.

$$\begin{cases} \dfrac{d^2 x_1}{dt^2} + \dfrac{g}{\ell} x_1 = \dfrac{g}{6\ell} \varepsilon^2 x_1^3 + \varepsilon^2 k(x_2 - x_1) \\ \dfrac{d^2 x_2}{dt^2} + \dfrac{g}{\ell} x_2 = \dfrac{g}{6\ell} \varepsilon^2 x_1^3 - \varepsilon^2 k(x_2 - x_1) \end{cases}$$

Let $t_1 = t$ $t_2 = \varepsilon^2 t$. Let,

$$\begin{cases} x_1 = A(t_2) e^{i\omega t_1} + A^*(t_2) e^{-i\omega t} \\ x_2 = B(t_2) e^{i\omega t_1} + B^*(t_2) e^{-i\omega t_1} \end{cases} \longrightarrow \text{Base oscillations.}$$

$$\omega^2 = g/\ell + \varepsilon^2 k.$$

Let $x_1 = x_1^{(0)} + \varepsilon^2 x_1^{(1)} + \ldots$

$x_2 = x_2^{(0)} + \varepsilon^2 x_2^{(1)} + \ldots$

$O(\varepsilon^2)$:

$$\frac{\partial x_2}{\partial t_1^2} + \left(2i\sqrt{\frac{g}{\ell}} \frac{\partial A}{\partial t_2} + *\right) e^{i\omega t} + \frac{g}{\ell} x_1 = \frac{g}{6\ell}\left(A^3 e^{3i\omega t} + 3A^2 A^* e^{i\omega t} + (*)\right)$$

$$+ k(B e^{i\omega t}) - k\varepsilon^2 A e^{i\omega t}$$

$\vdots$

To remove secular terms we must have that:

$$2i\sqrt{\frac{g}{\ell}} \frac{\partial A}{\partial t_2} = \frac{g}{6\ell} 3A^2 A^* + kB$$

$$2i\sqrt{\frac{g}{\ell}} \frac{\partial B}{\partial t_2} = \frac{g}{6\ell} 3B^2 B^* + kA$$

$$\Rightarrow \begin{cases} \dfrac{dA}{dt_2} = i\Delta k B + i\beta A^2 A^* \\ \dfrac{dB}{dt_2} = i\Delta k A + i\beta B^2 B^* \end{cases}, \quad \Delta k = \frac{k}{2\omega_0}, \quad \beta = \frac{g}{4\omega_0 \ell}.$$

Let

$$E = AA^* + BB^* \rightarrow \text{energy}$$
$$N = AA^* - BB^* \rightarrow \text{energy difference.}$$
$$P_+ = AB^* + A^*B \rightarrow \text{polarization}$$
$$P_- = i(AB^* - A^*B) \rightarrow \text{polarization.}$$

Calculating:

1. $$\frac{dE}{dt_2} = A\frac{dA^*}{dt} + A^*\frac{dA}{dt} + \frac{dB}{dt}B^* + \frac{dB^*}{dt} + B$$
$$= A(-i\Delta K B^* - i\beta A^{*2}A)$$
$$+ A^*(i\Delta K B + i\beta A^2A^*)$$
$$+ B(-i\Delta K A^* - i\beta B^{*2}B)$$
$$+ B^*(i\Delta K A + i\beta A^2A^*)$$
$$= 0.$$

2. $$\frac{dN}{dt_2} = -2iAB^*\Delta K + 2iA^*B\Delta K$$
$$= -2\Delta K P_-$$

3. $$\frac{dP_+}{dt_2} = \frac{dA}{dt_2}B^* + A\frac{dB^*}{dt_2} + \frac{dA^*}{dt_2}B + A^*\frac{dB}{dt_2}$$
$$= B^*(i\Delta K B + i\beta A^2A^*)$$
$$+ A(-i\Delta K A^* - i\beta B^{*2}B)$$
$$+ B(-i\Delta K B^* - i\beta A^{*2}A)$$
$$+ A^*(i\Delta K A + i\beta B^2B^*)$$
$$= i\beta B^*A^2A^* - i\beta BA^{*2}A + i\beta A^*B^2B^* - i\beta AB^{*2}B$$
$$= i\beta A^*B(BB^* - AA^*) + i\beta AB^*(AA^* - BB^*)$$
$$= \beta N P_-$$

4. $$\frac{dP_-}{dt_2} = 2\Delta K N - \beta N P_+.$$

$$\begin{cases} \dfrac{dE}{dt_2} = 0 \\ \dfrac{dN}{dt_2} = -2\Delta K P_- \\ \dfrac{dP_+}{dt_2} = \beta N P_- \\ \dfrac{dP_-}{dt_2} = 2\Delta k N - \beta N P_+ \end{cases}$$

Eliminate P_- to obtain:

$$\frac{dP_+}{dt_2} = -\frac{\beta}{2\Delta K} N \frac{dN}{dt_2}$$

$$\Rightarrow P_+ + \frac{\beta}{4\Delta K} N^2 = P_+^0 + \frac{\beta}{4\Delta K} N_0^2$$

We therefore have

$$\frac{d^2 N}{dt_2^2} = -\frac{\partial U}{\partial N}$$

where

$$U = \left(2(\Delta K)^2 - \beta \Delta K P_+^0 - \frac{\beta^2}{4} N_0^2\right) N^2 + \frac{\beta^2}{8} N^4.$$