If we assume $U = U_0 + \varepsilon U_1 + \dots$

$$\Rightarrow (1+(h_b')^2)\frac{\partial^2 U_0}{\partial \xi^2} + (\alpha^2+\beta^2)\frac{\partial U_0}{\partial \xi} = 0$$

$$\Rightarrow U_0 = A(r) + B(r)\exp\left(-\frac{(\alpha^2+\beta^2)\,\xi}{1+(h_b')^2}\right)$$

Boundary conditions imply:

$$U_0(0,r) = g$$

$$\lim_{\xi\to\infty} U(\xi, r) = \int_0^{h_b(r)} f(\alpha s+\beta r, \beta s-\alpha r)\,ds + g(x_0, y_0)$$

The solution can be pieced together as usual.

Burger's Equation

Suppose we want to model the densit $\rho(x,t)$ of cars on a stretch of highway.

$\rho(x,t)$ – traffic density (#cars / length)
$q(x,t)$ – traffic flow. (# cars / hour)

N of cars in (a,b) is

$$N = \int_a^b \rho(x,t)\,dx$$

$$\Rightarrow \frac{dN}{dt} = q(a) - q(b) \qquad \text{(Conservation Law).}$$

$$\Rightarrow \int_a^b \frac{\partial \rho(x,t)}{\partial t}\,dx = q(a) - q(b)$$

$$\Rightarrow \int_a^b \frac{\partial \rho(x,t)}{\partial t}\,dx = -\int_a^b \frac{\partial q}{\partial x}\,dx.$$

Take limit as $(b-a) \to 0$ we get

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x}$$

We need a consitutive law.

Let v be the velocity of cars.

$$\Rightarrow q = \rho \cdot v.$$

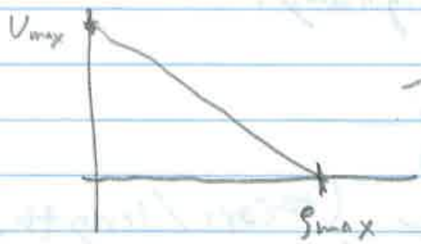
Assume $v = v(\rho)$, velocity only depends on density.

$$\Rightarrow \frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0.$$

This is like advection with a changing speed.
The function $q'(\rho)$ is the density wave velocity.

Simple Traffic Model.

1. v should be decreasing
2. $v(0) = v_{max}$ (speed limit)
3. $v(\rho_{max}) = 0$ (traffic jam).



$$\longrightarrow v = -\frac{\rho_{max}}{v_{max}}\rho + v_{max}$$

$$\Rightarrow q = -\frac{\rho_{max}}{v_{max}}\rho^2 + v_{max}\rho.$$

This gives us:

$$\frac{\partial \rho}{\partial t} + v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)\frac{\partial \rho}{\partial x} = 0.$$

Red light green light.

$$\frac{\partial \rho}{\partial t} + (1-2\rho)\frac{\partial \rho}{\partial x} = 0$$

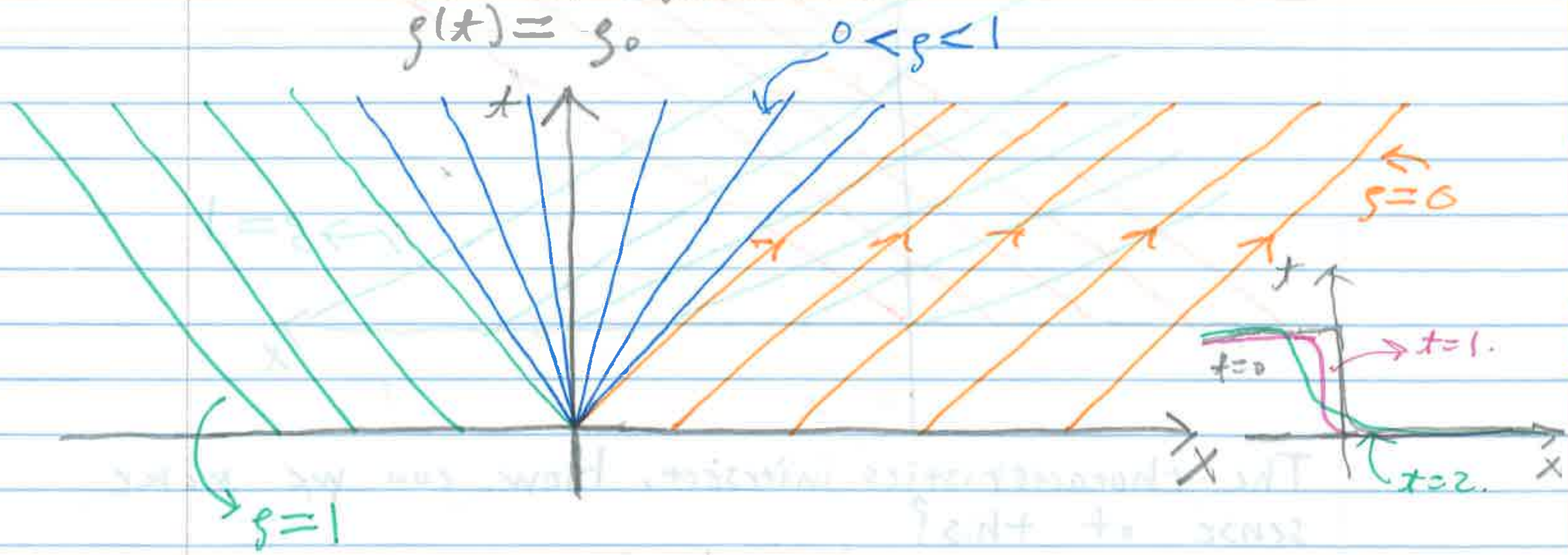
$$\rho(x,0) = \begin{cases} 1, & x<0 \\ 0, & x>0. \end{cases}$$

Characteristics:

$$\frac{dx}{dt} = 1-2\rho, \quad \frac{d\rho}{dt} = 0.$$

$\Rightarrow x(t) = (1-2\rho_0)t + x_0$

$\rho(t) = \rho_0$



What happens at $x=0$? ρ can take any value so the slope rotates from -1 to 1. This is known as a rarefraction wave.

Green light red light.

$$\frac{\partial\rho}{\partial t} + (1-2\rho)\frac{\partial\rho}{\partial x} = 0$$

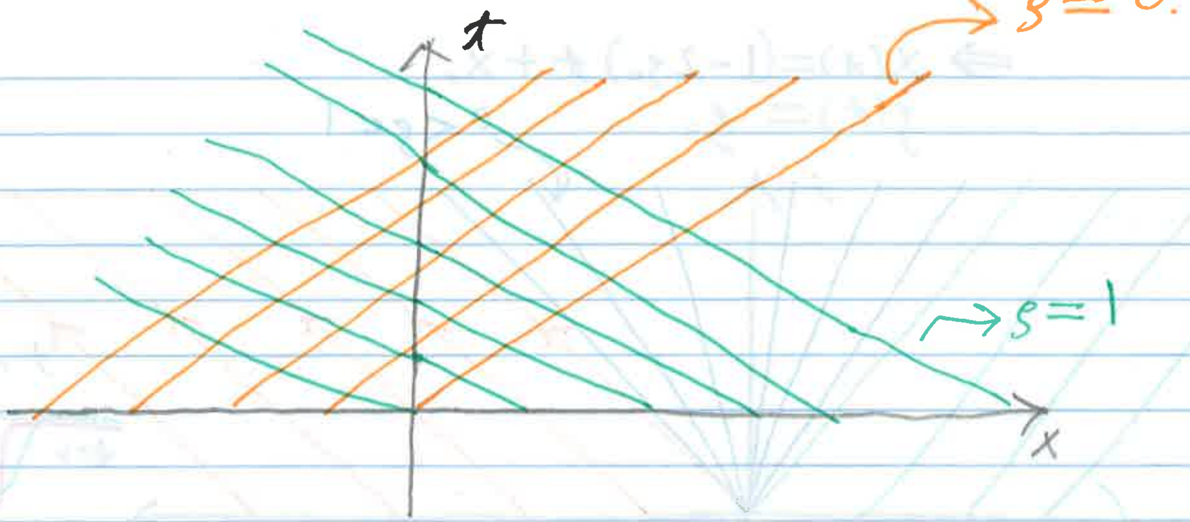
$$\rho(x,0) = 0, \quad x<0,$$
$$\rho(0,t) = 1, \quad t>0$$

Characteristics:

$$\frac{dx}{dt} = 1-2\rho, \quad \frac{d\rho}{dt} = 0$$

$$\Rightarrow x = (1-2\rho_0)t + x_0, \quad \rho = \rho_0$$

$$\text{or } t = \frac{1}{1-2\rho_0}x + t_0, \quad \rho = \rho_0$$

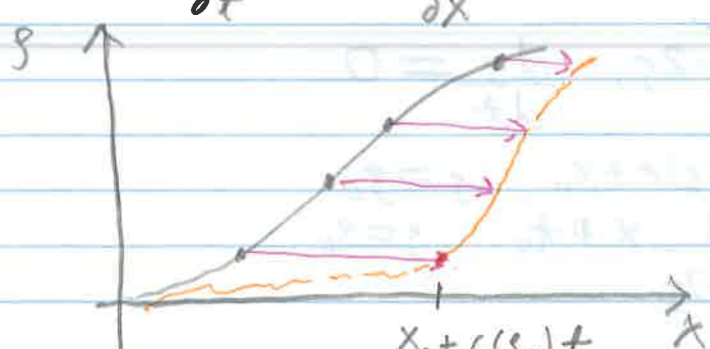


The characteristics intersect. How can we make sense of this?

Shock Waves.

Consider the general P.D.E.

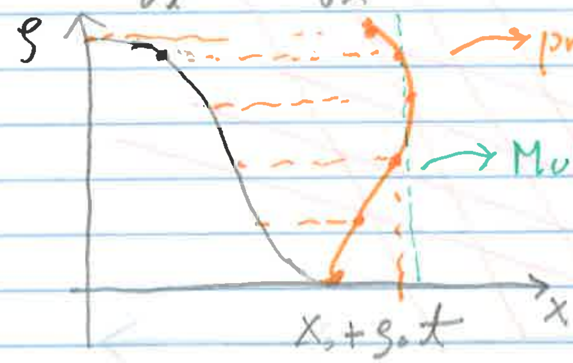
$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$$



$x_0 + c(\rho_0)t$

Lets look at

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0$$

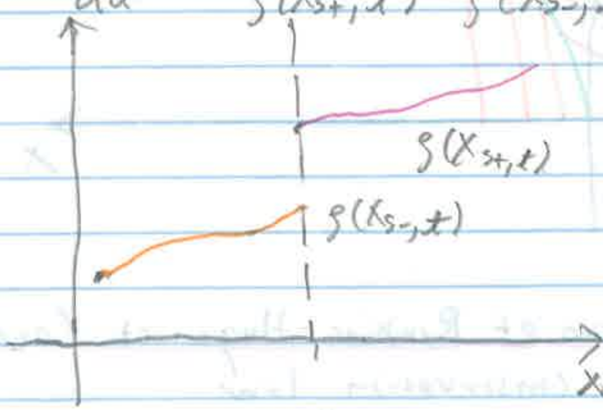


Physically, there cannot be a multivalued solution.

At a value $x_s(t)$ there should be a jump or a shock. Since ρ is conserved to the left and to the right of the schock we must have

$$\rho(x_{s-},t)\left[u(x_{s-},t) - \frac{dx_s}{dt}\right] = \rho(x_+,t)\left[u(x_{s+},t) - \frac{dx_s}{dt}\right]$$

$$\Rightarrow \frac{dx_s}{dt} = \frac{q(x_{s+},t) - q(x_{s-},t)}{\rho(x_{s+},t) - \rho(x_{s-},t)} = \frac{[q]}{[\rho]} \quad \text{(Rankine-Hugoniot condition)}$$



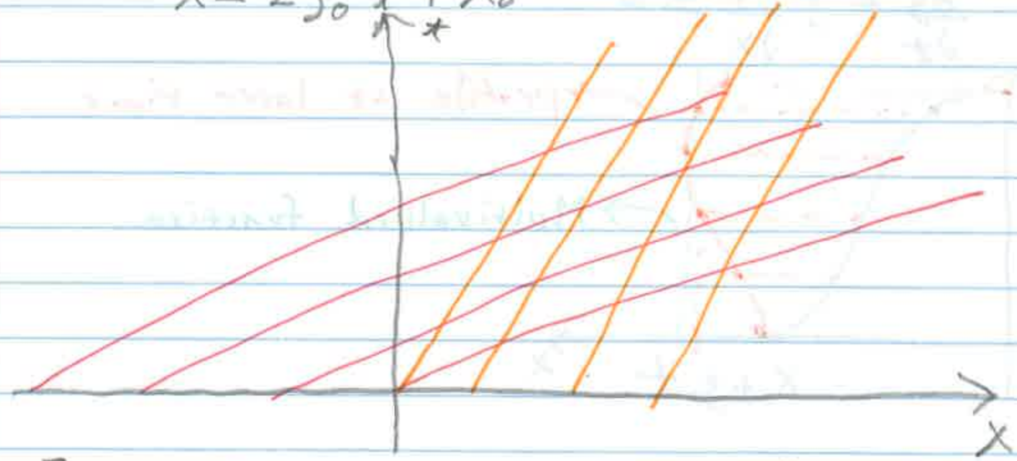
Example:

$$\frac{\partial \rho}{\partial t} + 2\rho \frac{\partial \rho}{\partial x} = 0$$

$$\rho(x,0) = \begin{cases} 4, & x<0 \\ 3, & x>0 \end{cases}$$

The characteristics are:

$x = 2\rho_0 t + x_0$

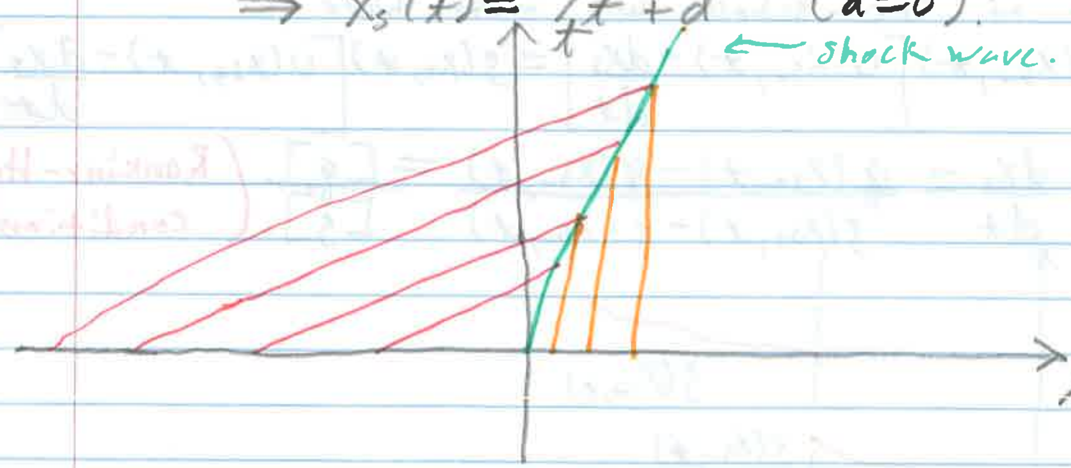


For our system we know $q'(\rho) = 2\rho \Rightarrow q(\rho) = \rho^2$.

The R-K Huigonet condition implies

$$\frac{dx_s}{dt} = \frac{q(4) - q(3)}{4 - 3} = 7$$

$$\Rightarrow x_s(t) = 7t + d \quad (d=0).$$



Alternate Derivation of Rankine-Hugoniet Condition.

We have the conservation law

$$\frac{d}{dt}\int_a^b \rho \, dx = q(a,t) - q(b,t)$$

Supposse we have a shock located at $x_s(t)$.

$$\frac{d}{dt}\left[\int_a^{x_s(t)} \rho(x,t)\,dx + \int_{x_s(t)}^{b} \rho(x,t)\,dx\right] = q(a,t) - q(b,t)$$

$$\Rightarrow \int_a^{x_s(t)} \frac{\partial \rho}{\partial t}\,dx + \frac{dx_s}{dt}\rho(x_s^-,t) + \int_{x_s(t)}^{b} \frac{\partial \rho}{\partial t}\,dx - \frac{dx_s}{dt}\rho(x_s^+,t)$$

$$= q(a,t) - q(b,t).$$

Now away from the shock

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x}$$

$$\Rightarrow q(a,t) - q(x_s^-,t) + \frac{dx_s}{dt}\rho(x_s^-,t) + q(x_s^+,t) - q(b,t)$$

$$- \frac{dx_s}{dt}\rho(x_s^+,t) = q(a,t) - q(b,t)$$

$$\Rightarrow \frac{dx_s}{dt} = \frac{q(x_s^-,t) - q(x_s^+,t)}{\rho(x_s^-,t) - \rho(x_s^+,t)}$$ (Rankine - Hugonoit Condition)

Viscosity Solutions.

One point of view is that discontinuous solutions cannot describe reality. There must be some small effect that removes non-smoothness.

$$u_t + u u_x = \varepsilon u_{xx}$$ (Burger's equation)

Conservation Law — Viscosity (diffussion, friction, etc.)

Example

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

Outer Solution:

$$\partial_t u_0 + u_0 \partial_x u_0 = 0$$

Characteristics:

$$\frac{dx}{dt} = u_0 \quad \Rightarrow x(t) = u_0 t + c$$

$$\frac{du_0}{dt} = 0 \quad \Rightarrow u_0(t) = d.$$

We know the characteristics intersect. We must add an inner layer to correct non-second differentiability.

Let

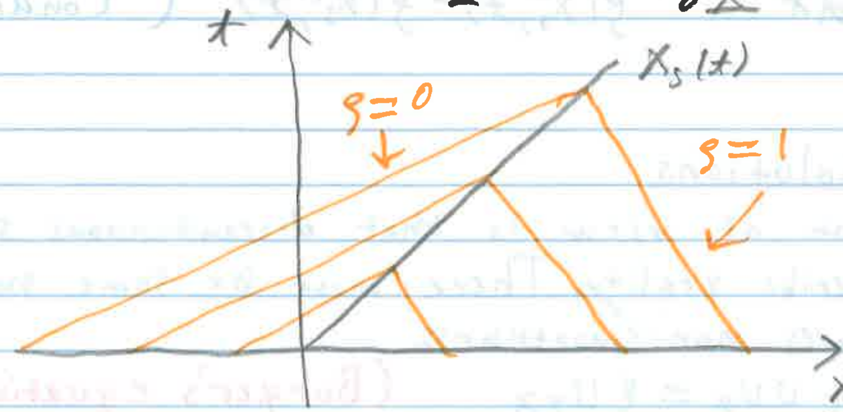
$$X = \frac{x - x_s(t)}{\varepsilon^\alpha} \Rightarrow \frac{\partial}{\partial t} = \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X}$$

$$\partial_t U_0 - \varepsilon^{-\alpha} s'(t)\partial_X U_0 + \varepsilon^{-\alpha} U_0 \partial_X U_0 = \varepsilon^{1-2\alpha}\partial_X^2 U_0$$

Letting $\alpha = 1$ we obtain:

$$-s'(t)\frac{\partial U_0}{\partial X} + U_0\frac{\partial U_0}{\partial X} = \frac{\partial^2 U_0}{\partial X^2}$$

$$\Rightarrow -s'(t)U_0 + \frac{1}{2}U_0^2 = \frac{\partial U_0}{\partial X} + A(t)$$



$\lim_{X \to -\infty} U_0 = 0$ and $\lim_{X \to \infty} U_0 = 1$

$$\Rightarrow 0 = A(t)$$

$$-s'(t) + \frac{1}{2} = 0$$

$\Rightarrow s'(t) = 1/2$. (Rankine Hugoniot Condition).

$\Rightarrow s(t) = 1/2\, t$.

So we now need to solve:

$$\tfrac{1}{2} U_0 + \frac{1}{2} U_0^2 = \frac{\partial U_0}{\partial X}$$

$$\Rightarrow U_0(X, t) = \frac{1}{1 + B(t)\, e^{-X/2}}$$

$B(t)$ has to be determined at $O(\varepsilon)$.