Matched Asymptotics in P.D.E.s

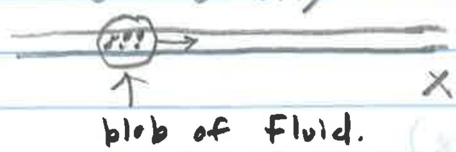
Method of Characteristics

Lets try to solve the advection equation.

$$\frac{\partial w}{\partial t} = -c\frac{\partial w}{dx} \quad *$$

$$\underset{\text{velocity}}{\uparrow} \qquad \underset{\text{slope}}{\uparrow}$$

Think of density of particles in a tube:



Let $x(t)$ denote the position of a test particle, and $w(x(t), t)$ denote the density as observed by the particle.

Lets calculate $\frac{dw}{dt}$ as observed by the particle

$$\frac{d}{dt} w = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{\partial x}$$

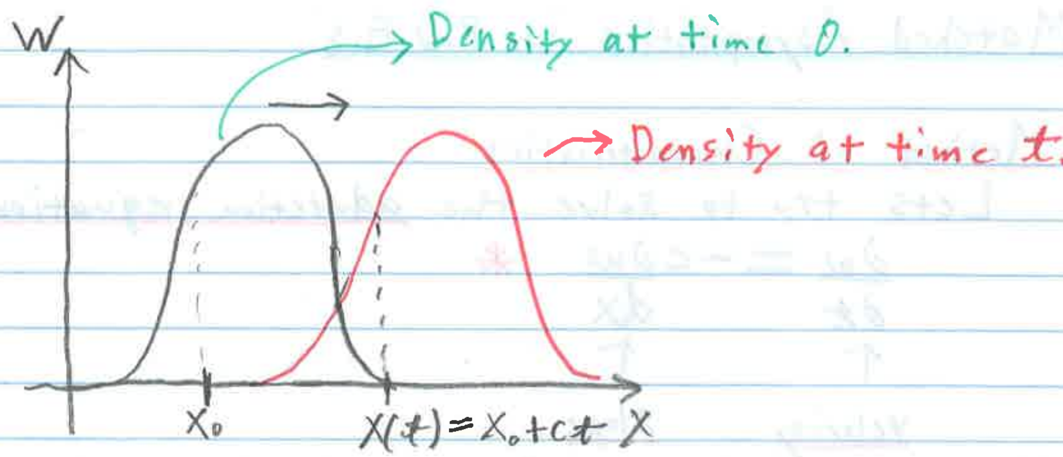
From $*$ it follows that if $\frac{dx}{dt} = c$, i.e. it moves with at the right speed then

$$\frac{dw}{dt} = 0 \Rightarrow w(0) = w(x(0), 0) = w(x_0, 0)$$

$$\frac{dx}{dt} = c \Rightarrow x = ct + x_0$$

What we learned:

$*$ The fluid density located at x_0 is transported to the right at speed c.



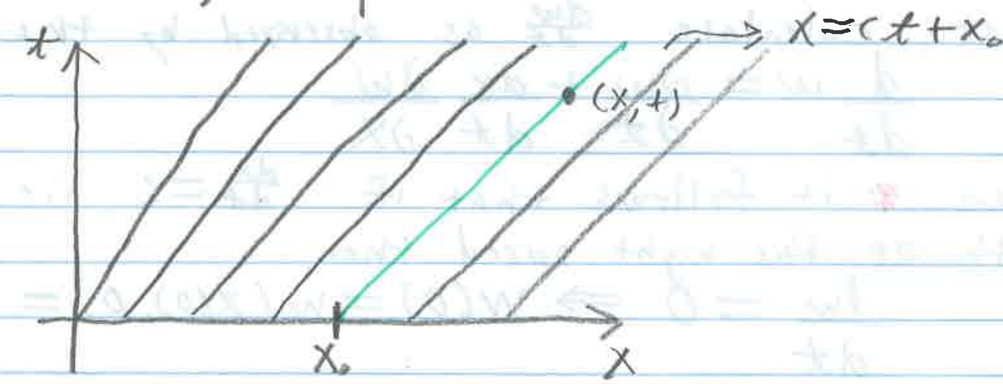
So, if w solves

$$w_t = -c\,w_x$$
$$w(x,0) = f(x)$$

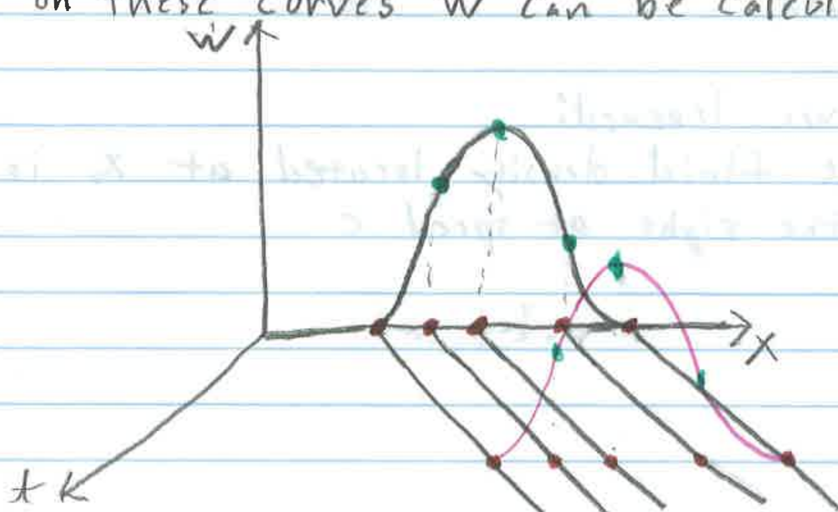
Then

$$w(x,t) = f(x-ct).$$

Another way to plot solutions:



The curves $x=ct+x_0$ are called characteristics and on these curves w can be calculated:



Example:

$$\frac{\partial w}{\partial t} + 3t^2 \frac{\partial w}{\partial x} = 2tw$$

Let $x(t)$ be the position of a test particle and $w(t)$ the density observed by the particle. Then,

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{\partial x}$$

If $\frac{dx}{dt} = 3t^2$ then,

$$\frac{dx}{dt} = 3t^2$$

$$\frac{dw}{dt} = 2tw.$$

The characteristics are given by:

$$x = t^3 + x_0 \quad *$$

Along characteristics we have that:

$$w = Ke^{t^2}$$

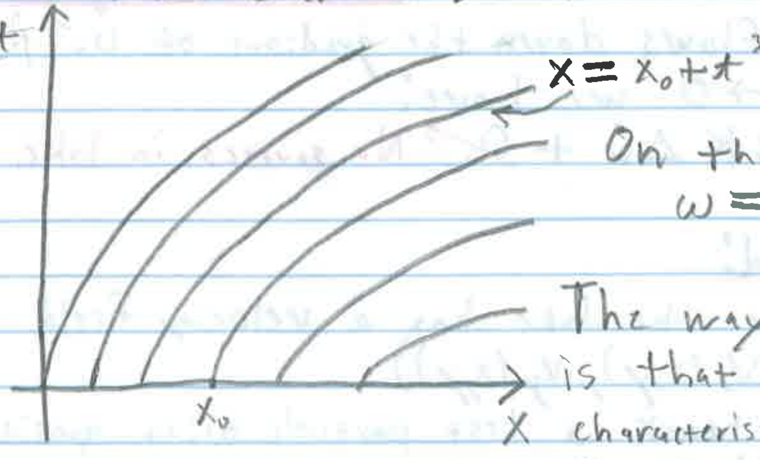
If we have initial data

$$w(x_0, 0) = f(x_0) \quad **$$

Then,

$$w = f(x_0)e^{t^2}$$

$$\Rightarrow w = f(x - t^3)e^{t^2}.$$



On the characteristic curve

$$w = f(x_0)e^{t^2}$$

The way I think about this is that the graph of the characteristics is the graph of the solution and $*$ and $**$ are the rules for reading the graph.

Example:

Diffusion of pollutant dominated by convection.

$u \sim$ concentration of pollution.

If we have just diffusion then

$$\frac{\partial u}{\partial t} = K \Delta u \quad (\text{velocity} = \nabla \cdot \nabla f)$$

Derivation:

tn Rate of change of total density $=$ $-$ flux boundary $+$ source terms.

$$\frac{\partial}{\partial t}\int_{\Omega'} u \, dx\,dy = -\int_{\partial\Omega'} \vec{\Phi}\cdot\vec{n}\, ds + \int_{\Omega'} Q \, dx\,dy$$

pollution per unit time per unit area.

$$\Rightarrow \frac{\partial}{\partial t}\int_{\Omega'} u\, dx\,dy = -\int_{\Omega'} \nabla\cdot\vec{\Phi}\, dx\,dy + \int_{\Omega'} Q\, dx\,dy$$

* Simple diffusion

$$\vec{\Phi} = -K\nabla u \quad (\text{constitutive equation})$$

Pollution flows down the gradiant of u.

Take $\Omega' \to 0$ we have:

$$u_t = K\Delta u + \cancel{Q}$$

No sources in lake.

Velocity Field:

Now suppose the lake has a velocity field

$$\vec{V} = (V_x(x,y), V_y(x,y)).$$

Again lets track a test particle with position $\vec{x} = (x(t), y(t))$. Then,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{d\vec{x}}{dt}\cdot\nabla u = K\Delta u$$

The governing equation is

$$\frac{\partial u}{\partial t} + \vec{v}\cdot\nabla u = \varepsilon \Delta u$$

Example:

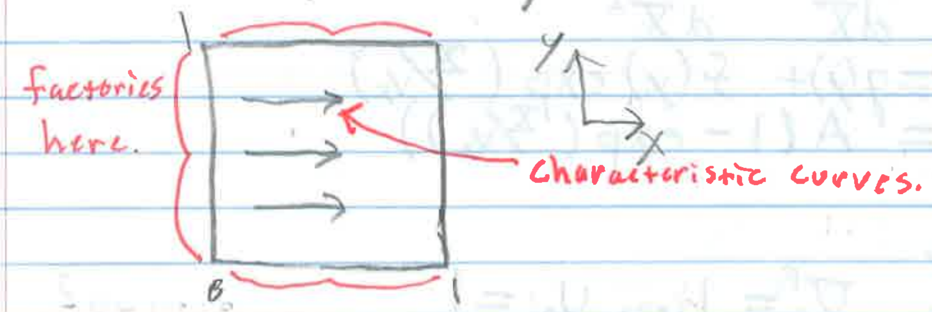
$$\frac{\partial u}{\partial t} + \vec{v}\cdot\nabla u = \varepsilon \Delta u$$

$$\Omega = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \end{cases}$$

$$u(0,y) = 1$$

$$u(1,y) = 0$$

$$u(x,1) = u(x,0) = 1$$



Suppose $\vec{v} = v_x \hat{e}_x$, and assume steady state $\Rightarrow \frac{\partial u}{\partial t} = 0$.

Outer Solution:

Suppose $u = u_0 + \varepsilon u_1 + \dots$

$O(1)$:

$$v_x \frac{\partial u_0}{\partial x} = 0$$

$$\Rightarrow u_0 = f(y)$$

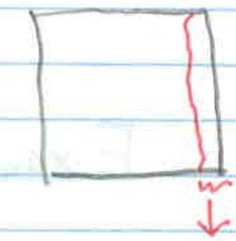
Boundary conditions at left

$$\Rightarrow u_0 = 1.$$

Right Inner Expansion:

$$X = \frac{x-1}{\varepsilon^{\alpha}}$$

$$\Rightarrow \varepsilon^{-\alpha} V_x \frac{dU_0}{dX} = \varepsilon^{1-2\alpha} \frac{d^2 U_0}{dX^2} + \varepsilon \frac{d^2 U_0}{dy^2}$$



width ε
b.l.

Balance:

$$\alpha = 1$$

Expand:

$$U_0 = U_0^R + \varepsilon U_1^R + \dots$$

$O(1)$:

$$V_x \frac{dU_0^R}{dX} = \frac{d^2 U_0^R}{dX^2}$$

$$U_0^R = g(y) + f(y)\exp(X/V_x)$$

$$U_0^R = A(1 - \exp(X/V_x))$$

Matching

$$\lim_{X \to -\infty} U_0^R = \lim_{x \to 1} U_0 = 1$$

$$\Rightarrow A = 1.$$

Composite Expansion:

$$U \sim 1 - \exp\left(\frac{x-1}{\varepsilon V_x}\right)$$

Does not quite satisfy b.c. in b.l.

Lower Right Layer

Rescale by $\Upsilon = \frac{y}{\varepsilon^{\beta}}$, $\Xi = \frac{x-1}{\varepsilon^{\alpha}}$.

$$\Rightarrow \varepsilon^{-\alpha} \frac{\partial v}{\partial \Xi} V_x = \varepsilon^{1-2\alpha} \frac{\partial^2 v}{\partial \Xi^2} + \varepsilon^{1-2\beta} \frac{\partial^2 v}{\partial \Upsilon^2}$$

Balance

$$-\alpha = 1-2\alpha \quad \text{and} \quad 1-2\alpha = 1-2\beta$$

$$\Rightarrow \alpha = 1 \quad \text{and} \quad \beta = 1$$

We need to solve:

$$V_x \frac{\partial U}{\partial \Xi} = \frac{\partial^2 U}{\partial \Xi^2} + \frac{\partial^2 U}{\partial \Upsilon^2}.$$

$$U(\Xi, 0) = -\exp\left(\frac{\Xi}{V_x}\right)$$

$$\lim_{\Xi \to -\infty} U = 0$$

$$\lim_{\Upsilon \to \infty} U = 0$$

Switch coordinates $\Xi' \to -\Xi \Rightarrow \frac{d}{d\Xi} = -\frac{d}{d\Xi'}$

$$-V_x \frac{\partial U}{d\Xi} = \frac{\partial^2 U}{\partial \Xi^2} + \frac{\partial^2 U}{\partial \Upsilon^2}$$

$$U(\Xi, 0) = -\exp\left(-\frac{\Xi}{V_x}\right)$$

Fourier Transform:

$$\mathcal{F}(U) = \hat{U} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\Xi} U(\Xi)\, d\Xi, \quad \mathcal{F}^{-1}(g) = \int_{-\infty}^{\infty} e^{-ik\Xi} g(k)\, dk$$

Extend U as an even function $\bar{U}$.

$$\bar{U}(\Xi, \Upsilon) = \int_{-\infty}^{\infty} g(k, \Upsilon) e^{-ik\Xi}\, dk \quad \text{(Fourier Decomposition).}$$

$$\Rightarrow ik V_x\, g(k, \Upsilon) = -k^2 g(k, \Upsilon) + \frac{\partial^2 g}{d\Upsilon^2}$$

$$\Rightarrow g(k,Y) = A(k)\, e^{-\sqrt{ik+k^2}\, y} + B(k)\, e^{\sqrt{ik+k^2}\, y}$$

$$\lim_{Y\to\infty} g(k,Y) = 0 \Rightarrow B(k) = 0.$$

We also have that

$$\bar{U}(X,0) = -\exp\left(-\frac{X}{V_x}\right) = \int_{-\infty}^{\infty} g(k,0)\, e^{-ikX}\, dk$$

$$\Rightarrow g(k,0) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \exp\left(\frac{-X}{V_x}\right) e^{ikX}\, dX$$

$$\Rightarrow g(k,0) = -\frac{V_x}{\pi}\,\frac{1}{(1+k^2V_x)}$$

The solution is:

$$\bar{U}(X,Y) = \int_{-\infty}^{\infty} -\frac{V_x}{\pi}\,\frac{1}{1+k^2V_x}\, e^{-\sqrt{ik+k^2}\, Y} e^{-ikX}\, dk.$$

Composite Expansion

$$u \sim 1 - \exp\left(\frac{x-1}{\varepsilon V_x}\right) - \int_{-\infty}^{\infty} \frac{V_x}{\pi}\,\frac{1}{1+k^2V_x}\, e^{-\sqrt{ik+k^2}\, y/\varepsilon}\, e^{ik(x-1)/\varepsilon}\, dk$$

$$- \int_{-\infty}^{\infty} \frac{V_x}{\pi}\,\frac{1}{1+k^2V_x}\, e^{-\sqrt{ik+k^2}\,(y-1)/\varepsilon}\, e^{ik(x-1)/\varepsilon}\, dk.$$

General Elliptic Problem:

$$\varepsilon \Delta u + \alpha \partial_x u + \beta \partial_y u = f(x,y)$$
$$u = g(x,y) \quad \text{on } \partial\Omega$$

Ω – simply connected, bounded, smooth boundary, convex.
α, β are constants.

* Diffusion and advection

Outer Expansion:

$u = u_0 + \varepsilon u_1 + \dots$

$$\Rightarrow \alpha \frac{\partial u_0}{\partial x} + \beta \frac{\partial u_0}{\partial y} = f(x,y)$$

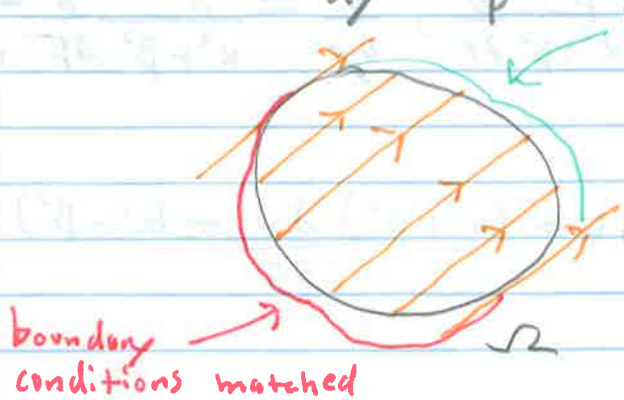
Treat y as time and apply method of characteristics.
Take test particle moving along curve $x(y)$.

$$\frac{du_0}{dy} = \frac{dx}{dy}\frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial y}$$

Set $\quad \dfrac{dx}{dy} = \dfrac{\alpha}{\beta} \quad *$

$$\Rightarrow \beta \frac{du_0}{dy} = \alpha \frac{\partial u_0}{\partial x} + \beta \frac{\partial u_0}{\partial y}$$

$$\Rightarrow \frac{du_0}{dy} = \frac{1}{\beta} f(x(y), y) \quad **$$



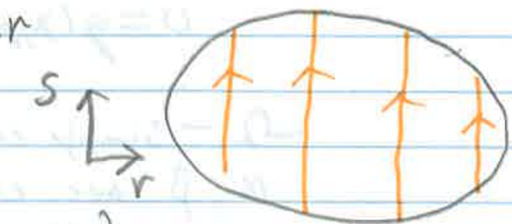
The characteristic curves have slope β/α.

$$x = \beta/\alpha\, y + x_0$$
$$u_0(x,y) = \int_{y_0}^{y} \frac{1}{\beta} f(\beta/\alpha\, y + x_0, y)\,dy + g(x_0,y_0)$$

The boundary conditions cannot be completely matched. To analyze we rotate our coordinate system so that the characteristic curves are vertical:

$$x = \alpha s + \beta r \quad , \quad y = \beta s - \alpha r$$

$$\Rightarrow \frac{\partial}{\partial s} = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$$



$$\Rightarrow \partial_s U_0 = f(\alpha s + \beta r, \beta s - \alpha r)$$

$$\Rightarrow U_0 = \int_0^s f(\alpha s + \beta r, \beta s - \alpha r)\, ds + g(x_0, y_0).$$

Inner Solution

Rescale by

$$\xi = \frac{s - h_b(r)}{\varepsilon}$$ ← The coordinate of the boundary layer must depend on r.

$$\frac{\partial}{\partial s} = \frac{\partial \xi}{\partial s} \frac{\partial}{\partial \xi} = \frac{1}{\varepsilon} \frac{\partial}{\partial \xi}$$

$$\frac{\partial^2}{\partial s^2} = \frac{1}{\varepsilon^2} \frac{\partial^2}{\partial \xi^2}$$

$$\frac{\partial}{\partial r} = -\frac{h_b'}{\varepsilon} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial r}$$

$$\frac{\partial^2}{\partial r^2} = -\frac{h_b''}{\varepsilon} \frac{\partial}{\partial \xi} + \frac{(h_b')^2}{\varepsilon^2} \frac{\partial^2}{\partial \xi^2} - \frac{2h_b'}{\varepsilon} \frac{\partial^2}{\partial \xi \partial r} + \frac{\partial^2}{\partial r^2}$$

Now,

$$\frac{\partial}{\partial x} = \frac{\alpha}{\alpha^2+\beta^2} \frac{\partial}{\partial s} + \frac{\beta}{\alpha^2+\beta^2} \frac{\partial}{\partial r} , \quad \frac{\partial}{\partial y} = \frac{\beta}{\alpha^2+\beta^2} \frac{\partial}{\partial s} - \frac{\alpha}{\alpha^2+\beta^2} \frac{\partial}{\partial r}$$

⇒ The P.D.E is

$$\varepsilon \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial r^2} \right) U + (\alpha^2+\beta^2) \frac{\partial U}{\partial s} = (\alpha^2+\beta^2) f$$

If we assume $U = U_0 + \varepsilon U_1 + \dots$

$$\Rightarrow (1+(h_b')^2)\frac{\partial^2 U_0}{\partial \xi^2} + (\alpha^2+\beta^2)\frac{\partial U_0}{\partial \xi} = 0$$

$$\Rightarrow U_0 = A(r) + B(r)\exp\left(-\frac{(\alpha^2+\beta^2)\xi}{1+(h_b')^2}\right)$$

Boundary conditions imply:

$$U_0(0,r) = g$$

$$\lim_{\xi \to \infty} U_0(\xi, r) = \int_0^{h_b(r)} f(\alpha s+\beta r, \beta s-\alpha r)\, ds + g(x_0, y_0).$$

The solution can be pieced together as usual.

Burger's Equation