Chapter 2: Matched Asymptotic Expansions

Uniformity

Lets try solving the equation

$$(y-1)(y-x) = \varepsilon y$$

Guess

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$$

$O(1)$:

$$(y_0-1)(y_0-x) = 0$$
$$\Rightarrow y_0 = 1, \quad y_0 = x$$

$O(\varepsilon)$

$$y_1(1-x) = 1, \quad (x-1)y_1 = x$$

$$\Rightarrow y_1 = \frac{1}{1-x}, \quad y_1 = \frac{x}{1-x}.$$

One solution curve is then asymptotic to:

$$y \sim 1 + \frac{\varepsilon}{1-x}.$$

This is an asymptotic approximation to the solution. However, it is not uniformly asymptotic.

*Fix x, then,

$$\left| y - 1 + \frac{\varepsilon}{1-x} \right| = o(\varepsilon)$$

function of x and δ.

This means, $\forall \delta > 0$, $\exists \varepsilon_2(x,\delta)$ such that $\varepsilon < \varepsilon_2(x,\delta)$ implies $\left| y - 1 + \frac{\varepsilon}{1-x} \right| \le \delta \varepsilon$.

*This is not a useful notion of approximations as ε may be small but it is fixed!

Definition - $f(x,\varepsilon)$ is uniformly asymptotic to $\phi(x,\varepsilon)$ on the interval I if $\forall \delta > 0$, $\exists \varepsilon_2(\delta)$ such that $0 < \varepsilon < \varepsilon_2(\delta)$
$\Rightarrow |f(x,\varepsilon)| \le \delta |\phi(x,\varepsilon)|$ for all $x \in I$.

Example - Find an asymptotic approximation to the solution of the equation

$$\varepsilon y'' + 2y' + 2y = 0$$
$$y(0) = 0$$
$$y(1) = 1$$

Generic solution is of the form:

$$y(x) = A\left[\exp\left(\frac{-1+\sqrt{1-2\varepsilon}}{\varepsilon}x\right) - \exp\left(\frac{-1-\sqrt{1-2\varepsilon}}{\varepsilon}x\right)\right]$$

$$\Rightarrow \frac{y}{A} \sim \exp(-x) - \exp\left(-\frac{2}{\varepsilon}x\right)$$

$$\Rightarrow \frac{y}{A} \sim \exp(-x).$$

This is not uniformly valid since.

$$|y(0) - A| = A \neq o(1).$$

Example:

Solve

$$y'' + 2\varepsilon y' + y = 0, \quad t > 0.$$
$$y(0) = 0,$$

$$\Rightarrow y(t) = e^{-\varepsilon t}\sin(t\sqrt{1-\varepsilon^2})$$
$$\Rightarrow y \sim (1 - \varepsilon t + \ldots)\sin(t(1 - \tfrac{1}{2}\varepsilon^2 + \ldots))$$
$$\Rightarrow y \sim (1 - \varepsilon t + \ldots)(\sin(t) - \tfrac{1}{2}\varepsilon^2 t\cos(t) + \ldots)$$
$$\Rightarrow y \sim \sin(t) - \varepsilon t\sin(t).$$ $\rightarrow$ very bad approximation.

Clearly not uniform on $(0, \infty)$ since

$$\lim_{t\to\infty} \sin(t) - \varepsilon t\sin(t) = -\infty.$$

Boundary Layers

Example:

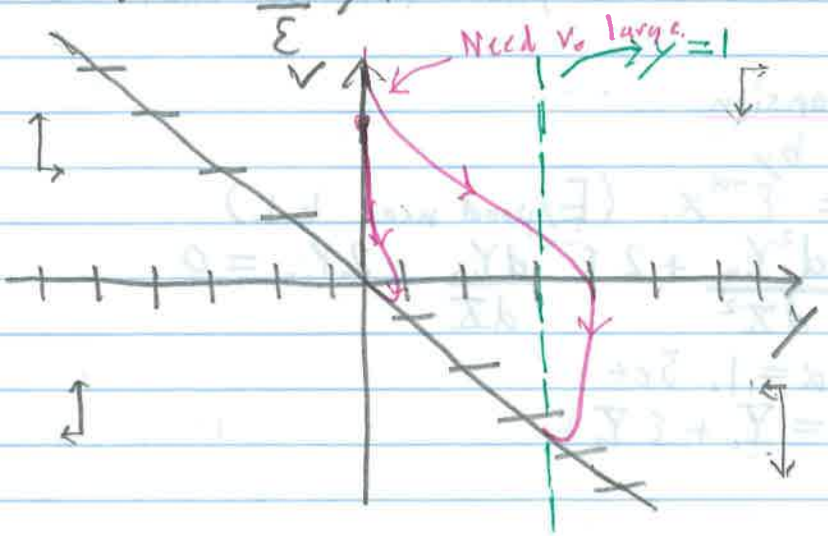
$\varepsilon y'' + 2y' + 2y = 0,$

$y(0) = 0, \quad y(1) = 1$

Lets draw the phase portrait. Letting $v = y'$ we have:

$y' = v$

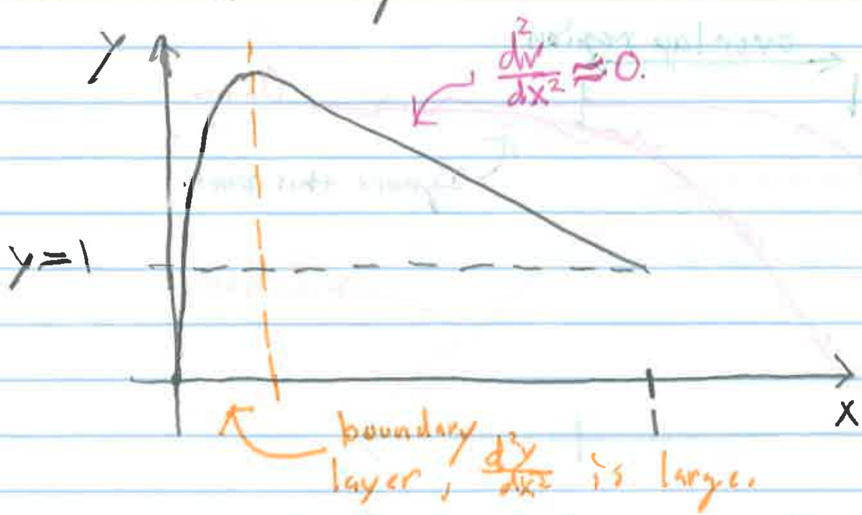
$v' = -\frac{2}{\varepsilon}(y+v)$



Initially v must be very large. Also,

$$\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{dv}{dy} v$$

which is also large.



Outer expansion.

$$y_{ot} = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$$

$$\Rightarrow \varepsilon(y_0'' + \varepsilon y_1'' + \dots) + 2(y_0' + \varepsilon y_1' + \dots) + 2(y_0 + \varepsilon y_1 + \dots) = 0.$$

$O(1)$:

$$y_0' + y_0 = 0 \Rightarrow y_0 = ae^{-x}$$

We know from the phase portrait that $a = e \Rightarrow y_0 = e^{1-x}$.

Inner Expansion.

Rescale by

$$X = \varepsilon^{-\alpha} x, \quad (\text{Expand near b.l.})$$

$$\Rightarrow \varepsilon^{1-2\alpha} \frac{d^2 Y_{in}}{dX^2} + 2\varepsilon^{-\alpha} \frac{dY_{in}}{dX} + 2Y_{in} = 0.$$

Balance $\rightarrow \alpha = 1$. Set

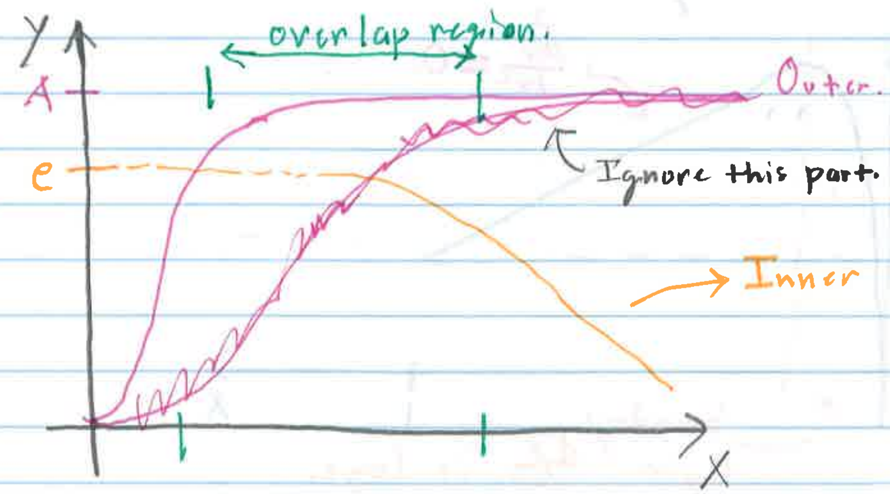
$$Y_{in} = Y_0 + \varepsilon Y_1 + \dots$$

$O(1)$:

$$\frac{d^2 Y_0}{dX^2} + 2\frac{dY_0}{dX} = 0.$$

$$Y_0(0) = 0$$

$$\Rightarrow Y_0(x) = A(1 - e^{-2X})$$



To match the two solutions $A = e$.

*The key point is that both expansions approximate the same function.

Composite Solution

Two solutions must be combined to form uniform asymptotic approximation. We simply add the two approximations and subtract the common part.

$$y(x) \sim y_0(x) + Y_0\left(\tfrac{x}{\varepsilon}\right) - y_0(0)$$

$$\sim e^{1-x} - e^{1-2x/\varepsilon}$$ → width of boundary layer.

This function has a boundary layer of width ε.

Second term:

Outer:

$$y_1' + y_1 = -\tfrac{1}{2} y_0'', \quad y(1) = 0$$

$$\Rightarrow y_1' + y_1 = -\tfrac{1}{2} e^{-1-x}$$

$$\Rightarrow (e^x y_1)' = -\tfrac{1}{2} e^{-1}$$

$$\Rightarrow e^x y_1 = -\tfrac{1}{2} e^{-1} x + c, \quad c = \tfrac{1}{2} e^{-1}$$

$$\Rightarrow y_1(x) = \tfrac{1}{2}(1-x) e^{1-x}.$$

Inner:

$$Y_1'' + 2Y_1' = -2Y_0, \quad Y_1(0) = 0$$

$$\Rightarrow Y_1'' + 2Y_1' = -2e(1 - e^{-2X})$$

$$Y_1' + 2Y_1 = -2e(X - e^{-2X}) + C$$

$$\Rightarrow (e^{2X} Y_1)' = -2eXe^{2X} + 2e + Ce^{2X}$$

$$\Rightarrow e^{2X} Y_1 = -eXe^{2X} + \tfrac{1}{2} e \cdot e^{2X} + 2eX + C\frac{e^{2X}}{2} + d$$

$$Y_1 = B(1 - e^{-2X}) - Xe^{1}(1 + e^{-2X})$$

How do we match B so that outer and inner layers overlap correctly?

The idea is to introduce an overlap scaling:

$$x_\eta = \frac{x}{\varepsilon^\beta}, \qquad 0 < \beta < 1$$

B.L. Outer solution.

$$y_{inner} = e^{1-\varepsilon^\beta x_\eta} + \frac{\varepsilon}{2}(1-\varepsilon^\beta x_\eta)e^{1-x_\eta\varepsilon^\beta} + \dots$$
$$= \underline{e^1 - \varepsilon^\beta x_\eta e^1} + \underline{\frac{\varepsilon}{2} e^1} + \frac{1}{2}\varepsilon^{2\beta} e^1 x_\eta^2 + \dots$$

$$y_{outer} = e^1(1-e^{-2x_\eta/1-\beta}) + \varepsilon\left[B(1-e^{-2x_\eta/1-\beta}) - \frac{x_\eta e^1}{\varepsilon^{1-\beta}}(1+e^{-2x_\eta/1-\beta})\right]$$
$$= \underline{e^1 - \varepsilon^\beta x_\eta e^1} + \underline{\varepsilon B} + \dots$$

Select $B = e^1/2$.

Composite Solution

Add y_1 and Y_1 and subtract common part.

$$\Rightarrow y \sim y_0 + \varepsilon y_1 + Y_0 + \varepsilon Y_1 - (e^1 - x_\eta e^1 \varepsilon^\beta + \frac{\varepsilon}{2} e^1).$$
$$\Rightarrow y \sim e^{1-x} - (1+x)e^{1-2x/\varepsilon} + \frac{\varepsilon}{2}\left[(1-x)e^{1-x} - e^{1-2x/\varepsilon}\right]$$

Example

$$\varepsilon y'' + (1+\varepsilon)y' + y = 0$$
$$y(0) = 0$$
$$y(1) = e^{-1}$$

Outer Solution

$$y_{ot} = y_0 + \varepsilon y_1 + \dots$$

$O(1)$:

$$y_0' + y_0 = 0$$
$$\Rightarrow y_0 = e^{-x}.$$

$\mathcal{O}(\varepsilon)$:

$$y_0'' + y_1' + y_0' + y_1 = 0$$
$$\Rightarrow y_1' + y_1 = 0$$

Inner Solution.

Let $X = \varepsilon^{-\alpha} x$

$$\Rightarrow \varepsilon^{1-2\alpha} \frac{d^2y}{dX^2} + (1+\varepsilon)\varepsilon^{-\alpha}\frac{dy}{dX} + y = 0$$

Balance $\rightarrow \alpha = 1$

$$Y_{in} = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \ldots$$

$$\Rightarrow \frac{d^2 Y_{in}}{dX^2} + (1+\varepsilon)\frac{dY_{in}}{dX} + \varepsilon Y_{in} = 0$$

$\mathcal{O}(1)$:

$$\frac{d^2 Y_0}{dX^2} + \frac{dY_{0}}{dX} = 0$$
$$\Rightarrow Y_0 = A(1 - e^{-X}) \rightarrow \text{Trivial Matching gives } A = 1.$$

$\mathcal{O}(\varepsilon)$:

$$\frac{d^2 Y_1}{dX^2} + \frac{dY_1}{dX} + \frac{dY_0}{dX} + Y_0 = 0$$
$$\Rightarrow \frac{d^2 Y_1}{dX^2} + \frac{dY_1}{dX} = -A$$
$$Y_1 = B(1 - e^{-X}) - AX$$

Matching:

Let $x_\eta = x/\varepsilon^\beta$, $0 < \beta < 1$.

$$y_{out} = e^{-\varepsilon^\beta x}$$
$$= 1 - \varepsilon^\beta x_\eta + \frac{\varepsilon^{2\beta} x_\eta^2}{2} \ldots$$

$$y_{in} = A\left(1-e^{-\varepsilon^{\beta-1}X_y}\right) + \varepsilon B\left(1-e^{-\varepsilon^{\beta-1}X_y}\right) - \varepsilon^{\beta-1}X_y A$$
$$= A\cancel{\varepsilon^{\beta-1}X_y} + \underbrace{\varepsilon^{\beta} B X_y}_{\text{common part.}} - \cancel{\varepsilon^{\beta-1}X_y A}$$

We must have that $B = -1$.

Composite Expansion

$$y = e^{-x} + (1-e^{-x/\varepsilon}) - \varepsilon(1-e^{-x/\varepsilon}) \overbrace{- x + \varepsilon^{\beta}X_y}^{\text{common parts.}} - 1$$

$$\Rightarrow y \sim e^{-x} + e^{-x/\varepsilon} - \varepsilon(1-e^{-x/\varepsilon})$$

Example:

$$\varepsilon^2 y'' + \varepsilon x y' - y = -e^{x}.$$
$$y(0)=2, \quad y(1)=1$$

Outer Expansion:

Try an expansion of the form

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$$

$O(1)$:

$$y_0 = e^{x}.$$

Apparently we cannot satisfy either boundary condition.
We need two boundary layers!

Left layer:

$$X = \varepsilon^{-\alpha} x$$
$$\Rightarrow \varepsilon^{2-2\alpha}\frac{d^2y}{dX^2} + \varepsilon X\frac{dy}{dX} - y = -e^{\varepsilon^{\alpha}X}$$

Balancing we have that:

$$2 - 2\alpha = 0$$
$$\Rightarrow \alpha = 1 \longrightarrow \text{Boundary layer of width } \varepsilon$$

$O(1)$:

$$\frac{d^2 y_\ell}{dX^2} - y_\ell = -1, \quad y_\ell(0) = 2$$

$$\Rightarrow y_\ell = 1 + A e^{-\bar{X}} + (1-A) e^{\bar{X}}$$

Right Layer

$$\tilde{X} = \frac{x-1}{\varepsilon^\beta}$$

$$\Rightarrow \varepsilon^{2-2\beta} \frac{d^2 y_r}{d\tilde{X}^2} + (1+\varepsilon^\beta \tilde{X}) \varepsilon^{1-\beta} \frac{d y_r}{d\tilde{X}} - y_r = -e^{1+\varepsilon^\beta \tilde{X}}$$

Balancing:

$$2-2\beta = 1-\beta$$

$$\Rightarrow \beta = 1.$$

$$\Rightarrow \frac{d^2 y_r}{d\tilde{X}^2} + \frac{d y_r}{d\tilde{X}} - y_r = -e$$

$$y_r(0) = 1$$

$$\Rightarrow y_r(\tilde{X}) = e + B e^{\left(\frac{-1+\sqrt{5}}{2}\right)\tilde{X}} + (1-e-B) e^{\left(\frac{-1-\sqrt{5}}{2}\right)\tilde{X}},$$

Matching

left layer $\Rightarrow A = 1$

right layer $\Rightarrow B = (1-e)$

Composite Expansion

$$y \sim e^x + 1 + e^{-x/\varepsilon} + e + (1-e) e^{-\left(\frac{1+\sqrt{5}}{2}\right)(1-x)/\varepsilon} - 1 - e$$

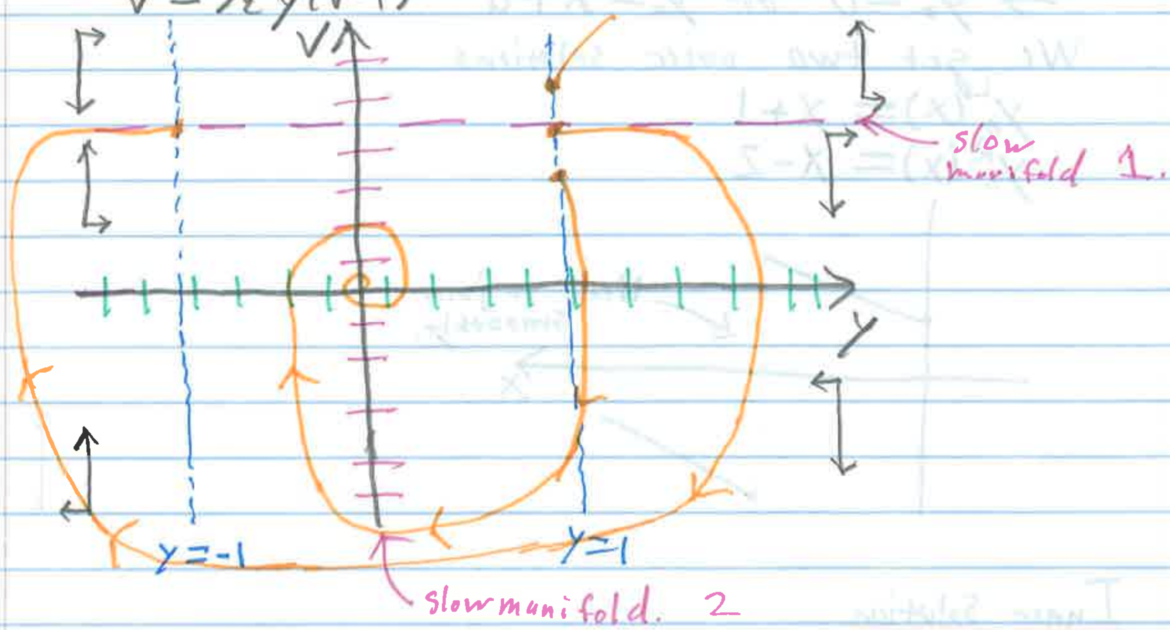
$$\Rightarrow y \sim e^x + e^{-x/\varepsilon} + (1-e) e^{-\left(\frac{1+\sqrt{5}}{2}\right)(1-x)/\varepsilon}.$$

Example (Interior Layer)

$$\varepsilon y'' = yy' - y$$
$$y(0) = 1,\; y(1) = -1$$

phase portrait.

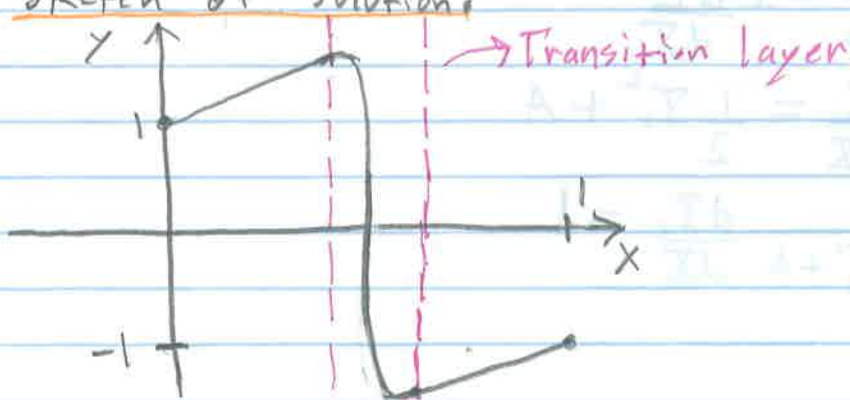
$$y' = v$$
$$v' = \tfrac{1}{\varepsilon} y(v-1)$$



On slow manifold 1 $v=1 \Rightarrow \frac{dy}{dx} = 1 \Rightarrow y = x + c$ is an exact solution. This means trajectories on the slow manifold stay on the slow manifold.

* Slow manifolds correspond to outer solutions. This tells us there will be a transition layer.

Sketch of solution:



Outer Solution.

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$$

$O(1)$:

$$y_0 y_0' - y_0 = 0$$

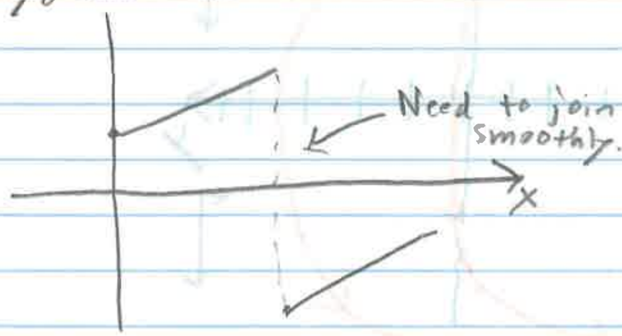
$$\frac{1}{2}\frac{d}{dx}(y_0)^2 = y_0$$

$$\Rightarrow y_0 = 0 \quad \text{or} \quad y_0 = x + a$$

We get two outer solutions

$$y_0^L(x) = x + 1$$

$$y_0^R(x) = x - 2$$



Inner Solution.

$$X = \frac{x - x_0}{\varepsilon^\alpha}, \quad x_0 - \text{to be determined.}$$

$$\Rightarrow \varepsilon^{1-2\alpha}\frac{d^2y}{dX^2} = \varepsilon^{-\alpha} y \frac{dy}{dX} - y$$

Balancing we have $\alpha = 1$. Expand!

$$y = Y_0 + \varepsilon Y_1 + \dots$$

$O(1)$:

$$\frac{d^2 Y_0}{dX^2} = Y \frac{dY}{dX}$$

$$\Rightarrow \frac{dY_0}{dX} = \frac{1}{2} Y_0^2 + A$$

$$\Rightarrow \frac{1}{\frac{1}{2}Y_0^2 + A}\frac{dY_0}{dX} = 1$$

$$\Rightarrow \int \frac{1}{\frac{1}{2}Y_0^2 + A}\, dY_0 = X + C$$

$$\Rightarrow Y_0 = A \tanh\left(\frac{AX}{2} + C\right)$$

$$\Rightarrow Y_0 = A \,\frac{e^{AX/2 + C} - e^{-AX/2 - C}}{e^{AX/2 + C} + e^{-AX/2 - C}}$$

$$= A\left(\frac{1 - Be^{-AX}}{1 + Be^{-AX}}\right)$$

$$\lim_{X \to \infty} Y_0(X) = \lim_{x \to x_0} y_0^R(x)$$

$$\Rightarrow A = x_0 - 2$$

$$\lim_{X \to -\infty} Y_0(X) = \lim_{x \to x_0} y_0^L(x)$$

$$\Rightarrow -A = x_0 + 1$$

$$\Rightarrow x_0 = \tfrac{1}{2}, \quad A = -\tfrac{3}{2}$$

So,

$$Y_0 = -\frac{3}{2}\tanh\left(\frac{-3}{4}\left(\frac{x - \frac{1}{2}}{\varepsilon}\right) + C\right)$$

Now, when $x = \frac{1}{2}$, we have $Y_0 = 0$ from the phase portrait.
This implies $C = 0$

$$\Rightarrow Y_0 = \frac{-3}{2}\tanh\left(-\frac{3}{4}\left(\frac{x - \frac{1}{2}}{\varepsilon}\right)\right)$$

$$= -\frac{3}{2}\tanh\left(\frac{3}{4}\left(\frac{\frac{1}{2} - x}{\varepsilon}\right)\right).$$

Composite Expansion.

We really need to do this in parts

$$y \sim \begin{cases} x + 1 - \frac{3}{2}\tanh\left(\frac{3}{4}\left(\frac{\frac{1}{2} - x}{\varepsilon}\right)\right) - \frac{3}{2} & x < \frac{1}{2} \\ x - 2 - \frac{3}{2}\tanh\left(\frac{3}{4}\left(\frac{\frac{1}{2} - x}{\varepsilon}\right)\right) + \frac{3}{2} & x > \frac{1}{2} \end{cases}$$

$$y \sim x - \tfrac{1}{2} - \tfrac{3}{2}\tanh\left(\frac{3}{4}\,\frac{(\frac{1}{2} - x)}{\varepsilon}\right)$$

Example (Corner Layer)

$$\varepsilon y'' + \left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right) y' - \left(x + \frac{1}{2}\right) y = 0$$

$$y(0) = 2$$
$$y(1) = 3$$

Outer Solution:

$$y = y_0 + \varepsilon y_1 + \dots$$

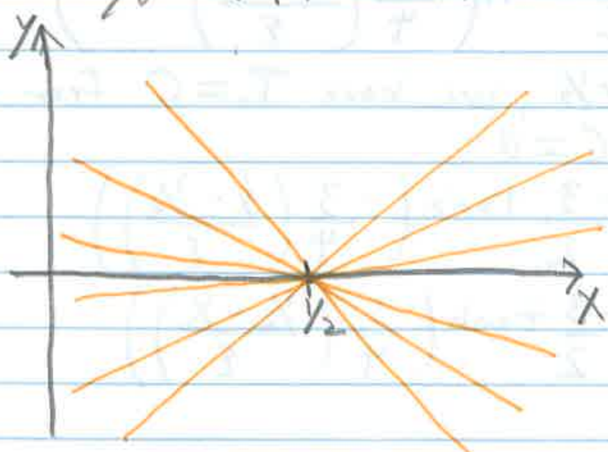
$O(1)$:

$$\left(x - \frac{1}{2}\right) y_0' - y_0 = 0$$

$$\frac{1}{y_0}\, dy_0 = \frac{1}{x - 1/2}$$

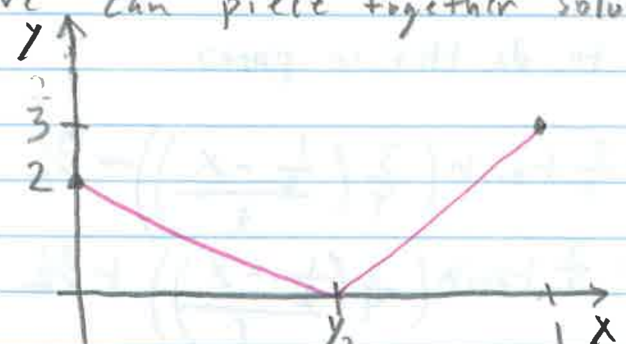
$$\ln(y_0) = \ln\left(x - \tfrac{1}{2}\right) + C$$

$$y_0 = A(x - 1/2).$$



Solutions are family of lines through $x = 1/2$

We can piece together solutions:



$$y_0 \sim \begin{cases} -4(x - 1/2) & \text{if } 0 \le x \le 1/2 \\ 6(x - 1/2) & \text{if } 1/2 \le x \le 1 \end{cases}$$

We can correct the discontinuity by adding a "corner" layer.

Inner Layer

Rescale by

$$X = \frac{x - 1/2}{\varepsilon^{\alpha}}$$

$$\Rightarrow \varepsilon^{1-2\alpha} \frac{d^2Y}{dX^2} + \varepsilon^{\alpha} X (1 + \varepsilon^{\alpha} X) \varepsilon^{-\alpha} \frac{dY}{dX} - \varepsilon^{\alpha} Y = 0$$

Balance:

$$1 - 2\alpha = \alpha$$

$$\Rightarrow \alpha = 1/2.$$

Let $Y = Y_0 + \varepsilon Y_1 + \ldots$

$O(1)$:

$$\frac{d^2Y_0}{dX^2} + X \frac{dY_0}{dX} - Y_0 = 0$$

$Y_0 = X$ is a solution, apply reduction of order.

$$Y_0 = AX + B\left[e^{X^2/2} + X \int_0^X e^{-s^2/2} ds \right].$$

$$Y_0 = \frac{A(x-1/2)}{\varepsilon^{1/2}} + B\left[e^{\frac{(x-1/2)^2}{2\varepsilon}} + \frac{(x-1/2)}{\varepsilon^{1/2}} \int_0^{(x-1/2)/\varepsilon} e^{-s^2/2} ds \right.$$

Matching

$$x_\eta = \frac{x - 1/2}{\varepsilon^{\beta}}, \quad 0 < \beta < 1/2$$

$$y_0 \sim \begin{cases} -4\varepsilon^{\beta} x_\eta, & x_\eta < 0 \\ 6\varepsilon^{\beta} x_\eta, & x_\eta > 0 \end{cases}$$

$$Y_0 = A\varepsilon^{\beta - 1/2} x_\eta + B e^{x_\eta^2 \varepsilon^{2\beta - 1}/2} + B\varepsilon^{\beta - 1/2} x_\eta \int_0^{x_\eta \varepsilon^{\beta - 1}} e^{-s^2/2} ds.$$

$$\sim A\varepsilon^{\beta - 1/2} x_\eta + \varepsilon^{\beta - 1/2} \operatorname{sgn}(x_\eta) \beta \sqrt{\frac{\pi}{2}}$$

We cannot match. We chose the wrong scaling.
We should have done

$$Y_{in} = \varepsilon^{1/2} Y_0 + \varepsilon^{\gamma} Y_1 + \dots$$

Do this, we get the matching condition:

$$A + B\sqrt{\frac{\pi}{2}} = 6$$

$$A - B\sqrt{\frac{\pi}{2}} = -4$$

$$\Rightarrow A = 1 \quad \text{and} \quad B = 5\sqrt{2/\pi}$$

Composite Expansion

$$y \sim -4(x-1/2) + \varepsilon^{1/2}\left[\frac{(x-1/2)}{\varepsilon^{1/2}} + 5\sqrt{\frac{2}{\pi}}\left(e^{-(x-1/2)^2/2\varepsilon} + \frac{(x-1/2)}{\varepsilon^{1/2}}\int_0^{(x-1/2)/\varepsilon^{1/2}} e^{-s^2/2}\,ds\right)\right] + 4(x-1/2)$$

overlap term

$$\Rightarrow y \sim (x-1/2) + 5\sqrt{\frac{2}{\pi\varepsilon}}\, e^{-(x-1/2)^2/2\varepsilon} + 5\sqrt{\frac{2}{\pi}}\,(x-1/2)\int_0^{(x-1/2)/\varepsilon^{1/2}} e^{-s^2/2}\,ds$$

$$\Rightarrow y \sim (x-1/2)\left[1 + 5\sqrt{\frac{2}{\pi}}\int_0^{\frac{(x-1/2)}{\varepsilon^{1/2}}} e^{-s^2/2}\,ds\right] + 5\sqrt{\frac{2}{\pi\varepsilon}}\exp\left(-\frac{(x-1/2)^2}{2\varepsilon}\right)$$