Theorem - If
$$f \sim a_1(x)\phi_1(\varepsilon) + a_2(x)\phi_2(\varepsilon)$$
and
$$\frac{d}{dx} f(x,\varepsilon) \sim b_1(x)\phi_1(\varepsilon) + b_2(x)\phi_2(\varepsilon)$$
then
$$b_1(x) = \frac{d}{dx} a_1(x), \quad b_2(x) = \frac{d}{dx} a_2(x).$$

Theorem - If
$$f \sim a_1(x)\phi_1(\varepsilon) + a_2(x)\phi_2(\varepsilon)$$
then
$$\int_a^b f(x,\varepsilon)\,dx \sim \left(\int_a^b a_1(x)\,dx\right)\phi_1(\varepsilon) + \left(\int_a^b a_2(x)\,dx\right)\phi_2(\varepsilon)$$

Example -

Find a asymptotic expansion for

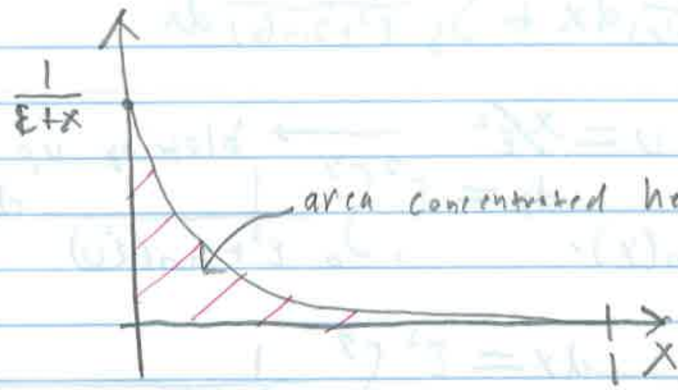
1. $f(\varepsilon) = \int_0^\varepsilon e^{-x^2}\,dx \qquad u = x/\varepsilon$

$$\Rightarrow f(\varepsilon) = \varepsilon \int_0^1 e^{-\varepsilon^2 u^2}\,du$$
$$\sim \varepsilon \int_0^1 \left(1 - \varepsilon^2 u^2 + \frac{\varepsilon^4 u^4}{2} + \dots\right)$$
$$\sim \varepsilon - \frac{\varepsilon^3}{3} + \frac{\varepsilon^5}{10} + \dots$$

2. $f(\varepsilon) = \int_0^1 \frac{1}{\varepsilon + x}\,dx$

$$= \ln(1+\varepsilon) - \ln(\varepsilon)$$
$$= \varepsilon - \ln(\varepsilon) + \dots$$
$$f(\varepsilon) \sim -\ln(\varepsilon) + \varepsilon + \dots$$

Lets try a different approach to get the asymptotic series.



Try Taylor expansion

$$f(\varepsilon)=\int_0^1 \frac{1}{1+\varepsilon/x}\,dx=\int_0^1 \left(1-\varepsilon/x+\varepsilon^2/x^2+\dots\right)dx$$

howerever, the integrals on the right are not finite valued. In fact for the right side to be well ordered we need $x \gg \varepsilon$. Set δ satisfying $\varepsilon \ll \delta \ll 1$.

$$f(\varepsilon)=\int_0^\delta \frac{1}{\varepsilon+x}\,dx+\int_\delta^1\left(\frac{1}{x}-\varepsilon/x^2+\varepsilon^2/x^3+\dots\right)dx$$

$$=\ln(\varepsilon+\delta)-\ln(\varepsilon)-\ln(\delta)+\varepsilon-\varepsilon/\delta+\dots$$
$$=\ln\left(\delta\left(1+\frac{\varepsilon}{\delta}\right)\right)-\ln(\varepsilon)-\ln(\delta)+\varepsilon-\varepsilon/\delta+\dots$$
$$=\ln(\delta)+\ln(1+\varepsilon/\delta)-\ln(\varepsilon)-\ln(\delta)+\varepsilon-\varepsilon/\delta+\dots$$
$$=\ln(\delta)+\varepsilon/\delta+\varepsilon^2/\delta^2-\ln(\varepsilon)-\ln(\delta)+\varepsilon-\varepsilon/\delta+\dots$$
$$=-\ln(\varepsilon)+\varepsilon+\dots$$

3. $f(\varepsilon)=\int_0^{\pi/3}\frac{dx}{\varepsilon^2+\sin(x)}$

$$\frac{1}{\varepsilon^2+\sin(x)}=\frac{1}{\sin(x)}\left(\frac{1}{1+\varepsilon^2/\sin(x)}\right)=\frac{1}{\sin(x)}\left(1-\frac{\varepsilon^2}{\sin(x)}+\dots\right)$$

(if $\varepsilon^2/\sin(x)\ll 1 \Longrightarrow \varepsilon^2/x \ll 1$.)

Set $\delta \gg \varepsilon^2$.

$$f(\varepsilon) = \int_0^{\delta} \frac{1}{\varepsilon^2 + \sin(x)}\,dx + \int_{\delta}^{1} \frac{1}{\varepsilon^2+\sin(x)}\,dx$$

Set $\eta = \delta/\varepsilon^2$, $\upsilon = x/\varepsilon^2$ $\longrightarrow$ blowing up the sing.

$$\Rightarrow \int_0^{\delta} \frac{1}{\varepsilon^2+\sin(x)}\,dx = \varepsilon^2 \int_0^{\eta} \frac{1}{\varepsilon^2 + \sin(\varepsilon^2\upsilon)}\,d\upsilon$$

$$\Rightarrow \int_0^{\delta} \frac{1}{\varepsilon^2+\sin(x)}\,dx = \varepsilon^2 \int_0^{\eta} \frac{1}{\varepsilon^2 + \varepsilon^2\upsilon - \frac{1}{6}\varepsilon^6\upsilon^3 + \ldots}\,d\upsilon$$

$$= \int_0^{\eta} \frac{1}{(1+\upsilon) - \frac{1}{6}\varepsilon^4\upsilon^3 + \ldots}\,d\upsilon$$

$$= \int_0^{\eta} \frac{1}{1+\upsilon}\left(\frac{1}{1 - \frac{\varepsilon^4\upsilon^3}{6(1+\upsilon)} + \ldots}\right)d\upsilon$$

$$= \int_0^{\eta} \frac{1}{1+\upsilon}\left(1 + \frac{\varepsilon^4\upsilon^3}{6(1+\upsilon)} + \ldots\right)d\upsilon$$

$$= \int_0^{\eta}\left(\frac{1}{1+\upsilon} + \frac{\varepsilon^4\upsilon^3}{6(1+\upsilon)^2} + \ldots\right)d\upsilon$$

$$\Rightarrow \int_0^{\delta} \frac{1}{\varepsilon^2+\sin(x)} \sim \ln(1+\eta) + \ldots$$

$$\sim \ln(\eta(1+1/\eta)) + \ldots$$

$$\sim \ln(\eta) + \ln(1+1/\eta) + \ldots$$

$$\sim \ln(\eta) + 1/\eta + \ldots$$

$$\sim \ln(\delta/\varepsilon^2) + 1/\eta + \ldots$$

$$\sim -2\ln(\varepsilon) + \ln(\delta) + 1/\eta + \ldots$$

Now lets do the other part

$$\int_{\delta}^{\pi/3} \frac{dx}{\varepsilon^2+\sin(x)}\,dx = \int_{\delta}^{\pi/3} \frac{1}{\sin(x)\left(1+\varepsilon^2/\sin(x)\right)}\,dx$$

$$= \int_{\delta}^{\pi/3} \frac{1}{\sin(x)}\left(1-\frac{\varepsilon^2}{\sin(x)}+\dots\right)dx$$

$$= \ln\left(\frac{1}{3}\sqrt{3}\right) - \ln\left(\tan\left(\frac{\delta}{2}\right)\right)$$

$$+ \varepsilon^2\left(\frac{1}{3}\sqrt{3} - \cot(\delta)\right) + \dots$$

$$\sim \ln\left(\frac{\frac{1}{3}\sqrt{3}}{\tan(\delta/2)}\right) + \dots$$

$$\sim \ln\left(\frac{\frac{2}{3}\sqrt{3}}{\delta}\right) + \dots$$

$$\sim \ln\left(\frac{2}{3}\sqrt{3}\right) - \ln(\delta) + \dots$$

Full asymptotic approximation:

$$f(\varepsilon) \sim -2\ln(\varepsilon) + \ln\left(\frac{2}{3}\sqrt{3}\right) + \dots$$

Example

Find a two-term asymptotic expansion, for small ε, for the solution of the following problem.

$$\nabla^2 u = 0 \quad \text{for } x^2+y^2<1,$$
$$u = e^{\varepsilon\cos\theta} \quad \text{for } x^2+y^2=1.$$

(membrane with boundary conditions)

We assume $u = U_0 + \varepsilon^2 U_1 + \dots$

$\mathcal{O}(1)$:

$$\nabla^2 U_0 = 0, \quad x^2+y^2<1,$$
$$U_0 = 1, \quad x^2+y^2=1.$$

Assume $U_0 = f(r)$

$$\Rightarrow f''(r) + \frac{1}{r} f'(r) = 0,$$

$$f(1) = 1, \quad r=1$$

$f=1$ is the solution

$$\Rightarrow U_0 = 1.$$

$\mathcal{O}(\varepsilon)$:

$$\nabla^2 U_1 = 0, \quad x^2+y^2<1$$
$$U_1 = \cos^2\theta, \quad x^2+y^2=1$$

The boundary conditions can be expressed as

$$U_1 = \tfrac{1}{2}(1+\cos(2\theta)), \quad x^2+y^2=1.$$

This motivates taking an ansatz of the form

$$U_1 = f_1(r) + f_2(r)\cos(2\theta)$$

$$\Rightarrow f_1'' + \frac{1}{r} f_1'(r) = 0, \qquad f_2'' + \frac{1}{r} f_2'(r) - \frac{4}{r^2} f_2 = 0$$
$$f_1(1) = 1/2 \qquad\qquad f_2(1) = 1/2.$$

$f_1(r) = \frac{1}{2}$, $\qquad f_2(r) = \frac{1}{2} r^{\beta}$

$$\Rightarrow \beta(\beta-1) + \beta - 4 = 0$$
$$\beta = \pm 2$$
$$\Rightarrow f_2(r) = \tfrac{1}{2} r^2.$$

$$\Rightarrow U_1(r) = \frac{1}{2} + \frac{r^2}{2}\cos(2\theta).$$

The full solution is then,

$$U(r) = 1 + \frac{\varepsilon}{2}(1 + r^2\cos(2\theta)) + \dots$$

Iterated Maps

1. Recall that a sequence is Cauchy if $\forall \varepsilon > 0$ $\exists M \in \mathbb{N}$ such that $n, m > M \Rightarrow |x_m - x_n| < \varepsilon$.

2. A sequence is Cauchy in $\mathbb{R}$ if and only if $\exists x^* \in \mathbb{R}$ such that
$$\lim_{n\to\infty} x_n = x^*$$

Suppose we want to solve the equation
$$f(x) = x.$$
where f is a smooth function. For a good initial guess x_0 we can look at iterated maps
$$x_{n+1} = f(x_n) = Tx_n = T^n x_0$$

Theorem - If $f'(x^*) < 1$ then $\exists \delta > 0$ such that $|x_0 - x^*| < \delta$
$$\Rightarrow \lim_{n\to\infty} T\overset{n}{x_0} = \lim_{n\to\infty} x^n = x^*.$$

~~proof~~:

Let $x_n = T^n x_n$ we will prove that x_n is Cauchy.
$$\begin{aligned} |x_m - x_n| &= |T^m x_m - T^n x_n| \\ &= |f(x_{m-1}) - f(x_{n-1})| \\ &\le \max_{x \in [x^*-\delta, x^*+\delta]} f'(x) |x_{m-1} - x_{n-1}|, \end{aligned}$$

where δ is selected so that $\max_{x \in [x^*-\delta, x^*+\delta]} f'(x) = M < 1$.
$$\begin{aligned} \Rightarrow |x_m - x_n| &\le M |x_{m-1} - x_{n-1}| \\ &\le M^n |x^{m-n} - x_0| \\ &\le M^n [|x^{m-n} - x^{m-n-1} + x^{m-n-1} - x_0|] \\ &\le M^n [|x^{m-n} - x^{m-n-1}| + |x^{m-n-1} - x_0|] \\ &\le M^n [|T^{m-n}x_0 - T^{m-n-1}x_0| + |T^{m-n-1}x_0 - T^{m-n-2}x_0| \\ &\quad + \dots + |Tx_0 - x_0|] \\ &\le M^n \left[\sum_{k=0}^{m-n} M^k |Tx_0 - x_0|\right] \end{aligned}$$

$$\Rightarrow |x_m - x_n| \le M^n \sum_{k=0}^{\infty} M^k |Tx_0 - x_0|$$
$$\le \frac{M^n}{1-M} |Tx_0 - x_0|$$

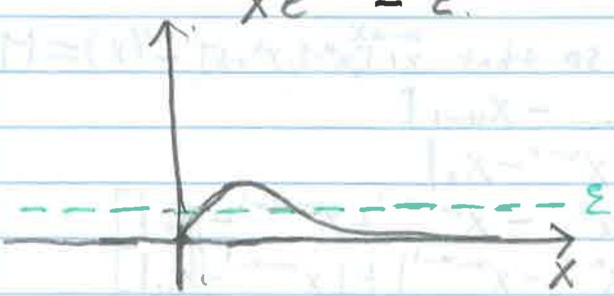
Therefore, x_n is Cauchy and consequently $\exists x^*$ such that $\lim_{n\to\infty} x_n = x^*$. From continuity we have that
$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_n) = f(x^*)$$
∎

Example:

Find a two term asymptotic solution to
$$x^2 + \varepsilon x - 1 = 0$$
Write as an iterated map
$$x_{n+1} = \pm\sqrt{1 - \varepsilon x_n}$$
Guess $x_0 = \pm 1$. Then,
$$x_1 = \pm\sqrt{1 \mp \varepsilon}$$
$$\rightarrow x_1 = \pm 1 \mp \varepsilon/2$$
Asymptotic approximation
$$x = 1 - \varepsilon/2 + \dots \quad \text{and} \quad x = -1 + \varepsilon/2 + \dots$$

Example:

Find an asymptote expansion for solutions to
$$xe^{-x} = \varepsilon.$$



Select $x_0 = \ln(1/\varepsilon)$ and consider the iterated map
$$x_{n+1} = \ln\left(\frac{1}{\varepsilon}\right) + \ln(x_n)$$
$$\Rightarrow x_1 = \ln\left(\frac{1}{\varepsilon}\right) + \ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right)$$

$$X_2 = \ln(1/\varepsilon) + \ln\left(\ln\left(\frac{1}{\varepsilon}\right) + \ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right)\right)$$

$$= \ln\left(\frac{1}{\varepsilon}\right) + \ln\left(\ln\left(\frac{1}{\varepsilon}\right)\left(1 + \frac{\ln(\ln(1/\varepsilon))}{\ln(1/\varepsilon)}\right)\right)$$

$$= \ln(1/\varepsilon) + \ln(\ln(1/\varepsilon)) + \ln\left(1 + \frac{\ln(\ln(1/\varepsilon))}{\ln(1/\varepsilon)}\right)$$

$$= \ln(1/\varepsilon) + \ln(\ln(1/\varepsilon)) + \frac{\ln(\ln(1/\varepsilon))}{\ln(1/\varepsilon)}$$

Therefore, the asymptotic approximation is given by

$$x = \ln(1/\varepsilon) + \ln(\ln(1/\varepsilon)) + \frac{\ln(\ln(1/\varepsilon))}{\ln(1/\varepsilon)}$$