APMA 1930M: Applied Asymptotic Analysis

Chapter 1: Introduction to Asymptotic Approximations.

Motivation - Many modern problems are nonlinear and need new techniques to solve them. Classical physical models are linear and a well developed set of tools have been developed to study such problems:
- linear algebra
- variation of parameters
- Fourier analysis

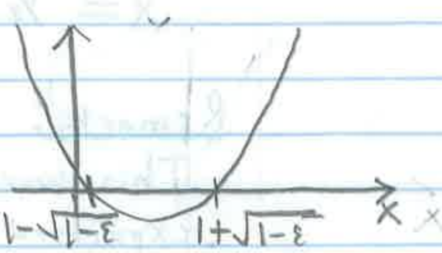
In this course we are interested in weakly nonlinear systems. Topics covered will include:
1. Matched asymptotics in ODEs and PDEs.
2. Method of multiple scales.
3. Nonlinear waves.
4. Homogenization.

*Remark - Although modern numerics can "solve" many problems, it tells us very little about the structure and physics of solutions. Moreover, without analysis there is no justification that numerical schemes have actually converged to a solution to the problem.

Example:

Let $0 < \varepsilon \ll 1$. Solve the equation

$$x^2 - 2x + \varepsilon = 0$$



Exact solution:

$$x = 1 \pm \sqrt{1-\varepsilon}$$
$$= 1 \pm \left(1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + \dots\right)$$ Taylor expansion valid for $|\varepsilon| < 1$

$$x = 2 - \tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + \dots, \quad x = \tfrac{1}{2}\varepsilon + \tfrac{1}{8}.$$

What if we did not know the solution beforehand?
Assume a power series of the form:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

Substitute in:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 - 2(x_0 + \varepsilon x_1 + \dots) + \varepsilon = 0$$

$$\Rightarrow x_0^2 + 2\varepsilon x_1 x_0 + \varepsilon^2(x_1^2 + 2x_1 x_2) - 2x_0 - 2\varepsilon x_1 - 2\varepsilon^2 x_2 + \varepsilon = 0.$$

Solve order by order:

$\mathcal{O}(1)$:

$$x_0^2 - 2x_0 = 0$$

$$\Rightarrow x_0 = 0, 2$$

$\mathcal{O}(\varepsilon)$:

$$2x_1 x_0 - 2x_1 + 1 = 0$$

If $x_0 = 0 \Rightarrow x_1 = 1/2$

If $x_0 = 2 \Rightarrow x_1 = -1/2$

$\mathcal{O}(\varepsilon^2)$:

$$x_1^2 + 2x_1 x_2 - 2x_2 = 0$$

If $x_0 = 0,\ x_1 = 1/2 \Rightarrow x_2 = 1/8$

If $x_0 = 2,\ x_1 = -1/2 \Rightarrow x_2 = -1/8$

We get back the correct series expansion:

$$x = \tfrac{1}{2}\varepsilon + \tfrac{1}{8}\varepsilon^2,\quad 2 - \tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2$$

Remark:

This worked because we knew the solutions could be expressed as a power series in ε. This is an example of a regular perturbation problem.

Example:

For $0<\varepsilon<<1$, solve the equation

$$\varepsilon x^3 - x + 1 = 0$$

Assume that

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$$

$$\Rightarrow \varepsilon(x_0 + \varepsilon x_1 + \ldots)^3 - (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots) + 1 = 0$$

$$\Rightarrow (\varepsilon x_0^3 + 3\varepsilon^2 x_0 x_1 + \ldots) - (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots) + 1 = 0$$

$O(1)$:

$$-x_0 + 1 = 0 \Rightarrow x_0 = 1$$

$O(\varepsilon)$:

$$x_0^3 - x_1 = 0$$

$$\Rightarrow x_1 = 1$$

$O(\varepsilon^2)$:

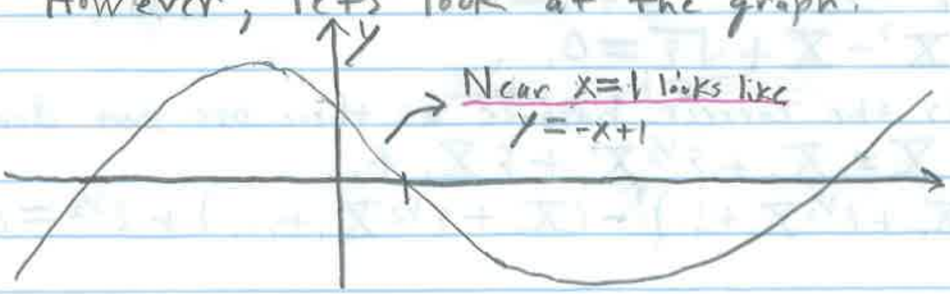
$$3x_0 x_1 - x_2 = 0$$

$$\Rightarrow x_2 = 3$$

The solution is

$$x = 1 + \varepsilon + 3\varepsilon^2 + \ldots$$

However, lets look at the graph:



How did we lose the other roots? We implicitely assumed that εx^3 is small near the roots. What if in fact εx^3 is large?

Lets rescale:

$$x = \varepsilon^{\alpha} X$$

$$\varepsilon^{1+3\alpha} X^3 - \varepsilon^{\alpha} X + 1 = 0$$

The key idea is that in order for a solution to exist in the form of a series two terms must "balance"

Possible balances:

1. $\varepsilon^{1+3\alpha} X^3 - \varepsilon^{\alpha} X + 1 = 0$

$\alpha = 0$, $1+3\alpha > 0$

$\Rightarrow$ Main balance

$-X + 1 = 0$ (already discussed)

2. $\varepsilon^{1+3\alpha} X^3 - \varepsilon^{\alpha} X + 1 = 0$

$1 + 3\alpha = 0 \Rightarrow \alpha = -1/3$

$\Rightarrow X^3 - \varepsilon^{-1/3} X + 1 = 0$

$\downarrow$

largest term so this can't be correct.

3. $\varepsilon^{1+3\alpha} X^3 - \varepsilon^{\alpha} X + 1 = 0$

$1 + 3\alpha = \alpha \Rightarrow \alpha = -1/2$

$\Rightarrow \varepsilon^{-1/2} X^3 - \varepsilon^{-1/2} X + 1 = 0$

$\Rightarrow X^3 - X + \sqrt{\varepsilon} = 0.$

This is the correct balance as there are two dominant terms.

Try $X = X_0 + \varepsilon^{1/2} X_1 + \varepsilon X_2 + \ldots$

$\Rightarrow (X_0 + \varepsilon^{1/2} X_1 + \ldots)^3 - (X_0 + \varepsilon^{1/2} X_1 + \ldots) + \varepsilon^{1/2} = 0$

$O(1)$:

$X_0^3 - X_0 = 0$

$\Rightarrow X_0 = 0, \pm 1.$

$O(\varepsilon^{1/2})$:

$3 X_0 X_1 - X_1 + 1 = 0$

If $X_0 = 0 \Rightarrow X_1 = -1$

If $X_0 = 1 \Rightarrow X_1 = -\frac{1}{2}$

If $X_0 = -1 \Rightarrow X_1 = \frac{1}{4}$

The three solutions are approximately given by:

$$x = \underbrace{1 + \varepsilon + 3\varepsilon^2}_{\text{Taylor series.}},\ \underbrace{\varepsilon^{-1/2} - \tfrac{1}{2},\ \varepsilon^{-1/2} + \tfrac{1}{4}}_{\text{Laurent series}}$$

Remark:

This calculation was very formal. This is an example of a singular perturbation problem. Can we rigorously justify what we did? Finding correct approximate solutions to nonlinear singular perturbation problems is a difficult art to master!

Example:

Lets estimate the probability $x \geq X_0$ for the distribution $\pi^{-1/2} e^{-x^2}$.

$$P(x \geq X_0) = \int_{X_0}^{\infty} e^{-x^2} dx \qquad (u = x^2 \Rightarrow du = 2x\,dx)$$

$$= \frac{1}{2} \int_{X_0^2}^{\infty} e^{-u} u^{-1/2} du$$

$$= -\frac{1}{2} \int_{X_0^2}^{\infty} \frac{d}{du}(e^{-u}) u^{-1/2} du$$

$$= \frac{e^{-X_0^2}}{2X_0} - \frac{1}{4} \int_{X_0^2}^{\infty} e^{-u} u^{-3/2} du$$

Continue formally to obtain:

$$P(x \geq X_0) \text{``}=\text{''} \frac{1}{2} \frac{e^{-X_0^2}}{X_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2X_0^2)^n}$$

However,

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n (2n-1)!!}{(2X_0^2)^n} \right| = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2X_0^2)}$$

$\Rightarrow$ The series diverges for all X_0!

How can we make sense of this? Note the following

$$X_0 e^{+X_0^2} P(x \geq X_0) - \sum (-1)^n \frac{(2n-1)!!}{(2X^2)^n} = \frac{X_0 e^{X_0^2}}{2^{N+1}} (2N+1)! \underbrace{\int_{X_0^2}^{\infty} e^{-U} U^{-N-3/2} du}_{\text{error}}$$

The error goes to zero for all N, but the full series diverges!

This is an example of an asymptotic sequence.

Example:

Projectile in a gravitational field:

$$\frac{d^2x}{dt^2} \simeq \frac{-gR^2}{(x+R)^2}$$

$$x(0) = 0$$

$$\left.\frac{dx}{dt}\right|_{t=0} = V_0$$

Rescale by

$$X = \frac{x}{L}, \quad \tau = \frac{t}{T} \Rightarrow \frac{d}{dt} = \frac{1}{T}\frac{d}{d\tau}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(LX) = \frac{L}{T^2}\frac{d^2X}{d\tau^2}$$

$$\Rightarrow \frac{L}{T^2}\frac{d^2X}{d\tau^2} = \frac{-gR^2}{(LX+R)^2}$$

$$\Rightarrow \frac{d^2X}{d\tau^2} = -\frac{gT^2}{L(L/R\,X+1)^2}$$

The initial conditions become

$$X(0) = 0$$

$$\frac{L}{T}\frac{dX}{d\tau} = V_0$$

Set $T = \frac{L}{V_0}$ → Sets intial velocity to 1.

Then we have that

$$\frac{d^2 X}{d\tau^2} = -\frac{gL}{v_0^2 (L/R\, X + 1)^2} \longrightarrow \text{set numerator to } 1.$$

Set

$$L = v_0^2/g \Rightarrow T = v_0/g.$$

Therefore we have that:

$$\frac{d^2 X}{d\tau^2} = -\frac{1}{(\varepsilon X + 1)^2}, \quad \varepsilon = \frac{v_0^2}{gR} << 1$$

$$X(0) = 0$$

$$\left.\frac{dX}{d\tau}\right|_{\tau=0} = 1.$$

Note that small parameters must be dimensionless.

Lets try solving by assuming

$$X(\tau) = X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + \ldots$$

$$\Rightarrow \frac{d^2 X_0}{d\tau^2} + \varepsilon \frac{d X_1}{d\tau^2} = -\frac{1}{(1 + \varepsilon(X_0 + \varepsilon X_1 + \ldots))^2}$$

$$= -1 + 2\varepsilon X_0 + \ldots$$

$$\frac{1}{(1+x)^2} = 1 - 2x + \ldots$$

Taylor Series

$O(1)$:

$$\frac{d^2 X_0}{d\tau^2} = -1$$

$$X_0 = 0, \quad \left.\frac{dX_0}{d\tau}\right|_{\tau=0} = 1$$

$$\Rightarrow X_0 = -\tfrac{1}{2}\tau^2 + \tau \rightarrow \text{Uniform gravitational field solution.}$$

$O(\varepsilon)$:

$$\frac{d^2 X_1}{d\tau^2} = 2\left(-\tfrac{1}{2}\tau^2 + \tau\right)$$

$$X_1(0) = 0, \quad \left.\frac{dX_1}{d\tau}\right|_{\tau=0} = 0$$

$$\Rightarrow X_1 = -\frac{\tau^4}{12} + \frac{\tau^3}{3}$$

Full Solution:

$$X(\tau) = -\tfrac{1}{2}\tau^2 + \tau + \varepsilon\left(-\frac{\tau^4}{12} + \frac{\tau^3}{3}\right) + \ldots$$

Order Symbols:

Suppose we want to approximate $\frac{1}{1-\varepsilon}$ for $0<\varepsilon\ll 1$.
We can expand in powers of ε:

$$\frac{1}{1-\varepsilon} = 1+\varepsilon+\frac{\varepsilon^2}{2}+\dots$$

An approximation is then given by

$$\frac{1}{1-\varepsilon} \approx 1+\varepsilon$$

The error in this approximation is bounded by:

$$E = \Big|\underbrace{\frac{1}{1-\varepsilon}}_{\text{exact}} - \underbrace{(1+\varepsilon)}_{\text{approximation}}\Big| \le \underbrace{C\varepsilon^2}_{\text{Scaling law in error}}, \quad (0<\varepsilon<\tfrac{1}{2})$$

The error in this approximation "scales like ε^2" and this result follows from Taylor's Theorem:

Theorem – Let $f(\varepsilon)$ be a function that is $n+1$ times continuously differentiable for $\varepsilon_1<\varepsilon<\varepsilon_2$. If $\varepsilon_0, \varepsilon \in (\varepsilon_1, \varepsilon_2)$ then $\exists \xi \in (\varepsilon_1, \varepsilon_2)$ such that

$$\boxed{f(\varepsilon) = f(\varepsilon_0) + (\varepsilon-\varepsilon_0) f'(\varepsilon_0) + \dots + \frac{1}{n!}(\varepsilon-\varepsilon_0)^n f^{(n)}(\varepsilon-\varepsilon_0) + \frac{1}{(n+1)!}(\varepsilon-\varepsilon_0)^{n+1} f^{(n+1)}(\xi)}$$

Taylor's theorem gives a natural technique for constructing approximations.

Remark: Another approximation would be

$$\frac{1}{1-\varepsilon} \approx 1+2\varepsilon$$

The error is

$$E = \left|\frac{1}{1-\varepsilon} - (1+2\varepsilon)\right| \le \varepsilon + C\varepsilon^2$$

The error still satisfies $\lim_{\varepsilon \to 0} E = 0$ but the error scales at the same rate as the approximation!

We need to define what is meant by "scaling".

Definition - $f = \mathcal{O}(\phi)$ as $\varepsilon \to \varepsilon_0$ means $\exists\, C, \varepsilon_1$ so that
$$|f(\varepsilon)| \leq C\,|\phi(\varepsilon)| \quad \text{for } \varepsilon_0 < \varepsilon < \varepsilon_1$$
We say "f is big Oh of ϕ" as $\varepsilon \to \varepsilon_0$.

Example:
1. $\sin(\varepsilon) - \varepsilon = \mathcal{O}(\varepsilon^2)$, as $\varepsilon \to 0$,
2. $\sin(\varepsilon) - \varepsilon \neq \mathcal{O}(\varepsilon^3)$, as $\varepsilon \to 0$,
3. $\sin(\varepsilon) - \varepsilon = \mathcal{O}(1)$, as $\varepsilon \to 0$,

Definition - $f = o(\phi)$ as $\varepsilon \to \varepsilon_0$ means $\forall \delta > 0$, $\exists\, \varepsilon_2 > 0$ so that
$$|f(\varepsilon)| \leq \delta\,|\phi(\varepsilon)| \quad \text{for } \varepsilon_0 < \varepsilon < \varepsilon_2$$
We say "f is little oh of ϕ"

Example:
1. $\sin(\varepsilon) - \varepsilon \neq o(\varepsilon^2)$
2. $\sin(\varepsilon) - \varepsilon \neq o(\varepsilon^3)$
3. $\sin(\varepsilon) - \varepsilon = o(\varepsilon)$
4. $\sin(\varepsilon) - \varepsilon = o(1)$
5. $e^{-1/\varepsilon} = o(\varepsilon^N)$, $\forall N \in \mathbb{N}$.
   This is an example of a transcendantly small term.
6. $\varepsilon = o(\ln|\varepsilon|)$, as $\varepsilon \to 0$

Remark:
1. Big Oh means "roughly the same size as" near ε_0
2. Little oh means "much small than" near ε_0

Theorem –

1. If,
$$\lim_{\varepsilon \to \varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = L,$$
then $f = \mathcal{O}(\phi)$ as $\varepsilon \to \varepsilon_0$

2. If,
$$\lim_{\varepsilon \to \varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = 0,$$
then $f = o(\phi)$ as $\varepsilon \to \varepsilon_0$

proof

1. If $\lim_{\varepsilon \to \varepsilon_0} f(\varepsilon)/\phi(\varepsilon) = L$ then $\forall \eta > 0$, $\exists \delta_\eta$ such that $|\varepsilon - \varepsilon_0| < \delta_\eta \Rightarrow |f(\varepsilon)/\phi(\varepsilon) - L| \leq \eta$. Therefore,
$$-\eta \leq f(\varepsilon)/\phi(\varepsilon) - L \leq \eta$$
$$\Rightarrow (L - \eta)\phi(\varepsilon) \leq f(\varepsilon) \leq (L + \eta)\phi(\varepsilon)$$
Setting $\eta = |L/2|$, then $\varepsilon < \delta_{L/2} + \varepsilon_0$ implies
$$|f(\varepsilon)| \leq \max\{(L + |L/2|), (L - |L/2|)\}\phi(\varepsilon).$$

2. If $\lim_{\varepsilon \to \varepsilon_0} f(\varepsilon)/\phi(\varepsilon) = 0$ then $\forall \delta > 0$, $\exists \varepsilon_2$ such that $|\varepsilon - \varepsilon_0| < \varepsilon_2 \Rightarrow |f(\varepsilon)/\phi(\varepsilon)| \leq \delta$.
$$\Rightarrow |f(\varepsilon)| \leq \delta |\phi(\varepsilon)|.$$

$\blacksquare$

Notation –

1. $f << \phi$ means $f = o(\phi)$
2. $\varepsilon << 1$ means $\varepsilon \to 0$.

Properties:

1. $f = \mathcal{O}(1) \Leftrightarrow f$ is bounded. as $\varepsilon \to \varepsilon_0$
2. $f = o(1) \Leftrightarrow f \to 0$ as $\varepsilon \to \varepsilon_0$.
3. $f = o(\phi) \Rightarrow f = \mathcal{O}(\phi)$.

proof

(should be done by reader).

Example:

Show that $\mathcal{O}(f)\mathcal{O}(g)=\mathcal{O}(fg)$. Note, this means that if $f=\mathcal{O}(\phi_1)$ and $g=\mathcal{O}(\phi_2)$ then $f\cdot g=\mathcal{O}(\phi_1\phi_2)$.

~~proof~~

Suppose $f=\mathcal{O}(\phi_1)$ and $g=\mathcal{O}(\phi_2)$. Then, $\exists\, \varepsilon_1, \varepsilon_2$ and constants C_1, C_2 such that

$$\varepsilon_0<\varepsilon<\varepsilon_1 \Rightarrow |f(\varepsilon)|\le C_1|\phi_1(\varepsilon)|$$

$$\varepsilon_0<\varepsilon<\varepsilon_2 \Rightarrow |g(\varepsilon)|\le C_2|\phi_2(\varepsilon)|$$

Let $\varepsilon^*=\min\{\varepsilon_1,\varepsilon_2\}$. Then, $\varepsilon<\varepsilon^*$ implies

$$|f(\varepsilon)|\cdot|g(\varepsilon)|\le C_1C_2|\phi_1(\varepsilon)|\cdot|\phi_2(\varepsilon)|$$

$$\Rightarrow |f(\varepsilon)\cdot g(\varepsilon)|\le C_1C_2|\phi_1(\varepsilon)\phi_2(\varepsilon)|.$$

■

Definition - We say $f=\mathcal{O}_s(\varepsilon^\alpha)$ if $f=\mathcal{O}(\varepsilon^\alpha)$ but $f\ne o(\varepsilon^\alpha)$. We say "$f$ is strictly order ε^α"

Asymptotic Expansions

Definition - Given $f(\varepsilon)$ and $\phi(\varepsilon)$ we say $\phi(\varepsilon)$ is an asymptotic approximation if

$$f(\varepsilon)-\phi(\varepsilon)=o(\phi).$$

In this case we write $f\sim\phi$.

* The above definition means the error goes to zero faster than the approximation.

Example

1. $\dfrac{1}{1-\varepsilon}\sim 1+\varepsilon$
2. $\dfrac{1}{1-\varepsilon}$ is not asymptotic to $1+2\varepsilon$.
3. $\sin(\varepsilon)\sim\varepsilon$
4. $\sin(\varepsilon)\sim\varepsilon+\sqrt{50}\,\varepsilon^2$ (This is a problem).

<u>Remark</u> - Asymptotic approximations are not unique and give little information about the accuracy.

<u>Definition</u> -

1. The functions $\phi_1, \phi_2, \ldots$ form an <u>asymptotic sequence</u> or are <u>well ordered</u> if and only if $\phi_{n+1} = o(\phi_n)$. The functions ϕ_n are called <u>gauge functions</u>.
2. If $\phi_1(\varepsilon), \phi_2(\varepsilon), \ldots$ is an asymptotic sequence then f has an asymptotic expansion to n terms if and only if
$$f - \sum_{k=1}^{m} a_k \phi_k = o(\phi_m)$$

In this case we write
$$f \sim a_1 \phi_1(\varepsilon) + \ldots + a_n \phi_n(\varepsilon)$$

*The coefficients can be determined by
$$a_1 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\phi_1(\varepsilon)}$$
$$a_2 = \lim_{\varepsilon \to 0} \frac{f - a_1 \phi_1(\varepsilon)}{\varepsilon}$$
$$\vdots$$

<u>Examples</u>

1. $1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots$ is an asymptotic sequence
2. $\varepsilon^{1/2}, \varepsilon^{6/11}, \varepsilon^2, \ldots$ is an asymptotic sequence.
3. $\varepsilon, -\varepsilon^2 \ln(|\varepsilon|), \varepsilon^2, \varepsilon^3, \ldots$ is an asymptotic sequence.
4. $\varepsilon, \varepsilon^2 \sin(x), \varepsilon^3, \ldots$ is an asymptiotic sequence.

Example:

Find asymptotic expansions of the following

1. $$\frac{\sin(\varepsilon)}{\varepsilon^{3/2}} = \frac{\varepsilon - \varepsilon^3/6 + \varepsilon^5/5! + \ldots}{\varepsilon^{3/2}}$$

$$= \varepsilon^{-1/2} - \varepsilon^{3/2}/6 + \ldots$$

$$\Rightarrow \frac{\sin(\varepsilon)}{\varepsilon^{3/2}} \sim \varepsilon^{-1/2} - \frac{\varepsilon^{3/2}}{6} + \ldots$$

2. $$f(\varepsilon) = \frac{\sqrt{1+\varepsilon}}{\sin(\sqrt{\varepsilon})}$$

$$= \frac{1 + \frac{1}{2}\varepsilon + \ldots}{\varepsilon^{1/2} - \varepsilon^{3/2}/6 + \ldots}$$

$$= \varepsilon^{-1/2}\frac{\left(1 + \frac{1}{2}\varepsilon + \ldots\right)}{\left(1 - \varepsilon^{1}/6 + \ldots\right)}$$

$$= \varepsilon^{-1/2}\left(1 + \frac{1}{2}\varepsilon + \ldots\right)\left(1 + (\varepsilon^{1}/6 + \ldots)^1 + (\varepsilon^{1}/6 + \ldots)^2 + \ldots\right.$$

$$\Rightarrow f(\varepsilon) \sim \varepsilon^{-1/2} + \frac{2}{3}\varepsilon^{1/2}$$