Homework #8

#8.1.5.

Prove that any zero-eigenvalue bifurcation in two dimensions, the null-clines intersect tangentially.

Solution:

Consider the system

$$\begin{cases} \dot{x} = f(x,y,\nu) \\ \dot{y} = g(x,y,\nu) \end{cases}$$

and suppose a zero bifurcation occurs at ν^*. Let N_1, N_2 denote the null-clines satisfying:

$$N1 = \{(x,y) : f(x,y,\nu) = 0\},$$
$$N2 = \{(x,y) : g(x,y,\nu) = 0\},$$

which are the zero level sets for the functions f and g. Now, at ν^* it follows that the vectors (f_x, f_y) and (g_x, g_y) are parallel ~~on the~~ set $\vec{X}^* = N1 \cap N2$. Since (f_x, f_y) and (g_x, g_y) are normal to level sets of f, g it follows that the null-clines intersect tangentially at ν^*.

#8.1.6

Complete a bifurcation analysis for the system:

$$\begin{cases} \dot{x} = y - 2x \\ \dot{y} = \nu + x^2 - y \end{cases}$$

Solution:

The null-clines are given by

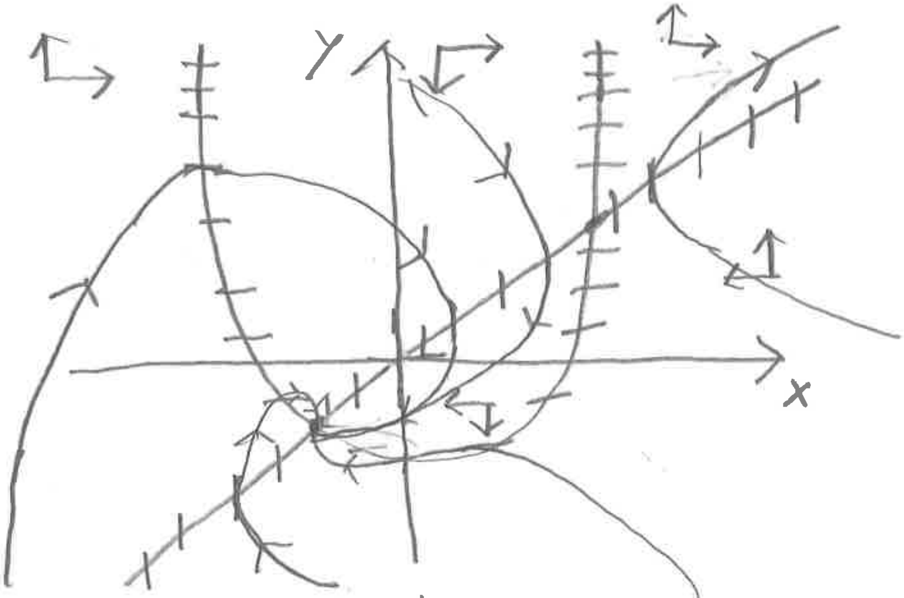
$$y = 2x, \quad \frac{dx}{dt} = 0$$
$$y = x^2 + \nu, \quad \frac{dy}{dt} = 0$$

We will clearly have a saddle-node bifurcation. On the next page we plot the phase portraits. The fixed points are given by:

$$x^2 + \nu = 2x$$

which has discriminant:

$$4 - 4\nu \Rightarrow \text{bifurcation at } \nu = 1$$



$\nu < 1$

$\nu > 1$

#8.1.11

Show that a saddle-node bifurcation occurs for the system

$$\begin{cases} \dot{u} = a(1-u) - uv^2 \\ \dot{v} = uv^2 - (a+k)v \end{cases}$$

Solution:

The null-clines are given by:

$$\begin{cases} u = a/a+v^2 \\ u = \dfrac{a+k}{v} \end{cases}$$

Therefore a fixed point (u^*, v^*) satisfies

$$av^* = (a+k)(a+v^{*2}).$$

The discriminant of this quadratic equation is given by:

$$\begin{aligned} D &= a^2 - 4(a+k)(a^2+ak) \\ &= a^2 - 4a^3 - 4a^2k - 4a^2k - 4ak^2 \end{aligned}$$

Solving $D = 0$ we have

$$0 = k^2 - 2ak + a^2 - a/4.$$

$$\Rightarrow k = \frac{2a \pm \sqrt{4a^2 - 4(a^2 - a/4)}}{2}$$

$$= a \pm \sqrt{2}/2.$$

The saddle-node bifurcation occurs at $a = \pm\sqrt{2}/2$.

#8.2.1

Consider the biased van-der Pol oscillator:

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = a.$$

Find the curves in (μ, a) space at which Hopf bifurcations occur.

Solution:

The fixed point occurs at $\dot{x}=0, x=a$. The Jacobian is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & -\mu(a^2-1) \end{pmatrix}.$$

Now,

$$\mathrm{Tr}(J) = -\mu(a^2-1), \quad \det(J) = 1.$$

Therefore, a Hopf-bifurcation occurs when

$$a=1 \text{ or } \mu=0.$$

#8.2.8.

Analyze the system

$$\begin{cases} \dot{x} = x[x(1-x)-y] \\ \dot{y} = y(x-a) \end{cases}$$

Solution:

The null-clines are given by:

1. $x=0, \frac{dx}{dt}=0$
2. $y=x(1-x), \frac{dx}{dt}=0$
3. $y=0, \frac{dy}{dt}=0$
4. $x=a, \frac{dy}{dt}=0$

The fixed points are then

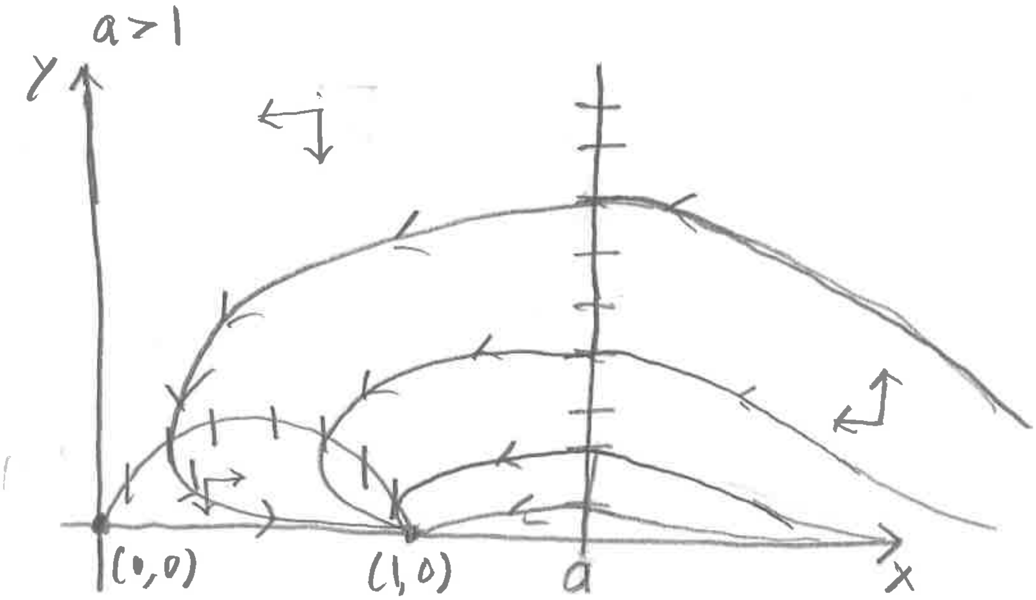
$$(0,0), (1,0), (a, a-a^2).$$

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 0 & -a \end{pmatrix}, \quad J(1,0) = \begin{pmatrix} 1 & -1 \\ 0 & 1-a \end{pmatrix}, \quad J(a,1-a) = \begin{pmatrix} -2a^2+a & -a \\ a-a^2 & 0 \end{pmatrix}$$

unclassified (Saddle) — Stable node/saddle. — Saddle/stable node/spirals.

Case 1:

$a > 1$



Now, we see that

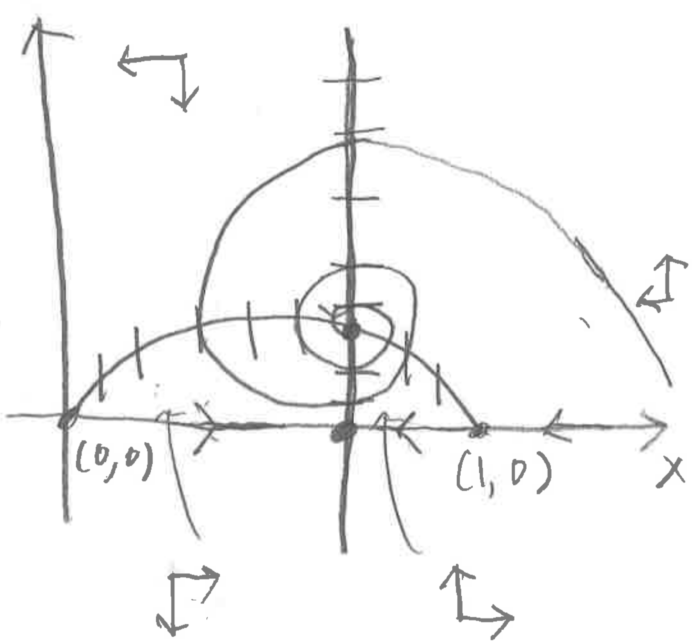
$$Tr(J(a, 1-a)) = -2a^2 + a$$

$$Det(J(a, 1-a)) = a^2(1-a)$$

Therefore, we have a Hopf-bifurcation at $a = 1/2$.

Case 2:

$1/2 < a < 1$



$Tr(-2a^2+a) > 0 \Rightarrow$ stable spiral.

Case 3: