Homework #7

#7.2.14

Sketch a phase portrait for the system

$$\begin{cases} \dot{x} = x^2 - y - 1 \\ \dot{y} = y(x-2) \end{cases}$$

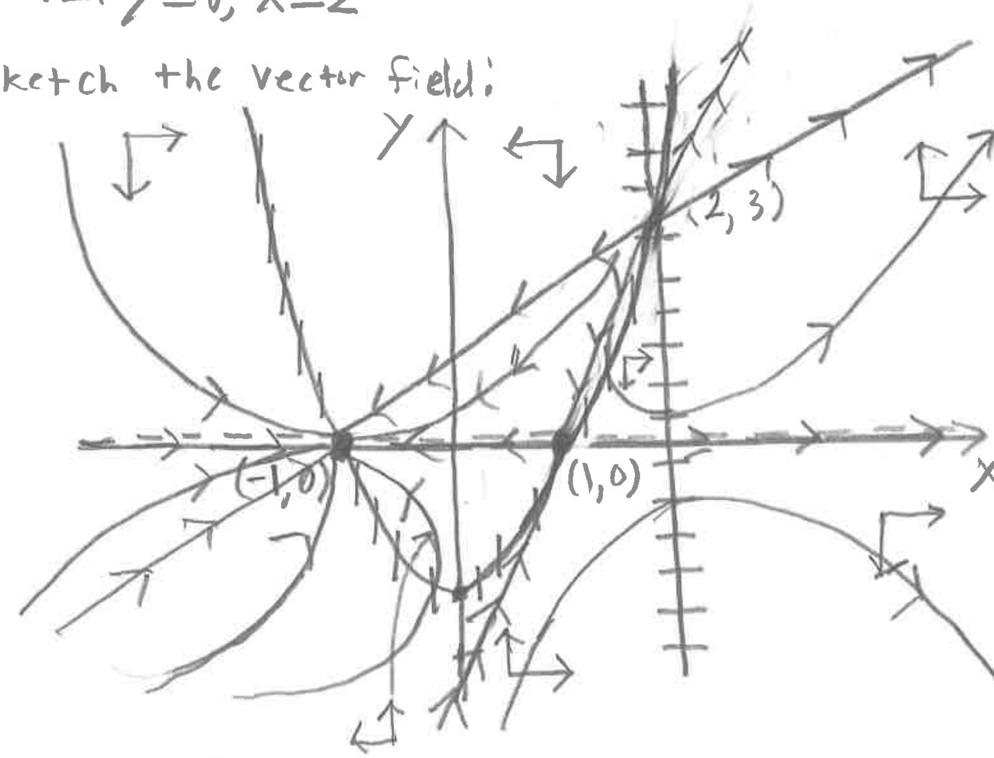
Solution:

The null-clines are given by:

N1: $y = x^2 - 1$

N2: $y = 0,\ x = 2$

Lets sketch the vector field:



Stability Analysis:

$$J(1,0) = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}, \quad J(-1,0) = \begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix}, \quad J(2,3) = \begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix}$$

$\Rightarrow (1,0)$ is a saddle $\quad \Rightarrow (-1,0)$ is a stable node $\quad \Rightarrow (2,3)$ is an unstable node.

Now, along the line $y = 3x - 3$ we have that

$$\frac{dy}{dx} = \frac{3(x-1)(x-2)}{x^2 - 3x + 3 - 1} = \frac{3(x-3)(x-2)}{x^2 - 3x + 2} = \frac{3(x-1)(x-2)}{(x-2)(x-1)} = 3.$$

This proves that this line is invariant under the flow. Also, the line $y = 0$ is invariant. The index of $(-1,0)$ is -1 and thus there can be no limit cycles in this system.

#7.3.4

Consider the system

$$\begin{cases} \dot{x} = x(1-4x^2-y^2) - \frac{1}{2}y(1+x) \\ \dot{y} = y(1-4x^2-y^2) + 2x(1+x). \end{cases}$$

Show that all trajectories approach the ellipse $4x^2+y^2=1$ as $t\to\infty$.

Solution:

Let $V = 4x^2+y^2$. Then

$$\dot{V} = 8x\dot{x} + 2y\dot{y} = 8x^2(1-V) - 4xy(1+x) + 2y^2(1-V) + 4xy(1+x)$$

$$\Rightarrow \dot{V} = 4V(1-V).$$

Consequently, $V=1$ is a stable fixed point and $V=0$ is unstable. Therefore,

$$\lim_{t\to\infty} V(x(t), y(t)) = 1,$$

which implies all solution trajectories approach the ellipse $4x^2+y^2=1$.

#7.3.5.

Show that the system

$$\begin{cases} \dot{x} = -x-y+x(x^2+2y^2) \\ \dot{y} = x-y+y(x^2+2y^2) \end{cases}$$

has at least one periodic solution.

Solution:

$$\dot{r} = \cos\theta(-r\cos\theta - r\sin\theta + r\cos\theta(r^2+r^2\sin^2\theta)$$
$$+ \sin\theta(+r\cos\theta - r\sin\theta + r\sin\theta(r^2+r^2\sin^2\theta)$$
$$= -r + r^3(1+\sin^2\theta)$$
$$= r(-1+r^2(1+\sin^2\theta)).$$

For $r>10$, $\dot{r}>0$ and for $r<1/\sqrt{2}$ $\dot{r}<0$ hense there is an unstable limit cycle surrounding the fixed point $(0,0)$.

#7.3.7

Analyze the limit cycles for the system

$$\begin{cases} \dot{x} = y + ax(1-2b-r^2) \\ \dot{y} = -x + ay(1-r^2) \end{cases}$$

Solution:

$$\dot{r} = \cos\theta(r\sin\theta + ar\cos\theta(1-2b-r^2)$$
$$+\sin\theta(-r\cos\theta + ar\sin\theta(1-r^2)$$
$$= ar(1-r^2) - 2abr\cos^2\theta$$
$$= ar[(1-r^2) - 2b\cos^2\theta]$$

For $r>1$ it follows that $\dot{r}<0$ and if $r^2<1-2b$ then $\dot{r}<0$. Consequently the system has at least one limit cycle. Now,

$$\dot{\theta} = \frac{-1}{r}\sin\theta\left(r\sin\theta + ar\cos\theta(1-2b-r^2)\right)$$
$$+\frac{1}{r}\cos\theta(-r\cos\theta + ar\sin\theta(1-r^2))$$

$$= -1 + 2ab\sin\theta\cos\theta$$
$$= -1 + ab\sin(2\theta)$$

Therefore,

$$\frac{d\theta}{dt} = -1 + ab\sin 2\theta$$

which implies that the period of the limit cycles are given by:

$$T = \left| \int_0^{2\pi} \frac{1}{-1+ab\sin 2\theta}\, d\theta \right|.$$

Now, if $b=0$ then

$$\begin{cases} \dot{r} = ar(1-r^2) \\ \dot{\theta} = -1 \end{cases}$$

which only has $r=1$ as a limit cycle.

#7.3.11

Plot the phase portrait for the system

$$\begin{cases} \dot{r} = r(1-r^2)(r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2) \\ \dot{\theta} = r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2 \end{cases}$$

Solution:

$\dot{r} = 0$ along the curve $r = 1$. Furthermore $\dot{r} = \dot{\theta} = 0$ when

$$r^2\sin^2\theta = -(r^2\cos^2\theta - 1)^2$$

which is only satisfied when $r = 1$ and $\theta = 0, \pi$. Consequently, the phase portrait is given by