Homework #6.

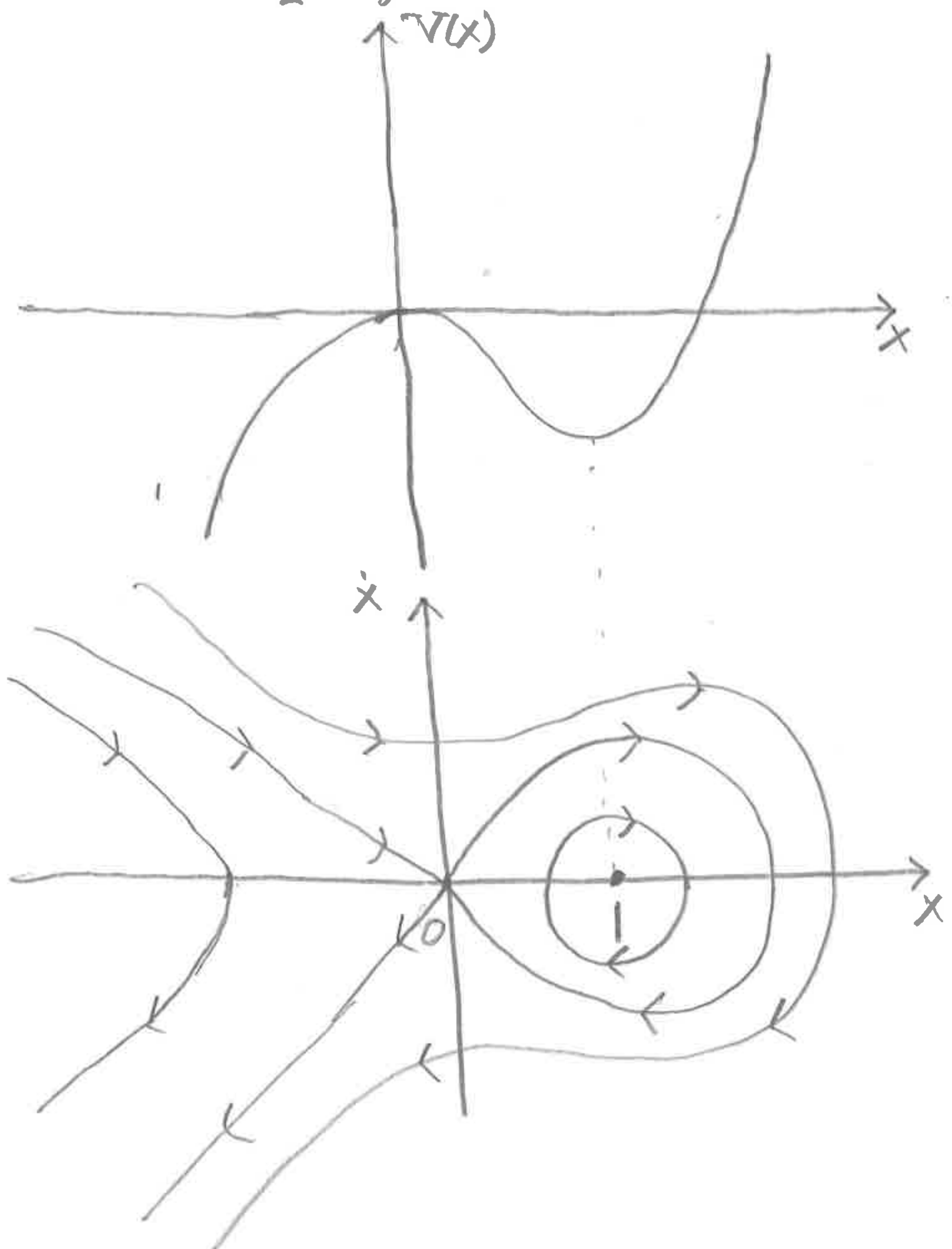
#6.5.2.

Consider the system $\ddot{x} = x - x^2$. Sketch the phase portrait and find equation for the homoclinic orbit.

Solution:

The energy for this system is

$$E = \frac{1}{2}\dot{x}^2 - \frac{x^2}{2} + \frac{x^3}{3} = \frac{1}{2}\dot{x}^2 + V(x).$$



The equation for the homoclinic orbit is given by:

$$\dot{x} = \pm\sqrt{x^2 - \frac{2x^3}{3}} = \pm x\sqrt{1 - \frac{2x}{3}} \quad (\text{for } 0 \le x \le 3/2)$$

#6.5.12

Consider the system $\dot{x}=xy$, $\dot{y}=-x^2$. Show that $(0,0)$ is not a nonlinear center despite the fact that $E=x^2+y^2$ is conserved.

Solution:

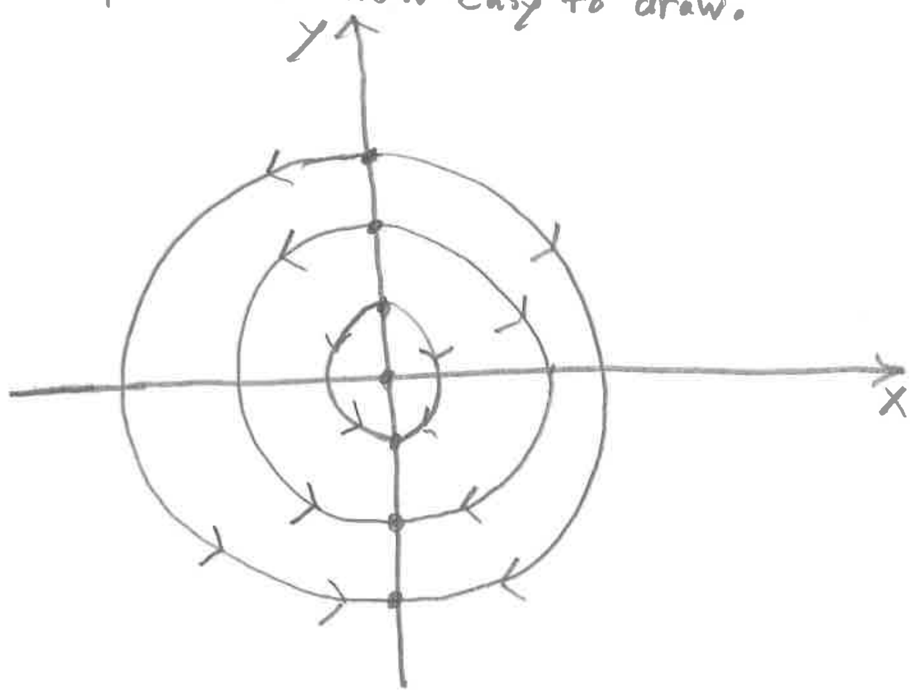
Lets convert the system to polar coordinates.

$$\dot{r} = \cos\theta\,\dot{x} + \sin\theta\,\dot{y}$$
$$\dot{\theta} = -\frac{1}{r}\sin\theta\,\dot{x} + \frac{1}{r}\cos\theta\,\dot{y}$$

$$\Rightarrow \dot{r} = r^2\cos^2\theta\sin\theta - r^2\cos^2\theta\sin\theta$$
$$\dot{\theta} = -\frac{1}{r}\cos\theta\sin^2\theta - \frac{1}{r}\cos^3\theta$$

$$\Rightarrow \begin{cases} \dot{r} = 0 \\ \dot{\theta} = -\frac{1}{r}\cos\theta \end{cases}$$

The phase portrait is now easy to draw:



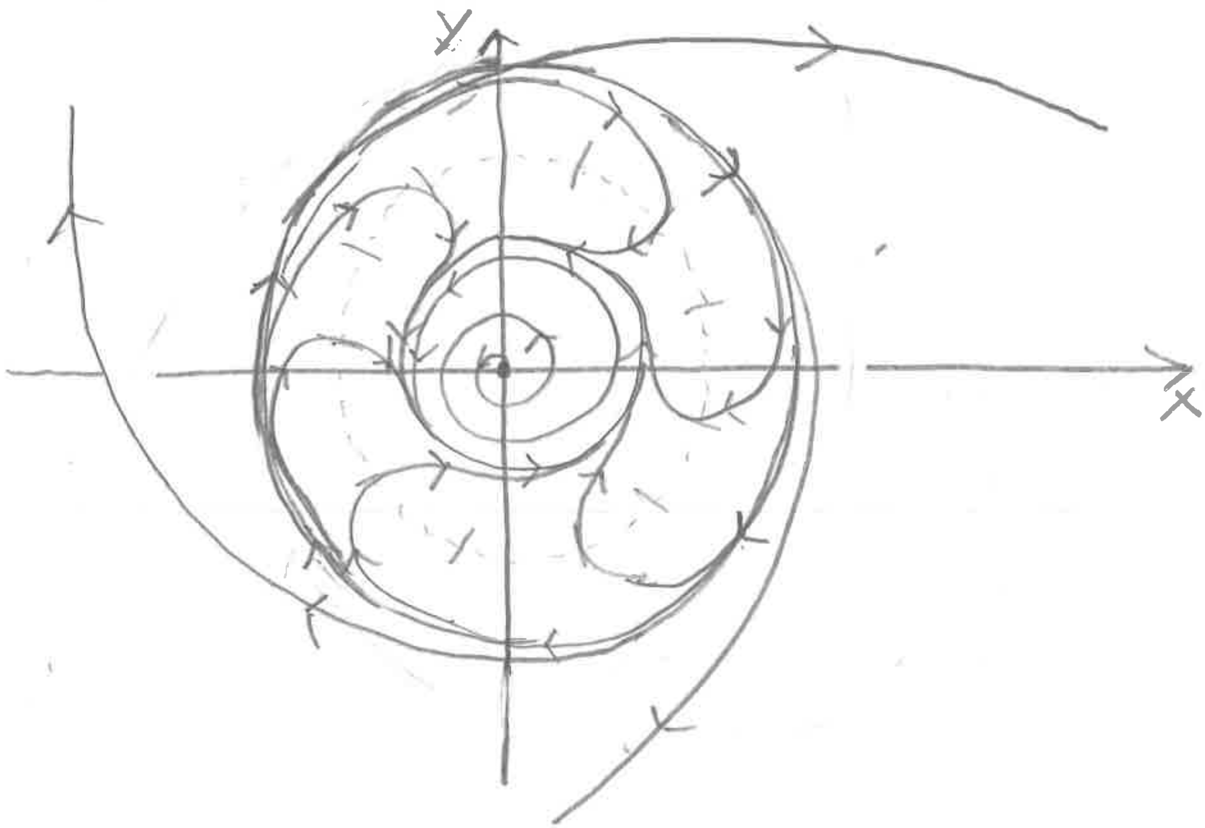
#6.8.9.

A smooth vector field is known to have two limit cycles one inside of the other. If one runs clockwise and the other counterclockwise then there must be one fixed point in between the cycles.

Solution:

False, consider the following system in polar coordinates:

$$\begin{cases} \dot{r} = (1-r)(3-r) \\ \dot{\theta} = 2-r \end{cases}$$

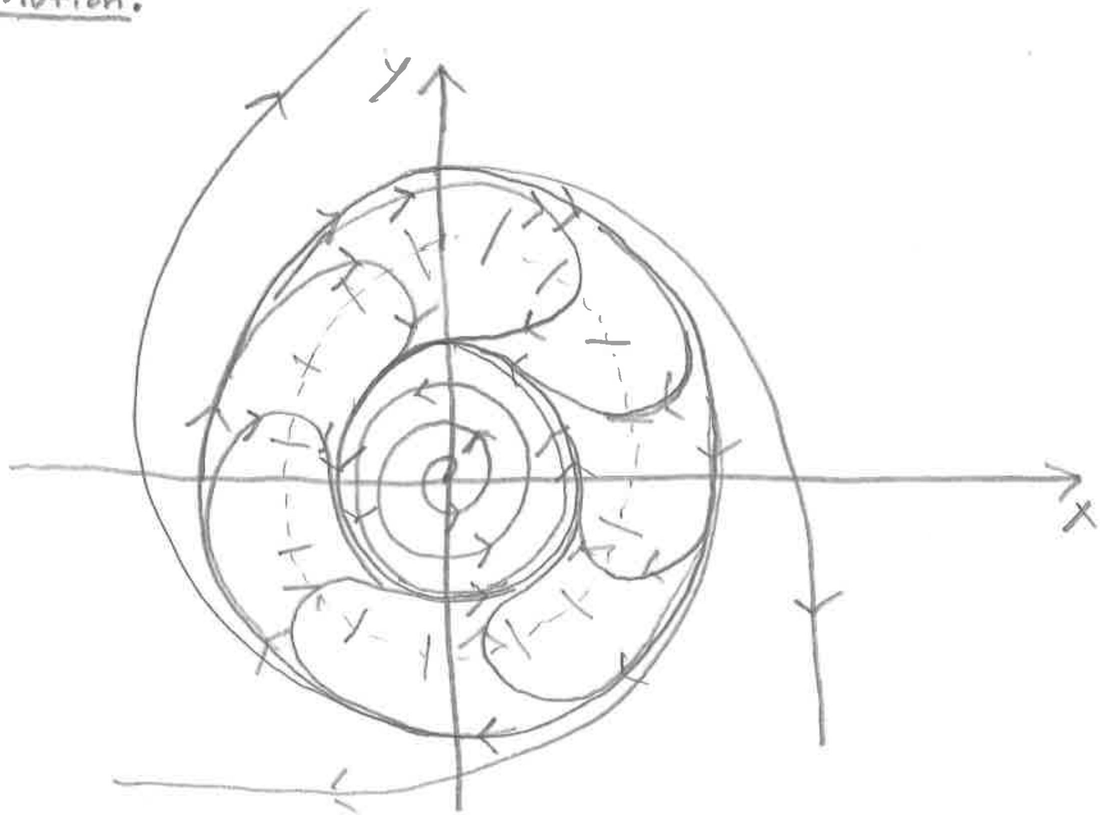


#7.1.3

Sketch the phase portrait for:

$$\begin{cases} \dot{r} = r(1-r^2)(4-r^2) \\ \dot{\theta} = 2-r^2 \end{cases}$$

Solution:



#7.2.10

Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits.

Solution:

Let $V = ax^2 + by^2$. Calculating we have that

$$\begin{aligned} \frac{dV}{dt} &= 2ax\dot{x} + 2by\dot{y} \\ &= 2ax(y-x^3) + 2by(-x-y^3) \\ &= 2axy - 2byxy - 2ax^4 - 2by^4. \end{aligned}$$

If $a=b=1$ it follows that V is a Lyapunov function for the system. Therefore, the system has no closed orbits.