Homework #6

#6.1.4

Sketch a phase portrait for

$$\begin{cases} \dot{x} = y \\ \dot{y} = x(1+y) - 1. \end{cases}$$

Solution:

The null-clines are

1. $y = 0$ $(\dot{x} = 0)$
2. $y = \frac{1-x}{x}$.

The single fixed point is given by

$$(x^*, y^*) = (1, 0).$$

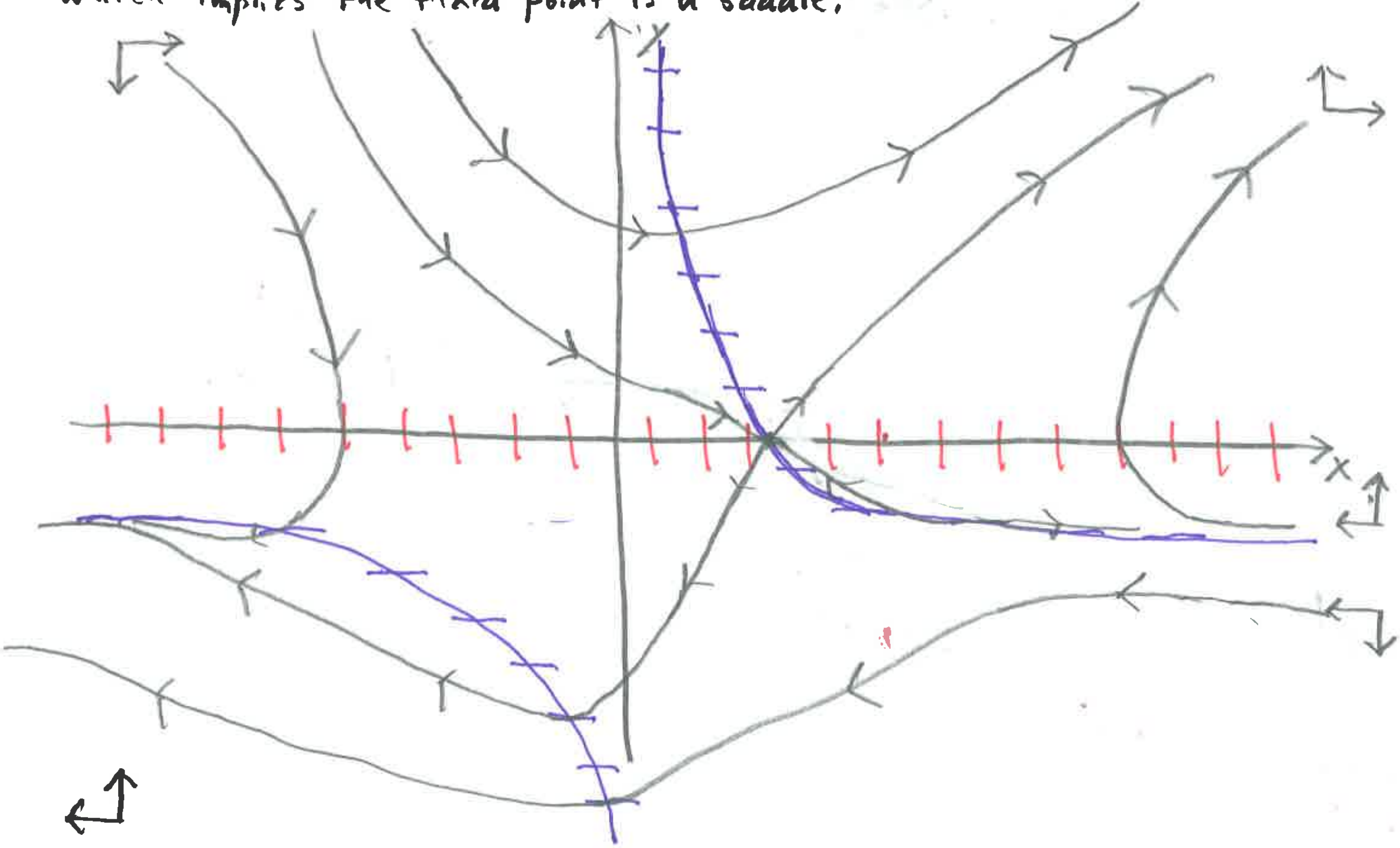
The Jacobian at $(1,0)$ is given by:

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues are given by

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Which implies the fixed point is a saddle.



#6.3.9

Consider the system
$$\begin{cases} \dot{x} = y^3 - 4x \\ \dot{y} = y^3 - y - 3x \end{cases},$$

Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ and sketch a phase portrait.

Solution:

Let $z = x - y$. It follows that
$$\dot{z} = -z. \quad *$$

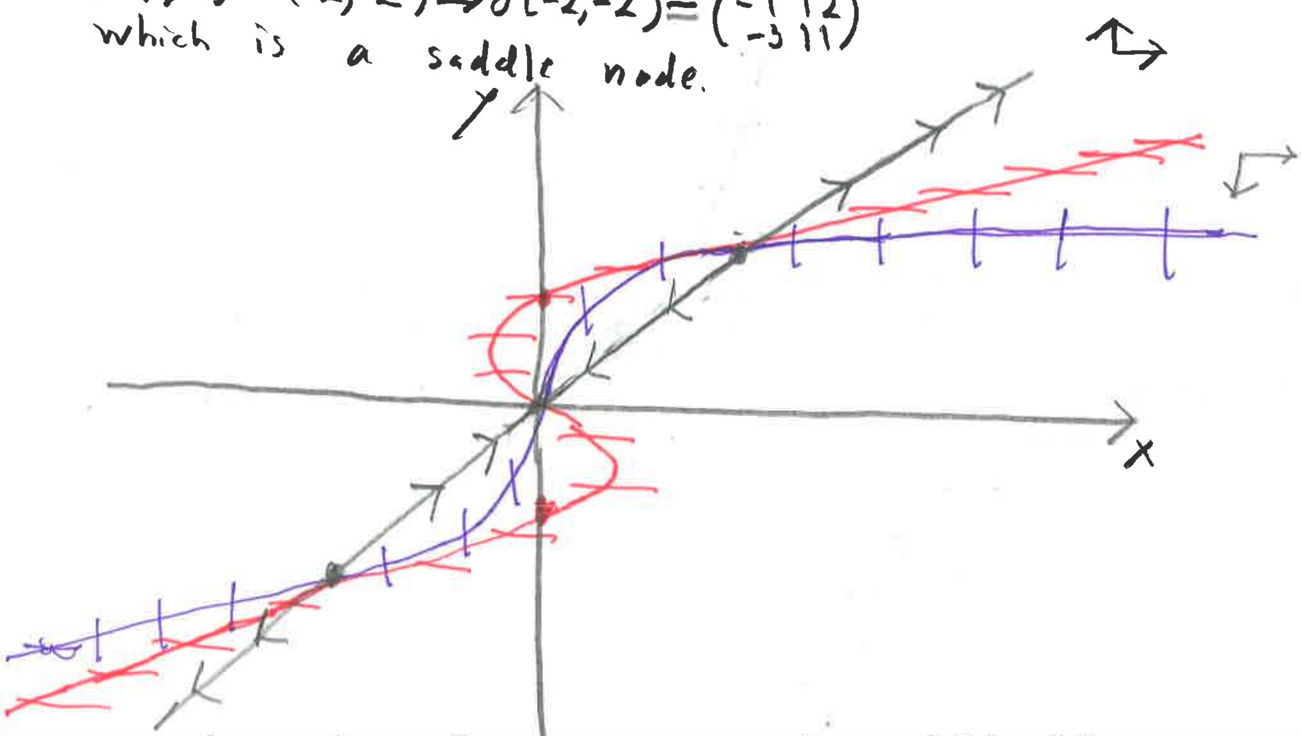
Consequently,
$$\lim_{t\to\infty} z(t) = 0 \Rightarrow |x(t) - y(t)| \to 0 \text{ as } t \to \infty.$$

Moreover $z=0$ is a solution to $*$ which proves that the line $y=x$ is invariant to the flow. The null-clines for this system are given by:

1. $x = y^3/4 \; \left(\frac{dx}{dt} = 0\right)$
2. $x = \frac{1}{3}y(y^2-1)$.

The fixed points are

1. $(x^*, y^*) = (0,0) \Rightarrow J(0,0) = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix}$
   which is a stable node.
2. $(x^*, y^*) = (2,2) \Rightarrow J(2,2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix}$
   which is a saddle node.
3. $(x^*, y^*) = (-2,-2) \Rightarrow J(-2,-2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix}$
   which is a saddle node.



#6.3.10

Sketch a phase portrait for

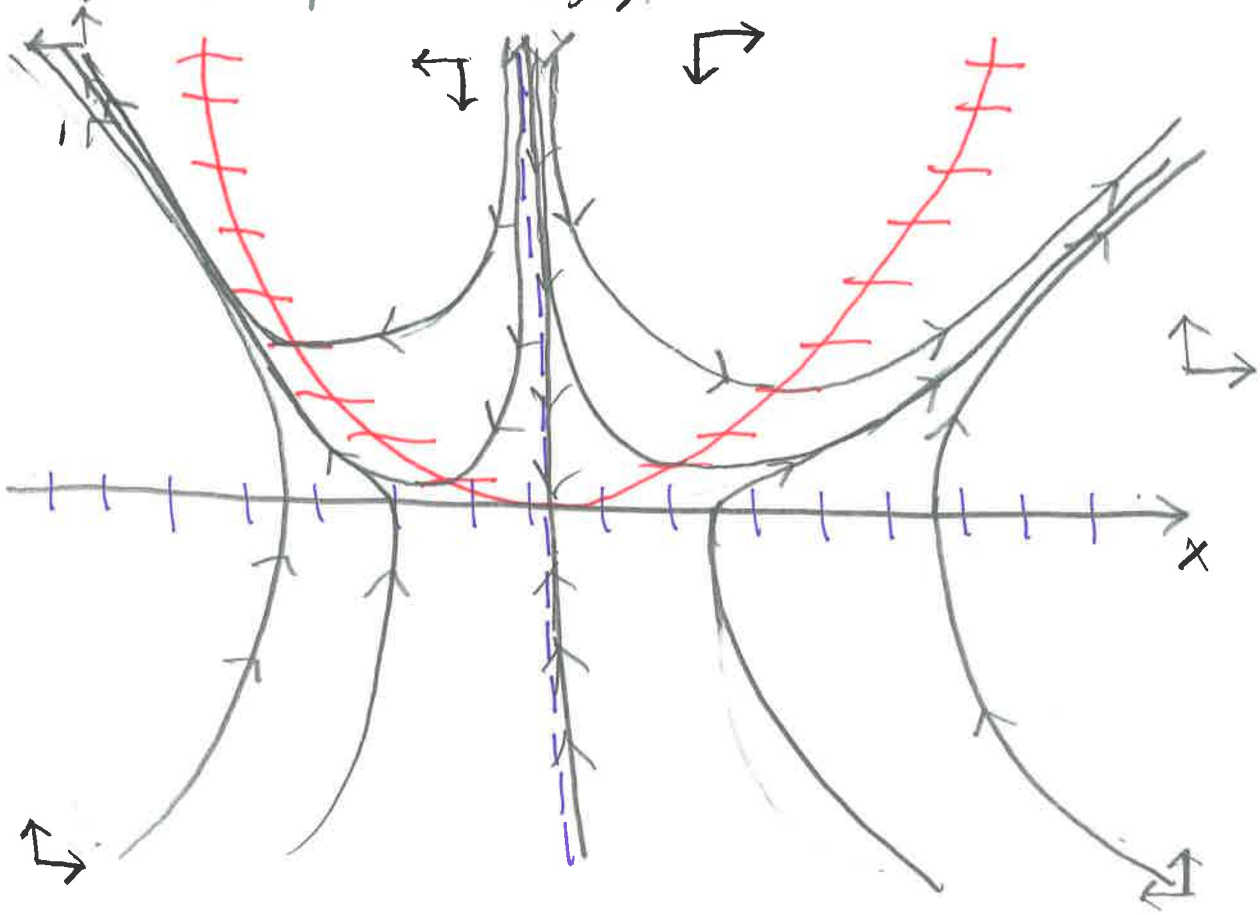
$$\begin{cases} \dot{x} = xy \\ \dot{y} = x^2 - y \end{cases}$$

Solution:

Null-clines

1. $y=0$ $\left(\frac{dx}{dt}=0\right)$
2. $x=0$ $\left(\frac{dx}{dt}=0\right)$
3. $y=x^2$ $\left(\frac{dy}{dt}=0\right)$

The only fixed point is $(0,0)$.



The fixed point behaves like a saddle.

#6.3.11

Consider the system in polar coordinates given by:

$$\begin{cases} \dot{r} = -r \\ \dot{\theta} = 1/\ln(r) \end{cases}$$

Show that $r \to 0$ as $t \to \infty$ and convert the system to Cartesian coordinates.

Solution:

$r(t) = r_0 e^{-t}$.

Consequently,

$$\dot{\theta} = \frac{1}{\ln(r_0) - t}$$

$$\Rightarrow \theta(t) = \ln\left(\frac{\ln(r_0)}{\ln(r_0) - t}\right) + \theta_0$$

Converting to Cartesian coordinates:

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta}$$
$$\dot{y} = \dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta}$$

$$\Rightarrow \dot{x} = -r\cos\theta - \frac{r\sin\theta}{\ln(r)}$$
$$\dot{y} = r\sin\theta + \frac{r\cos\theta}{\ln(r)}$$

$$\Rightarrow \begin{cases} \dot{x} = -x - \dfrac{2y}{\ln(x^2+y^2)} \\ \dot{y} = y + \dfrac{2x}{\ln(x^2+y^2)} \end{cases}$$

6.3.13

Consider the system

$$\begin{cases} \dot{x} = -y - x^3 \\ \dot{y} = x. \end{cases}$$

Show that the origin is a spiral.

Solution:

Converting to polar coordinates we have that

$$\begin{aligned} \dot{r} &= r\cos\theta\,\dot{x} + r\sin\theta\,\dot{y} \\ &= r\cos\theta(-r\sin\theta - r^3\cos^3\theta) + r\sin\theta \cdot r\cos\theta \\ &= -r^4\cos^4\theta. \end{aligned}$$

$$\begin{aligned} \dot{\theta} &= -\frac{1}{r}\sin\theta\,\dot{x} + \frac{1}{r}\cos\theta\,\dot{y} \\ &= +\sin^2\theta + r^2\cos^3\theta\sin\theta + .\cos^2\theta \\ &= 1 + r^2\cos^3\theta\sin\theta \end{aligned}$$

Now,

$$\frac{dr}{dt} = 0 \Rightarrow \theta = \pi/2, 3\pi/2.$$

However, for these values $\frac{d\theta}{dt} \neq 0$. Consequently,

$$\lim_{t\to\infty} r(t) = 0,$$

Moreover for $r<1$, $\frac{d\theta}{dt} \geq 0$ which indicates spiraling behavior.