Homework #3

#3.4.12

Construct an example in which the system $\dot{x} = f(x, r)$ has no fixed points for $r < 0$ and four branches for $r > 0$.

Solution:

$$\dot{x} = r + (x-1)^2(x+1)^2 \text{ works.}$$

#3.5.8

Show that the equation $\dot{u} = au + bu^3 - cu^5$ can be rewritten as

$$\frac{dx}{d\tau} = rx + x^3 - x^5.$$

Solution:

Let $\tau = \alpha t$ and $x = \beta u$ where α, β are to be selected later. Computing we have that

$$\frac{du}{dt} = \frac{d\tau}{dt}\frac{du}{d\tau} = \frac{\alpha}{\beta}\frac{dx}{d\tau}.$$

Therefore,

$$\frac{\alpha}{\beta}\frac{dx}{d\tau} = \frac{a}{\beta}x + \frac{b}{\beta^{1/3}}x^3 - \frac{c}{\beta^{1/5}}x^5$$

$$\Rightarrow \frac{dx}{d\tau} = \frac{a}{\alpha}x + \frac{b\beta^{2/3}}{\alpha}x^3 - \frac{c\beta^{4/5}}{\alpha}x^5$$

Consequently, if we set

$$\frac{b\beta^{2/3}}{\alpha} = 1 \quad \text{and} \quad \frac{c\beta^{4/5}}{\alpha} = 1$$

$$\Rightarrow c\beta^{4/5} = b\beta^{2/3} \quad \text{and} \quad \alpha = c\beta^{4/5}$$

$$\Rightarrow \beta^{2/15} = b/c \quad \text{and} \quad \alpha = c\beta^{4/5}$$

$$\Rightarrow \beta = (b/c)^{15/2} \quad \text{and} \quad \alpha = \frac{b^6}{c^5}$$

then

$$\frac{dx}{d\tau} = rx + x^3 - x^5$$

with

$$r = \frac{ac^5}{b^6}$$

#3.7.5.

Consider the system

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}.$$

a.) Show the system can be put in the dimensionless form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}.$$

Solution:

Let $\tau = \alpha t$ and $g = \beta X$. Then,

$$\frac{dg}{dt} = \alpha\beta \frac{dX}{d\tau} = k_1 s_0 - k_2 \beta X + \frac{k_3 \beta^2 X^2}{k_4^2 + \beta^2 X^2}$$

$$\Rightarrow \frac{dX}{d\tau} = \frac{k_1 s_0}{\alpha\beta} - \frac{k_2}{\alpha} X + \frac{k_3 \beta X^2}{\alpha(k_4^2 + \beta^2 X^2)}$$

$$= \frac{k_1 s_0}{\alpha\beta} - \frac{k_2}{\alpha} X + \frac{k_3}{\alpha\beta} \cdot \frac{X^2}{(k_4^2/\beta^2 + X^2)}.$$

Set

$$\frac{k_3}{\alpha\beta} = 1 \quad \text{and} \quad \frac{k_4^2}{\beta^2} = 1$$

$$\Rightarrow \beta = k_4 \quad \text{and} \quad \alpha = \frac{k_3}{k_4}.$$

In this units

$$\frac{dX}{d\tau} = s - rX + \frac{X^2}{1+X^2}$$

where

$$s = \frac{k_1 s_0}{k_3} \quad \text{and} \quad r = \frac{k_2}{k_3}.$$

b.) Show that if $s=0$ there are two fixed points if $r<r_c$.

Solution:

If $s=0$ the system is given by

$$\frac{dx}{d\tau}=x\left(-r+\frac{x}{1+x^2}\right)=f(x)$$

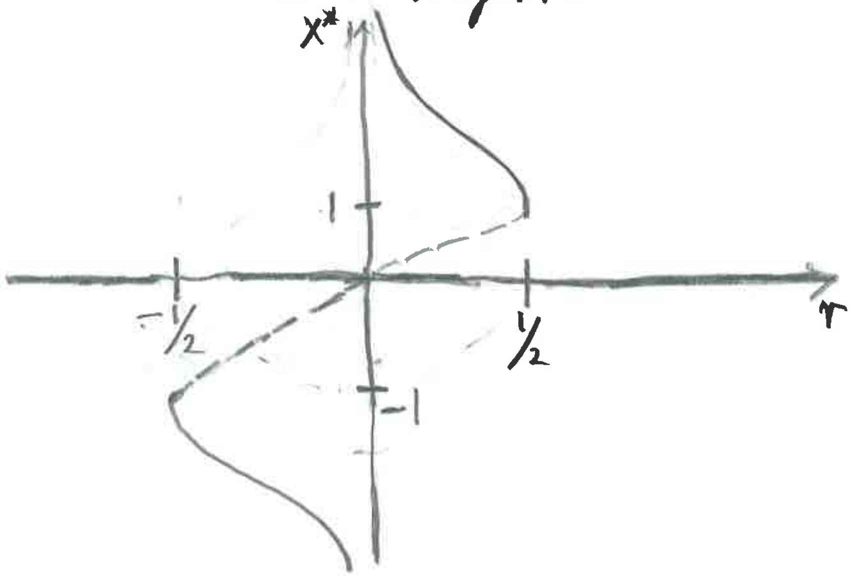
Fixed points are then given by

$$x=0 \quad \text{and} \quad x=\frac{\frac{1}{r}\pm\sqrt{\frac{1}{r^2}-4}}{2}$$

The condition that the other fixed point exists is then $-\frac{1}{2}<r<\frac{1}{2}$
The stability of the fixed point at the origin can be determined from:

$$f'(0)=-r,$$

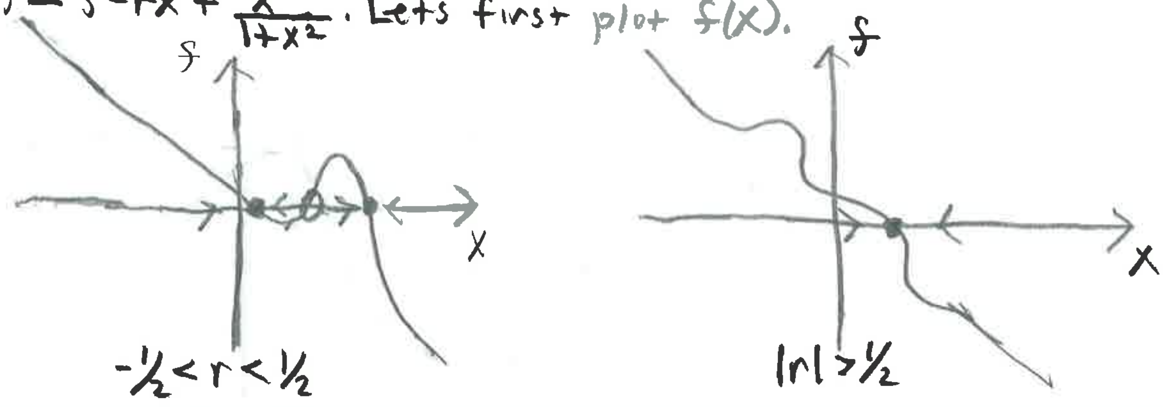
i.e. 0 is always stable which implies the other fixed points are unstable.
Below is a biforcation diagaram.



C.) What happens if $g(0)=0$ and $s=\varepsilon<<1$ and then s suddenly becomes zero. Does the gene turn off?

Solution:

Let $f(x)=s-rx+\frac{x^2}{1+x^2}$. Lets first plot $f(x)$.



Consequently, if $-\frac{1}{2}<r<\frac{1}{2}$ the gene may not return. In particular if $S>\left(\frac{1}{r}-\sqrt{\frac{1}{r^2}-4}\right)/2$ then we expect the system to shift equilibrium to the fixed point $X=\left(\frac{1}{r}+\sqrt{\frac{1}{r^2}-4}\right)/2$.

#4.3.6

On S^1 draw the phase portrait for $\dot{\theta}=\nu+\sin(\theta)+\cos(2\theta)$ and classify bifurcations that occur as ν is varied.

Solution:

$$\dot{\theta}=\nu+\sin(\theta)+\cos(2\theta)=\nu+\sin(\theta)+1-2\sin^2(\theta)$$

Therefore, for $|\nu|<2$ fixed points satisfy:

$$\sin(\theta)=\frac{-1\pm\sqrt{1+8(\nu+1)}}{4}$$

$$=\frac{1\pm\sqrt{9+8\nu}}{4}$$

The further condition for the existence of the roots is then

$$\nu>-8/9$$

Consequently, for $-8/9<\nu<2$, we have four roots. However, we now need to check when these roots are in the interval $(-1,1)$.

$$-1<\frac{1\pm\sqrt{9+8\nu}}{4}<1$$

$$\Rightarrow -5<\pm\sqrt{9+8\nu}<3$$

$$\Rightarrow 9+8\nu<25 \quad \text{or} \quad 9+8\nu<9$$

$$\Rightarrow \nu<2 \quad \text{or} \quad \nu<0$$

In summary:

1. If $\nu<-8/9$ or $\nu>2$ then there are no fixed points.
2. If $-8/9<\nu<0$ there are four fixed points.
3. If $0<\nu<2$ there are two fixed points.

The phase portraits are as follows:

$\nu < -8/9$

$-8/9 < \nu < 0$

$0 < \nu < 2$

$2 < \nu$

#4.4.1

Find the condition on the coefficients for which it is valid to approximate $mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$ by the equation $b\dot{\theta} + mgL\sin\theta = \Gamma$.

Solution:

Let $\tau = \alpha t$. Then

$$mL^2\alpha^2 \frac{d^2\theta}{d\tau^2} + b\alpha\frac{d\theta}{d\tau} + mgL\sin\theta = \Gamma$$

$$\Rightarrow \frac{L^2\alpha^2}{gL}\frac{d^2\theta}{d\tau^2} + \frac{b\alpha}{mgL}\frac{d\theta}{d\tau} + \sin\theta = \frac{\Gamma}{mgL}$$

Set $\alpha = \frac{mgL}{b}$. Then the condition is that:

$$\frac{L^2}{gL}\left(\frac{mgL}{b}\right)^2 = \frac{L^3 \cdot g \cdot m^2}{b^2} \ll 1.$$