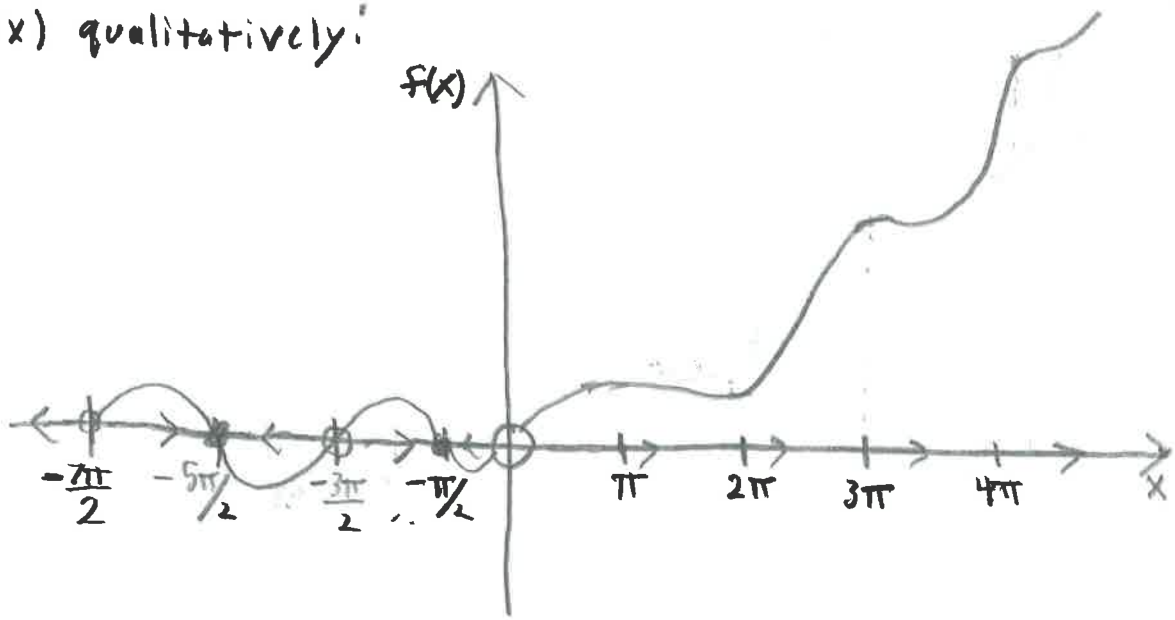
Homework #1, 1/30/15

#2.2.7

Analyze the equation $\dot{x}=e^{x}-\cos(x)$ graphically. Sketch the vector field on the real line, find all fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions.

Solution:

For positive values of x we have that $e^{x}>1$ and hence for $x>1$ $e^{x}-\cos(x)>0$, i.e. there are no fixed points in this regime. Consequently for $x>0$, $e^{x}-\cos(x)$ will "look lik" an oscillating exponential. If insted $x<0$ then $e^{x}\approx 0$ for $|x|>>1$ and $e^{x}-\cos(x)\approx-\cos(x)$. We can use this information to sketch $e^{x}-\cos(x)$ qualitatively:



The vector field can be drawn in as above and we can approximate the location of the fixed points and analyze their stability. We get an alternating sequence of stable fixed points approximately given by:

$$x\approx-\pi/2,-5\pi/2,-9\pi/2,\ldots$$

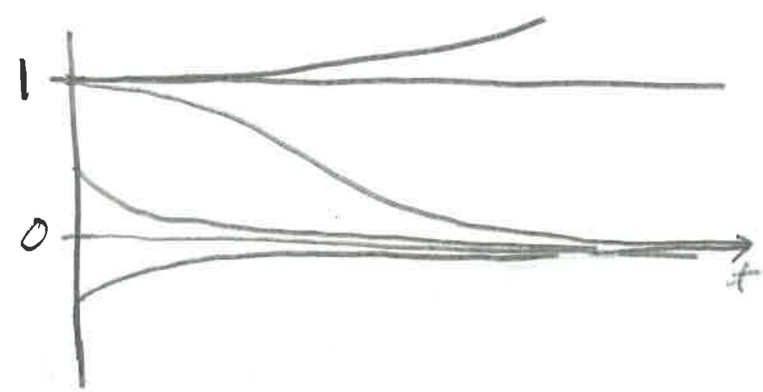
and unstable fixed points at:

$$x\approx-3\pi/2,-7\pi/2,-11\pi/2,\ldots.$$

On the following page I plot several representative solution curves.

#2.2.9

Find an equation $\dot{x}=f(x)$ whose solutions are consistent with the figure below.



Solution:

There is a stable fixed point at $x=0$ and unstable fixed point at $x=1$. From the figure it is clear that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$. The polynomial $f(x)=x(x-1)$ satisfies this criteria.

#2.2.10

Find an equation $\dot{x}=f(x)$ with the stated properties.

a.) Every real number is a fixed point.

b.) Every integer is a fixed point and no others.

c.) There are precisely 3 fixed points all of which are stable.

d.) There are no fixed points.

e.) There are precisely 100 fixed points.

Solution:

a.) $f(x)=0$.

b.) $f(x)=\sin(\pi x)$

c.) This is not possible since the function would necessarily satisfy $f'(x)<0$ at all fixed points.

d.) $f(x)=1$.

e.) $f(x)=\prod_{i=1}^{100}(x-i)$.

#2.3.5

Suppose X and Y are two species that reproduce exponentially fast: $\dot{X}=aX$ and $\dot{Y}=bY$ with $a>b>0$. Letting $x(t)=X(t)(X(t)+Y(t))^{-1}$ show that x increases monotonically and satisfies $\lim_{t\to\infty}x(t)=1$.

Solution:

Differentiating we have that

$$\dot{x}=\frac{(X+Y)\dot{X}-X(\dot{X}+\dot{Y})}{(X+Y)^2}$$

$$=\frac{\dot{X}Y-X\dot{Y}}{(X+Y)^2}$$

$$=\frac{(a-b)XY}{(X+Y)^2}$$

$$=(a-b)x\frac{Y}{X+Y}$$

$$=(a-b)x(1-x).$$

Consequently since $x=1$ is the only stable fixed point and $0<x_0<1$ it follows that x monotonically increases to 1.

#2.3.6

Consider the following simplified model of the proportion of a population speaking a language:

$$\dot{x} = s(1-x)x^a - (1-s)x(1-x)^a$$

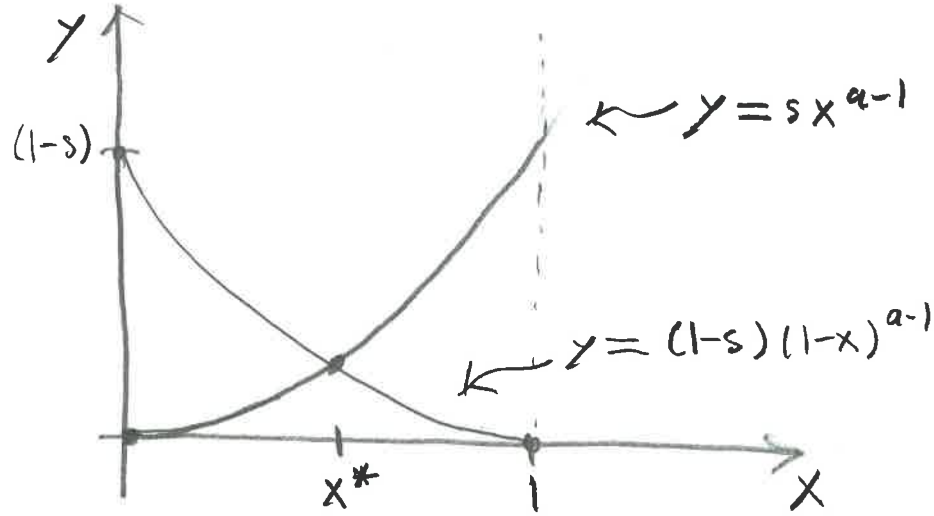
where $0 \le s \le 1$ is the attractiveness of a language and $a \ge 1$. Analyze the stability of the fixed points.

Solution:

Lets first determine the fixed points:

$$0 = s(1-x)x^a - (1-s)x(1-x)^a$$
$$= x(1-x)\left(s x^{a-1} - (1-s)(1-x)^{a-1}\right)$$

Clearly $x=0$ and $x=1$ are fixed points. Moreover, graphing $s x^{a-1}$ and $(1-s)(1-x)^{a-1}$ it is clear there is a third fixed point x^* satisfying $0<x^*<1$; see the figure below.



Let $f(x) = s(1-x)x^a - (1-s)x(1-x)^a$. Differentiating we have that

$$f'(x) = -s x^a + s(1-x)a x^{a-1} - (1-s)(1-x)^a + a(1-s)x(1-x)^{a-1}.$$

Therefore,

$$f'(0) = -(1-s) \text{ and } f'(1) = -s.$$

Therefore, 0 and 1 are both stable and consequently x^* must be unstable.