Chapter 8: Bifurcations Part Deux'

$$\dot{x} = f(x, y, \mu)$$
$$\dot{y} = g(x, y, \mu)$$

- A bifurcation point μ_* is a point where the topology of the phase portrait changes.

1. Let (x^*, y^*) denote an equilibrium point
2. Let λ_1, λ_2 denote the eigenvalues associtiated with λ_1, λ_2.

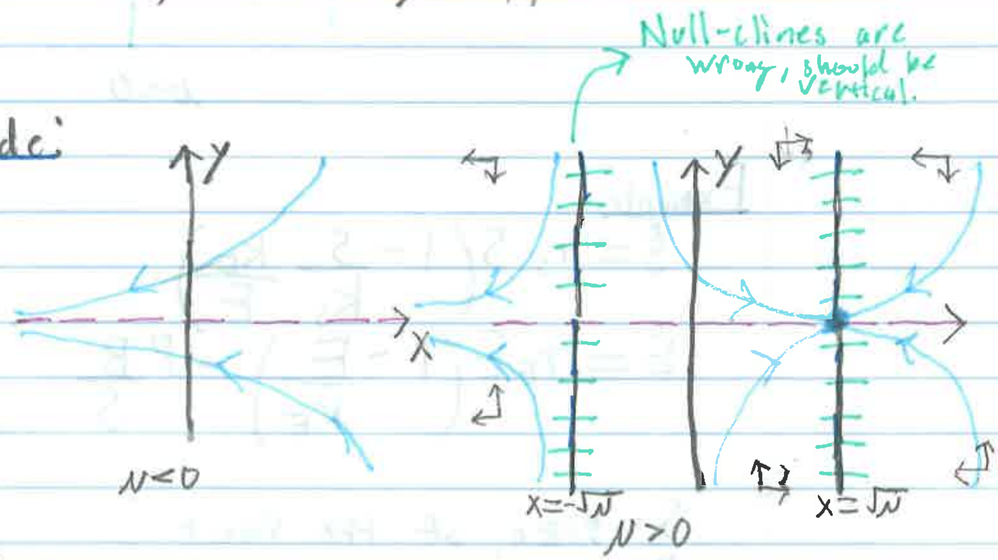
Bifurcations occur if one or both of the eigenvalues λ_1, λ_2 lie on the imaginary axis.

1. $\lambda_{1,2} = \pm i\omega \rightarrow$ Hopf bifurcation (new stuff)
2. $\lambda_1 = 0,\ \lambda_2 \neq 0 \rightarrow$ 1-D bifurcation
3. $\lambda_{1,2} = 0$, $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; never really happens.

1-D - Bifurcations

1. Saddle Node:

$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$



More Generally: One null-cline slips through another

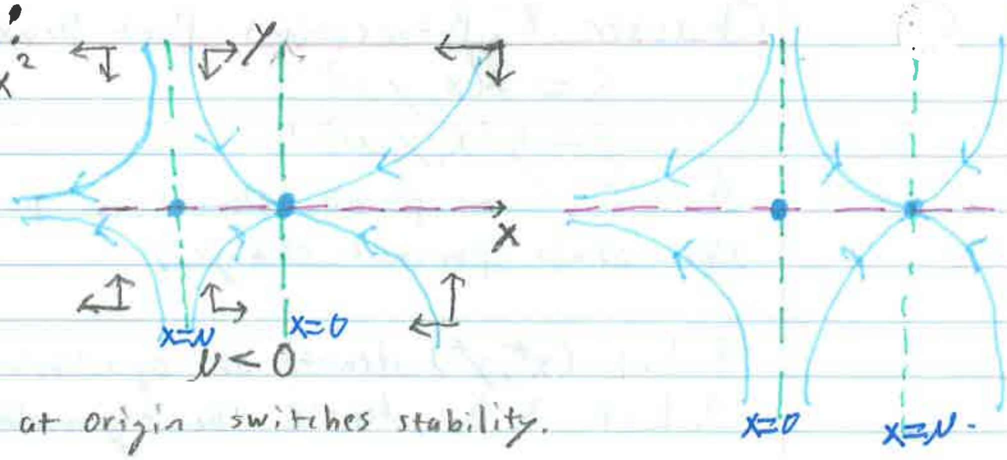
$\mu < \mu_*$

$\mu > \mu_*$

(blue null-cline intersects pink one)

2. Transcritical:

$$\begin{cases}\dot{x} = \mu x - x^2 \\ \dot{y} = -y\end{cases}$$

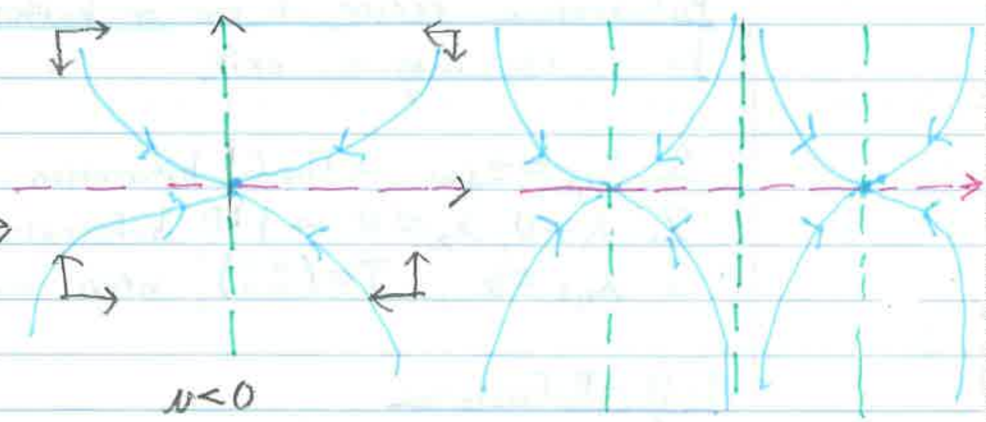


*Fixed point at origin switches stability.

3. Pitchfork:

$$\dot{x} = \mu x \pm x^3$$
$$\dot{y} = -y$$

Take − sign →



Example

$$\dot{S} = r_S S\left(1 - \frac{S}{K_S}\frac{K_E}{E}\right)$$

$$\dot{E} = r_E E\left(1 - \frac{E}{K_E}\right) - P\frac{B}{S}$$

$S \sim$ size of the forest
$B \sim$ worm population
$K_S \sim$ spruce tree carryoning capacity, when $E = K_E$
$K_E \sim$ energy resevere carraying capacity.
$B \sim$ bud worm population
$P \sim$ Rate energy reserve is eaten by worms.
$r_S \sim$ Growth rate of forest
$r_E \sim$ Growth rate of energy reserves.

Rescale

$x = S/K_S$

$y = E/K_E$

$\tau = r_S t$

$$\Rightarrow \frac{dx}{d\tau} = x\left(1 - \frac{x}{y}\right)$$

$$\frac{dy}{d\tau} = \alpha y(1-y) - \frac{\beta}{x}$$

$\alpha = r_E / r_S$

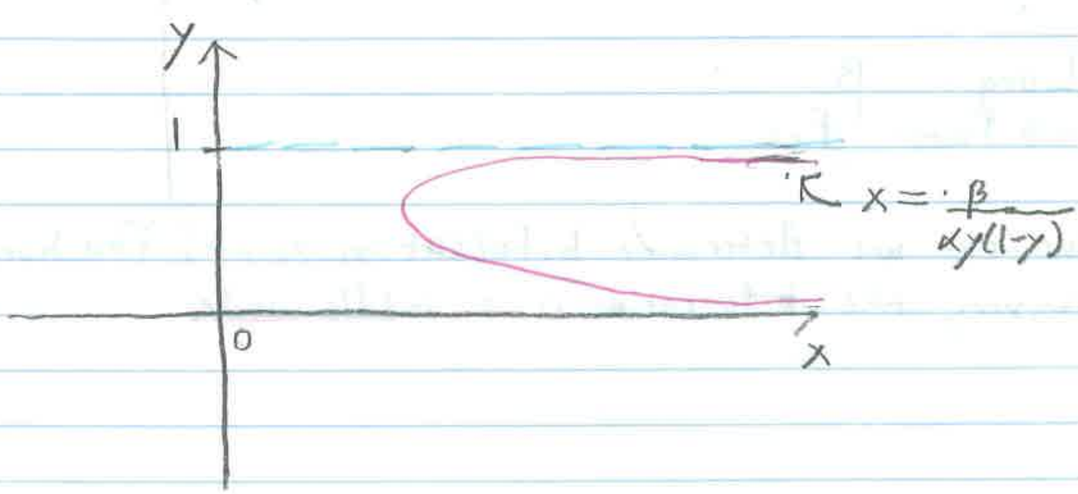
$\beta = \dfrac{PB}{K_S r_S}$

Null-clines

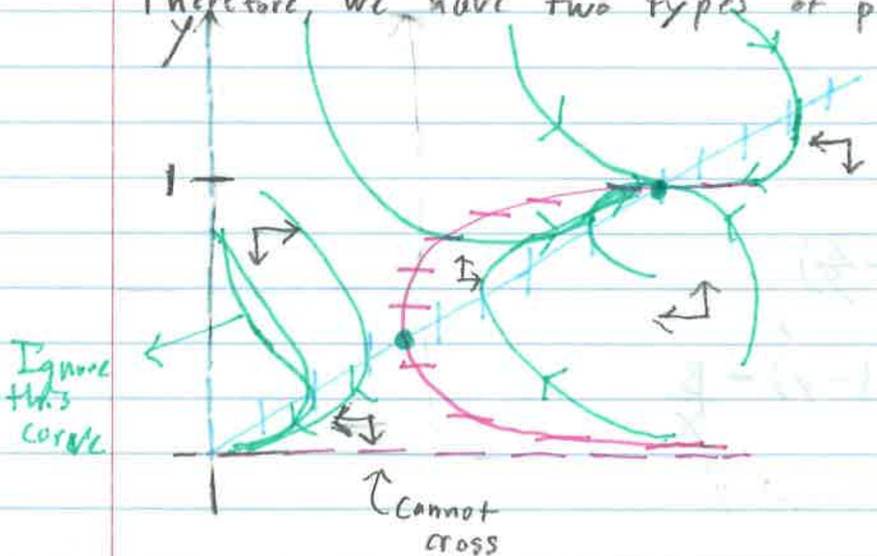
$\frac{dx}{d\tau} = 0, \quad x = 0, \quad y = x$

$\frac{dy}{d\tau} = 0, \quad x = \dfrac{\beta}{\alpha y(1-y)}$

Lets sketch the null-clines for this system. First lets figure out what the non-trivial null-cline looks like.



Therefore, we have two types of phase portraits:



Small β.
→ Forest can live with tree beds

Large β.
→ Forest dies.

How can we determine bifurcation point? Too hard. . .
However, the bifurcation is a saddle-node

Hopf-Bifurcation

Let $\omega > 0$

$$\dot{x} = \mu x - \omega y \pm (x^3 + x y^2) + b(-x^2 y - y^3)$$
$$\dot{y} = \underbrace{\omega x + \mu y}_{\text{linear rotations.}} \pm \underbrace{(x^2 y + y^3) + b(x^3 + x y^2)}_{\text{Cubic nonlinearity}}$$

$J(0,0) = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$, eigenvalues $\lambda_{1,2} = \mu \pm i\omega$

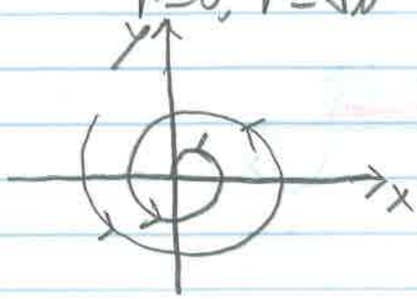
Convert to polar coordinates:

$$\begin{cases} \dot{r} = \mu r \pm r^3 \\ \dot{\theta} = \omega + b r^2 \end{cases}$$

1. Supercritical Hopf-bifurcation for "$-$" sign

Fixed points:

$r = 0$, $r = \sqrt{\mu}$



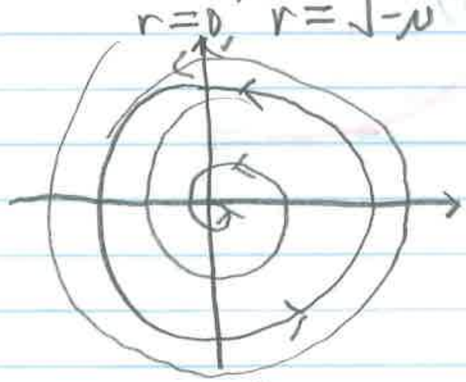
$\mu < 0$

$\mu > 0$

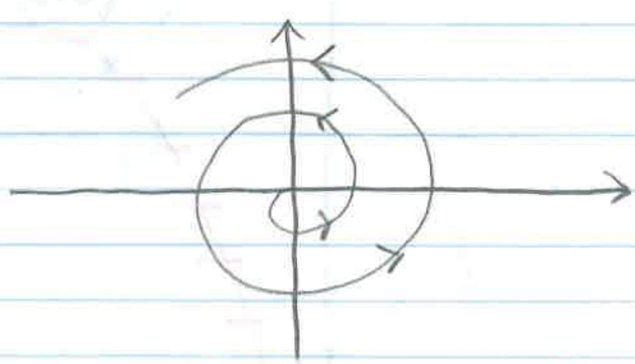
2. Subcritical Hopf-bifurcation for "$+$" sign

Fixed points:

$r = 0$, $r = \sqrt{-\mu}$



$\mu < 0$



$\mu > 0$

Example

"Chemical reaction"

$$\dot{x} = a - x + x^2 y$$
$$\dot{y} = b - x^2 y$$

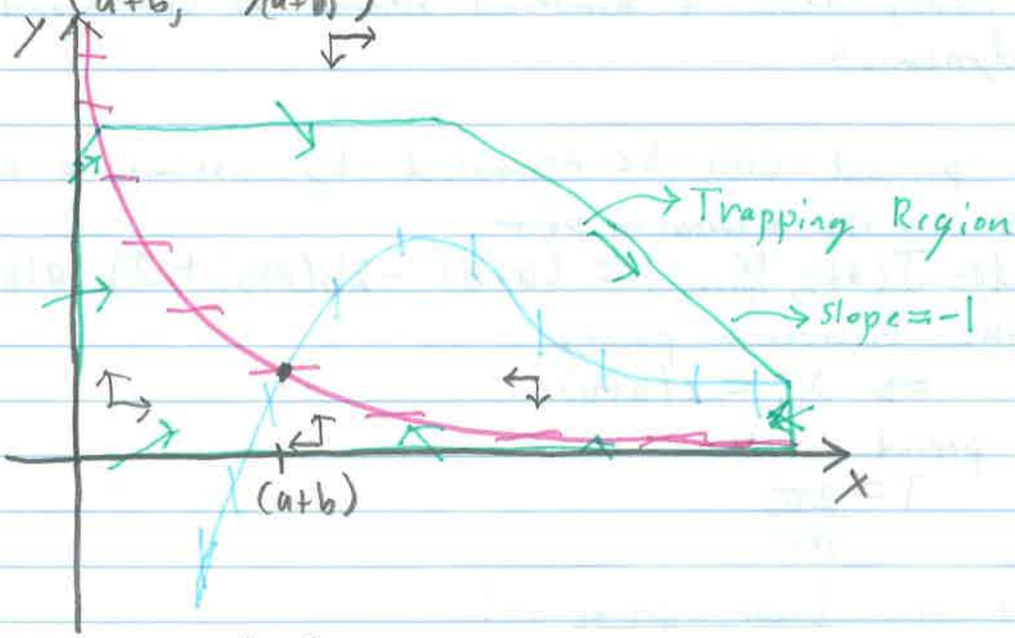
Null-clines

$$\frac{dx}{dt} = 0: \; y = \frac{x-a}{x^2}$$

$$\frac{dy}{dt} = 0: \; y = \frac{b}{x^2}$$

Fixed Point:

$$\left(a+b, \; b/(a+b)^2\right)$$



$$\frac{dy}{dx} \sim -1 \text{ for large } x$$

Solve $\frac{dy}{dx} < -1 \Rightarrow \frac{b - x^2 y}{a - x + x^2 y} < -1 \Rightarrow x > b + a$

$$\frac{dy}{dx} \sim \frac{b}{a} \text{ near } x = a$$

Solve:

$$\frac{b - x^2 y}{a - x + x^2 y} < \frac{b}{a} \Rightarrow y > \frac{1}{x(a+1)}$$

Since we have a trapping region we now check stability of the fixed point.

$$J = \begin{pmatrix} -1+2xy & x^2 \\ -2xy & -x^2 \end{pmatrix}$$

$$J(a+b, b/(a+b)^2) = \begin{pmatrix} -1+\frac{2b}{a+b} & (a+b)^2 \\ -2\frac{b}{a+b} & -(a+b)^2 \end{pmatrix}$$

Hopf-bifurcation occurs when

$$Tr[J(a+b, b/(a+b)^2] = 0$$

$$\Rightarrow -1 + 2\frac{b}{a+b} - (a+b)^2 = 0$$

$$\Rightarrow \quad b-a = (a+b)^3$$

This is a super-critical Hopf bifurcation.

* Really need a numerical scheme to understand complete dynamics.

The period can be estimated by assuming a circular orbit at the bifurcation point.

$$\det J(a+b, b/(a+b)^2) = (a+b)^2 - 2b(a+b) + 2b(a+b) = -\lambda_1^2$$

at the bifurcation point,

$$\Rightarrow \lambda_1 = i(a+b).$$

The period is then

$$T = \frac{2\pi}{a+b}$$

Since the linear solution is:

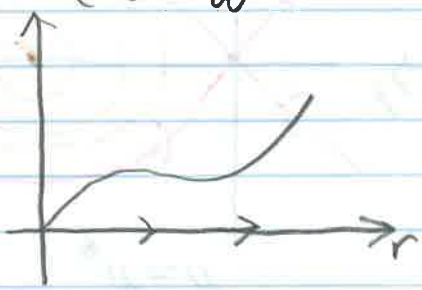
$$x = A\cos(\lambda_1 t)$$

$$y = A\sin(\lambda_1 t)$$

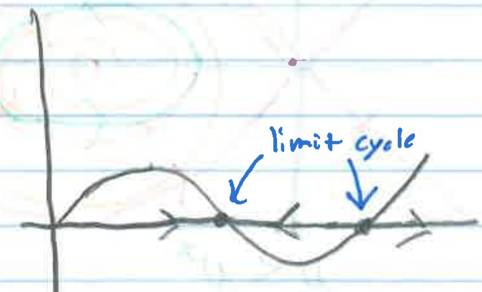
Other Periodic Bifurcations

1. Saddle Node:

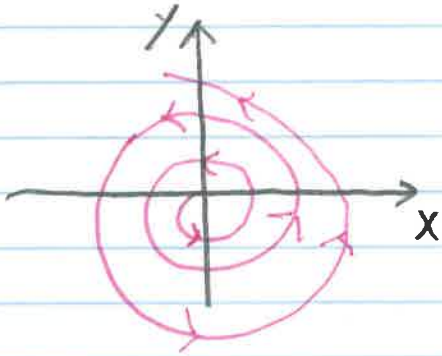
$$\begin{cases} \dot{r} = \mu r - r^3 + r^5 \\ \dot{\theta} = \omega \end{cases} \rightarrow \text{Quintic normal form}$$



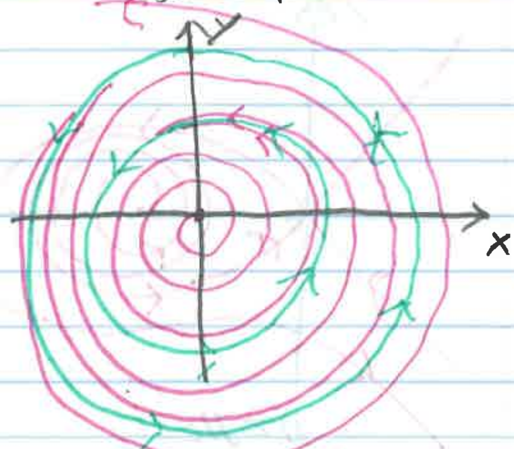
$\mu > 1/4$



$\mu < 1/4$



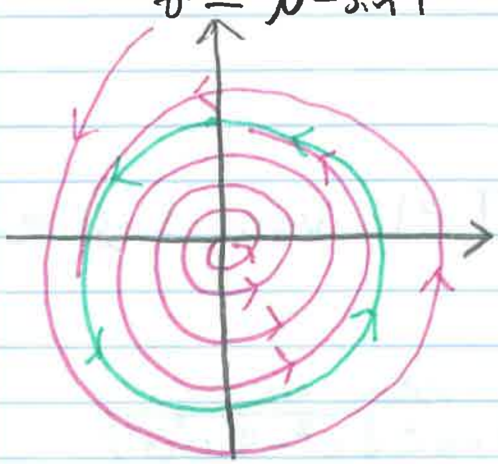
$\mu > 1/4$



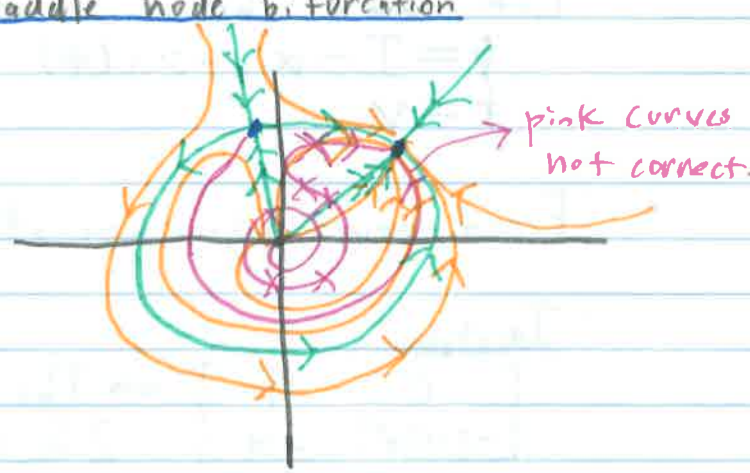
$\mu < 1/4$.

2. Infinite Period:

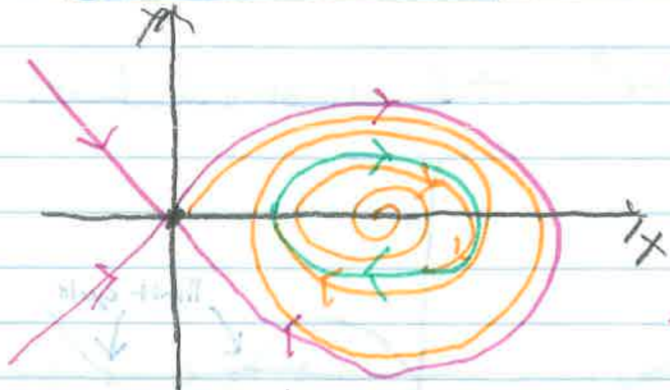
$$\dot{r} = r(1-r^2)$$
$$\dot{\theta} = \mu - \sin\theta \rightarrow \text{Saddle node biforcation}$$



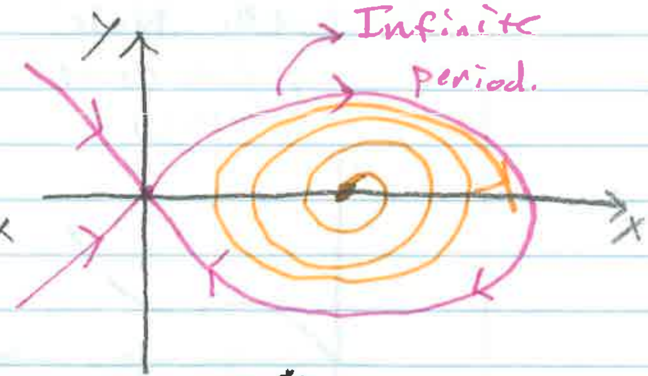
$\mu > 1$



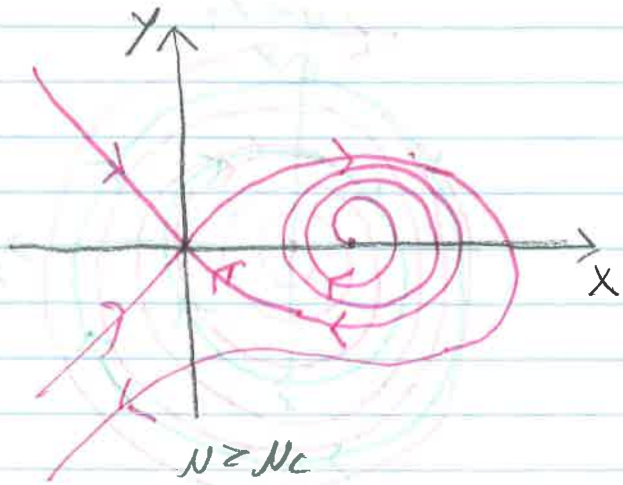
3. Homoclinic Bifurcation.



$\mu < \mu^*$



$\mu = \mu^*$



$\mu \geq \mu_c$

Example (Forced Pendulum With Friction)

$\ddot{\phi} + \alpha\dot{\phi} + \sin(\phi) = I, \qquad I, \alpha > 0.$

$\phi \in S^1$

Let $v = \dot{\phi}$ then

$\dot{v} = I - \alpha v - \sin(\phi)$

$\dot{\phi} = v$

Fixed points only exist if $I < 1$; $v = 0$, $\sin(\phi) = I$.

Jacobian:

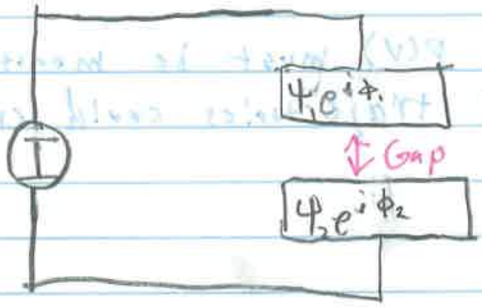
$$\begin{bmatrix} 0 & 1 \\ -\cos(\phi^*) & -\alpha \end{bmatrix} \Rightarrow \text{The eigenvalues are}$$

$$2\lambda_1, \lambda_2 = -\alpha \pm \sqrt{\alpha^2 - 4\cos(\phi^*)}$$

$$= -\alpha \pm \sqrt{\alpha^2 \pm 4\sqrt{1 - I^2}}$$

$\Rightarrow$ 1 fixed point is a saddle, the other a stable fixed point of some type.

Analogue with Josephenson junction



$\phi = \phi_1 - \phi_2 \rightarrow$ phase difference.

If $I<1$, junction acts as if it had zero resistance. There is a phase difference between the states.

If $I>1$, junction acts as a resistor. The voltage is given by:

$$V = \frac{\hbar}{2e} \langle \dot{\phi} \rangle$$

Case 1: $I>1$

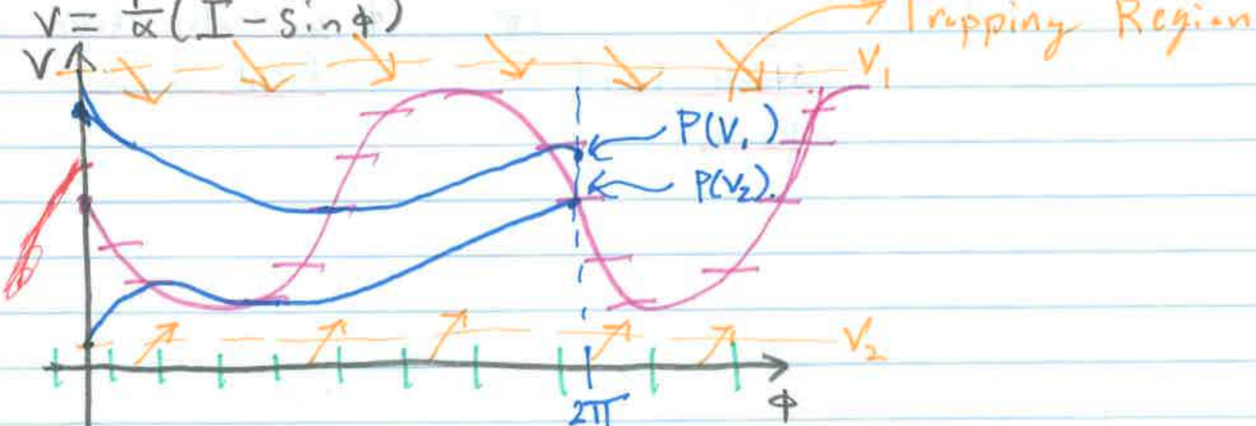
$\dot{\phi} = v$

$\dot{v} = I - \alpha v - \sin(\phi)$

Null-cline

$v = 0$

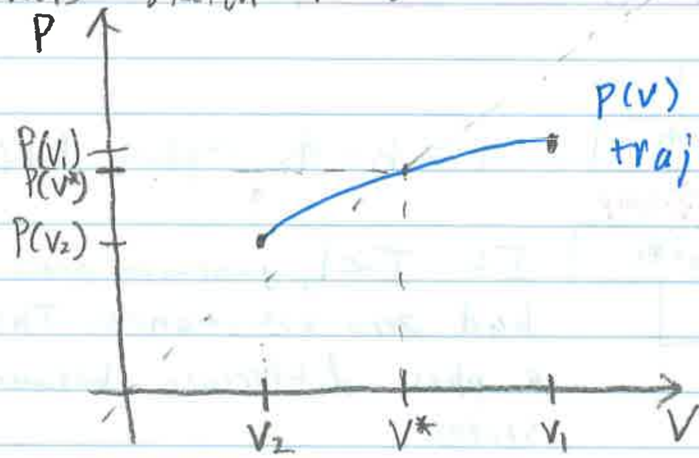
$v = \frac{1}{\alpha}(I - \sin\phi)$



The Poincare map $P: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ maps initial v at $\phi=0$ to value of v at $\phi = 2\pi$ for a solution trajectory.

1. $P(V_1) < V_1$
2. $P(V_2) > V_2$

Lets sketch $P(V)$:

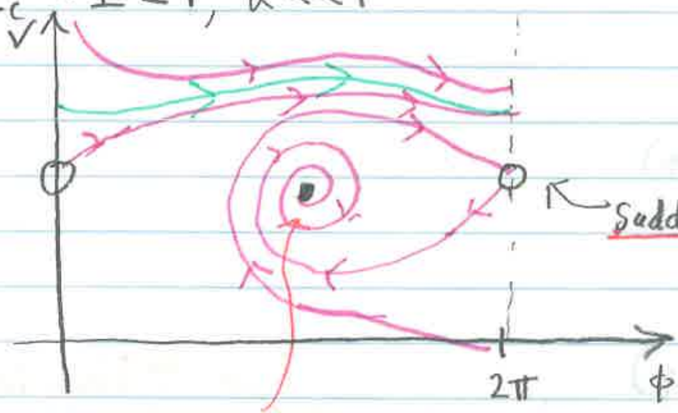


$P(V)$ must be monotone otherwise trajectories could cross.

$\Rightarrow \exists V^*$ such that $P(V^*) = V^*$. This implies the existence of a limit cycle.

Case 2:

$I_c < I < 1$, $\alpha << 1$



Saddle fixed point.

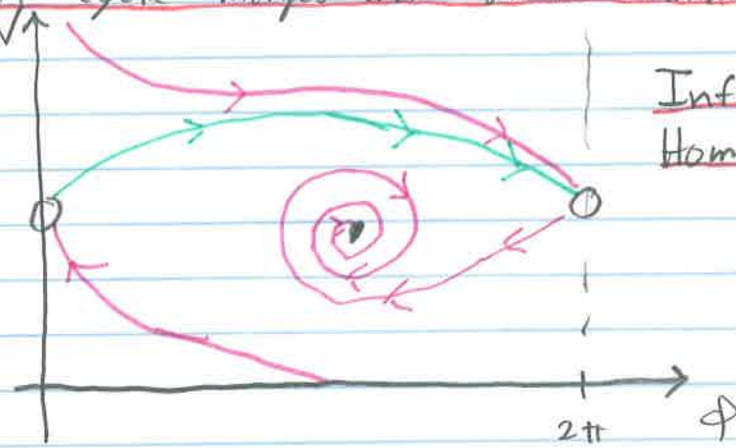
Stable fixed point.

Limit cycle still exists, however we also have a stable fixed point.

Case 3:

$I = I_c < 1, \ \alpha << 1.$

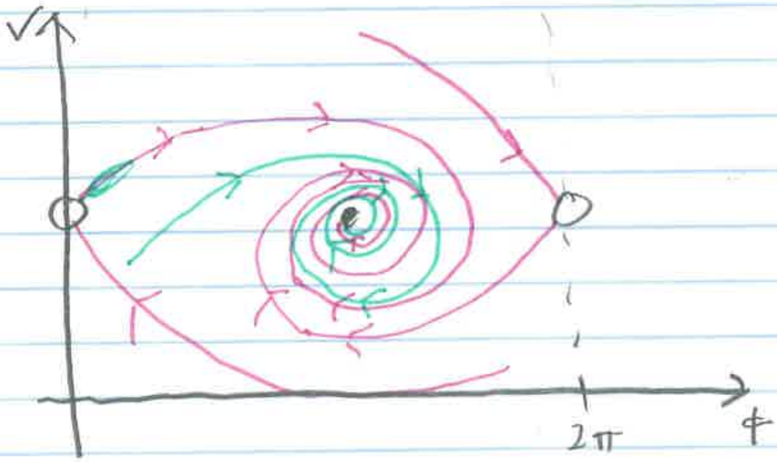
Limit cycle merges with stable manifold.



Infininite period bifurcation
Homoclinic bifurcation

Case 4:

$\alpha >> 1, \ I < 1.$



No limit cycle. Pendulum dies down.