Chapter 7: Limit Cycles

Definition - A closed trajectory is a limit cycle if it is seperated from all other closed trajectories.

a.) A limit cycle is stable if there is a tubular neighborhood such that trajectories that enter the neighborhood approach the limit cycle as $t \to \infty$.

b.) A limit cycle is unstable if it is not stable.

Example:

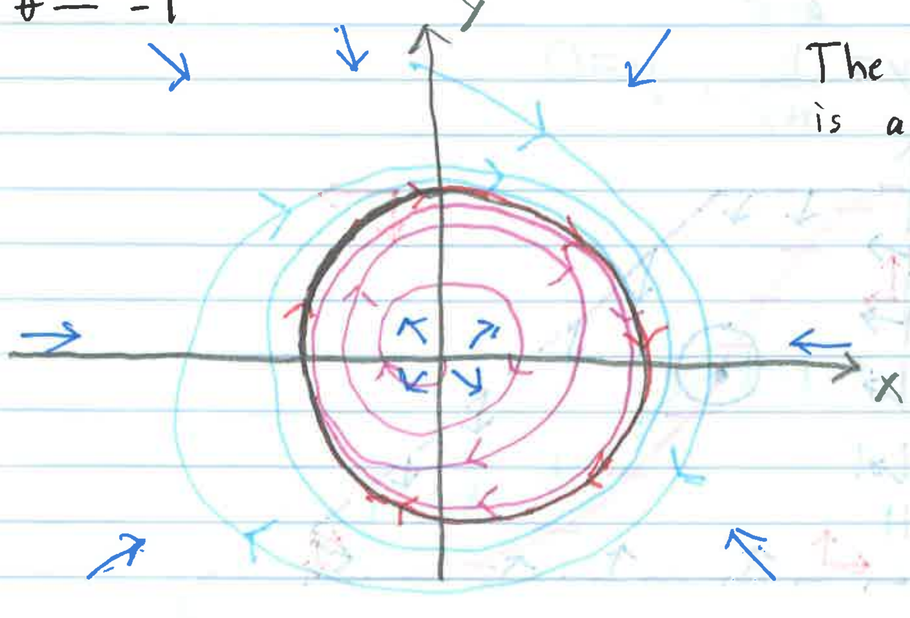
$$\dot{x} = -y + x\left(1 - \sqrt{x^2+y^2}\right), \quad x = r\cos\theta$$
$$\dot{y} = x + y\left(1 - \sqrt{x^2+y^2}\right) \quad y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$
$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \frac{1}{r}\begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$\Rightarrow \dot{r} = \cos\theta\left(-r\sin\theta + r\cos\theta(1-r)\right) + \sin\theta\left(r\cos\theta + r\sin\theta(1-r)\right)$$
$$\dot{\theta} = -\frac{\sin\theta}{r}\left(-r\sin\theta + r\cos\theta(1-r)\right) + \frac{\cos\theta}{r}\left(r\cos\theta + r\sin\theta(1-r)\right)$$

$$\Rightarrow \dot{r} = r(1-r)$$
$$\dot{\theta} = -1$$



The curve $x^2+y^2=1$ is a stable limit cycle.

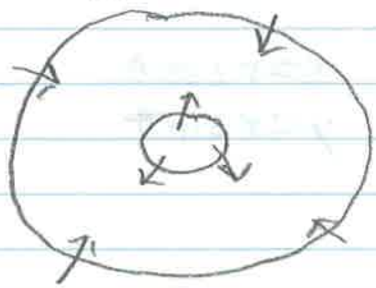
Poincare-Bendixon Theorem - Consider $\dot{\vec{x}} = F(\vec{x})$, with F continuously differentiable. Assum $R \subset \mathbb{R}^2$ is closed and bounded.

(i) R does not contain any fixed points.

(ii) There exists $\vec{x}(0) \in R$ so that $\vec{x} \in R$ for all $t \geq 0$.

Then R contains a limit cycle.

Typical Application:



R = trapping region.

Example:

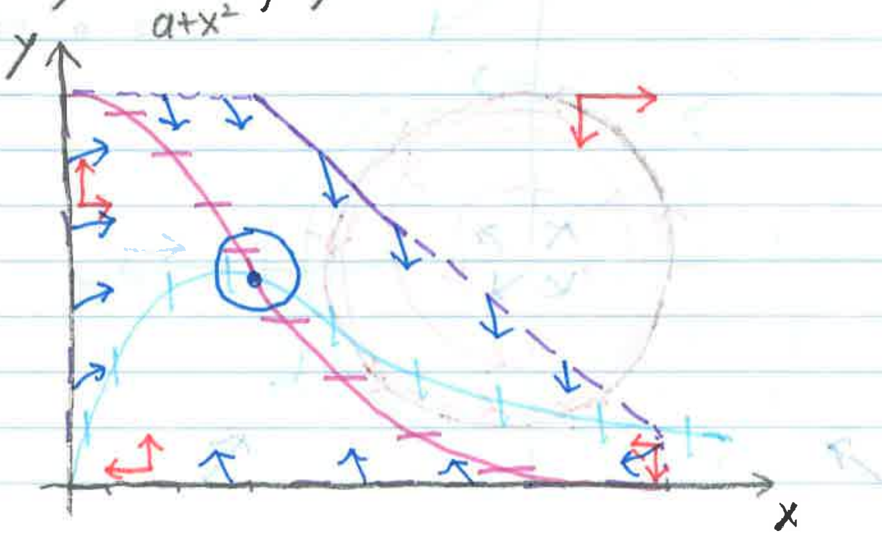
$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

Global Analysis:

Null clines:

$$y = \frac{x}{a+x^2}, \quad \dot{x} = 0$$

$$y = \frac{b}{a+x^2}, \quad \dot{y} = 0$$



When is $\frac{dy}{dx} < -1$

$$\Rightarrow b - ay - x^2 y < +x - ay - x^2 y$$
$$\Rightarrow b < x$$

Therefore, if $x > b$ we know $\frac{dy}{dx} < -1$.

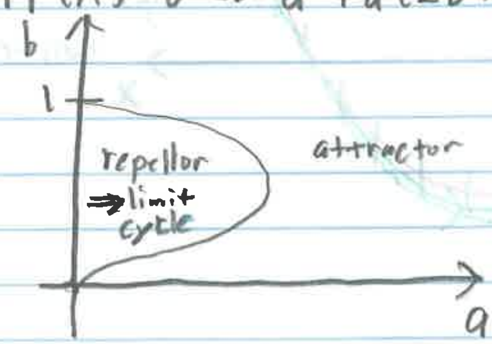
The fixed point is at $x = b$, $y = \frac{b}{a+b^2}$.

$$J = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -a - x^2 \end{pmatrix}$$

$$J\big|_{(x^*,y^*)} = \begin{pmatrix} -1 + \frac{2b}{a+b^2} & a + b^2 \\ -\frac{2b}{a+b^2} & -a - b^2 \end{pmatrix} = A$$

$$\det(A) = a + b^2 > 0$$
$$\operatorname{Tr}(A) = -1 - a - b^2 + \frac{2b^2}{a+b^2}$$

$$\operatorname{Tr}(A) = 0 \Leftrightarrow a^2 + a(2b^2+1) + b^2(b^2-1) = 0$$



Van der Pol Oscillator

$$\ddot{x} + \frac{1}{\varepsilon}(x^2 - 1)\dot{x} + x = 0$$

* $\frac{1}{\varepsilon} x^2 - 1$ is like a damping or pumping term.

- If $x^2 < 1$ energy is pumped in (forcing)
- If $x^2 > 1$ energy is pumped out (friction)

In fact:

$$\boxed{\frac{d}{dt} E = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{x^2}{2}\right) = -\frac{1}{\varepsilon}(x^2 - 1)\dot{x}^2}$$

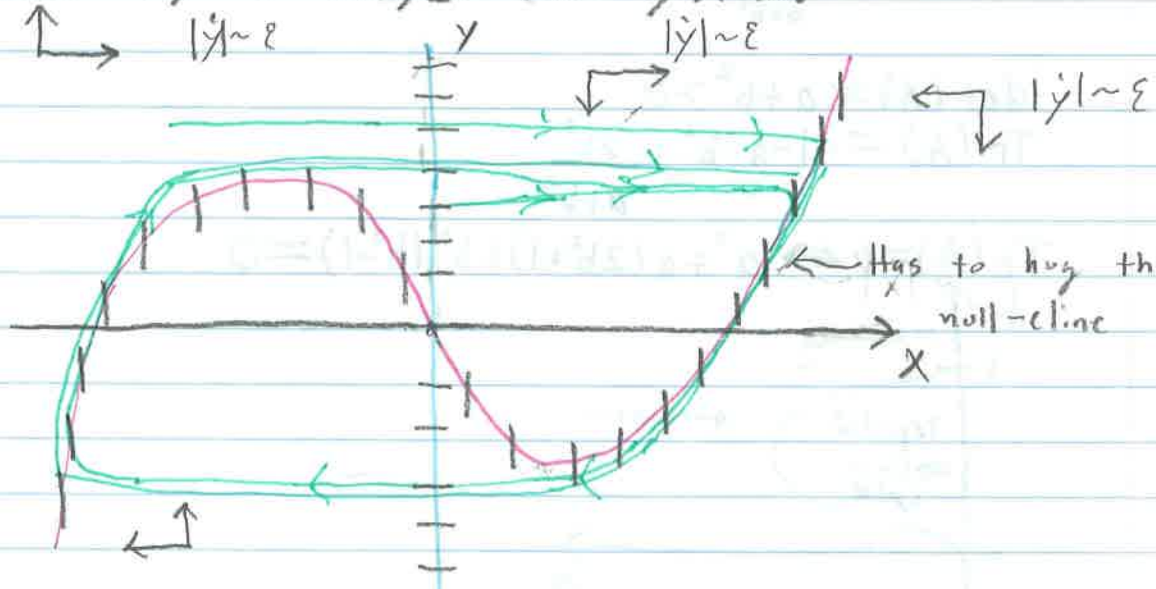
→ We expect the damping and pumping to lead to oscillations.

$$\ddot{x} + \frac{1}{\varepsilon}(x^2-1)\dot{x} = \frac{d}{dt}\left(\dot{x} + \frac{1}{\varepsilon}\left(\frac{x^3}{3} - x\right)\right)$$

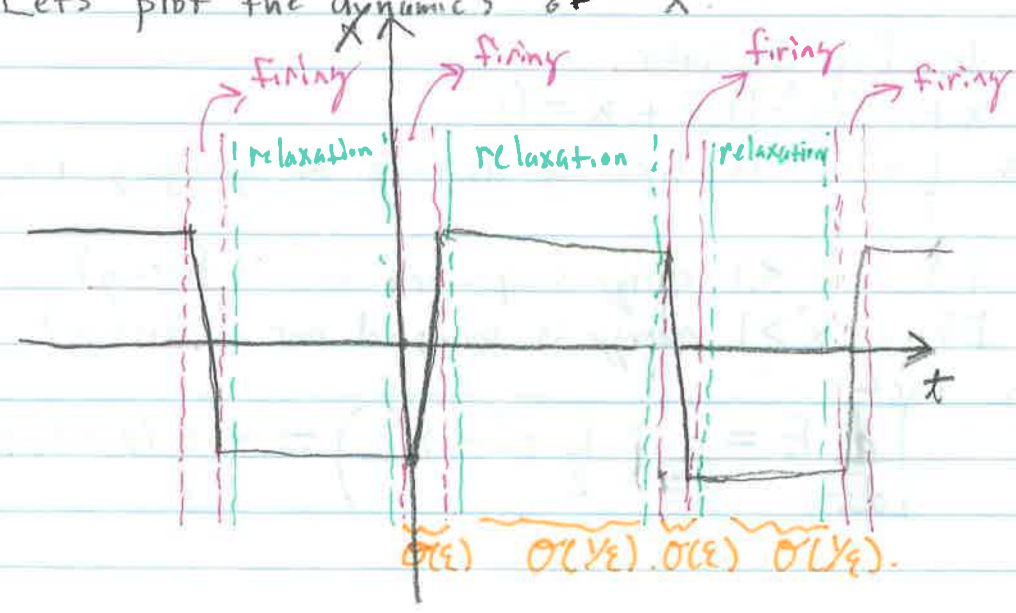
Let $F(x) = \frac{x^3}{3} - x$, $y = \varepsilon\dot{x} + \frac{x^3}{3} - x$

$$\Rightarrow \boxed{\begin{cases} \dot{x} = \frac{1}{\varepsilon}(y - F(x)), & 0 < \varepsilon << 1 \\ \dot{y} = -\varepsilon x \end{cases}}$$

Lets try to analyze this system:



Lets plot the dynamics of x.



Can we estimate the time scales; Essentially:

$\dot{x} \sim 1/\varepsilon$ away from the null-cline.

At the null-cline

$\dot{x} \sim \varepsilon, \; \dot{y} \sim \varepsilon$

as the two terms balance each other.

Example:

$$\ddot{x} = (1-3x^2-2\dot{x}^2)\dot{x} - x$$

Let $v = \dot{x}$,

$$\dot{x} = v$$
$$\dot{v} = (1-3x^2-2v^2)v - x.$$

The phase portrait analysis is done in Mathematica.
How can we prove there is a limit cycle?

Convert to polar coordinates!

$$\dot{r} = \cos\theta\,\dot{x} + \sin\theta\,\dot{v}$$
$$\dot{r} = \cos\theta \cdot r\sin\theta + [(1-3\cos^2\theta r^2 - 2r^2\sin^2\theta)r\sin\theta - r\cos\theta]\sin\theta$$

$$\Rightarrow \dot{r} = \sin\theta[r\cos\theta + (1-r^2-\cos^2\theta r^2)r\sin\theta - r\cos\theta]$$
$$\dot{r} = r\sin^2\theta[1-2r^2-r^2\cos^2\theta]$$

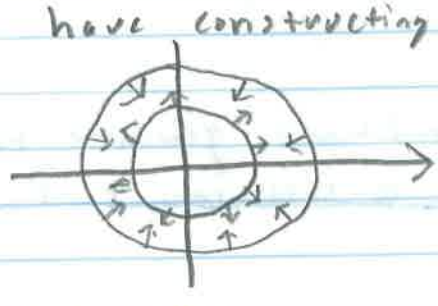
Let $r = 1/2$

Then $\dot{r} = \frac{1}{2}\sin^2\theta[1 - \frac{1}{2} - \frac{1}{2}\cos^2\theta] \geq 0$

Let $r = 1/\sqrt{2}$

Then $\dot{r} = \frac{1}{\sqrt{2}}\sin^2\theta[-\frac{1}{2}\cos^2\theta] \leq 0$

Therefore, we have constructing a trapping region



$\Rightarrow$ A limit cycle exists!

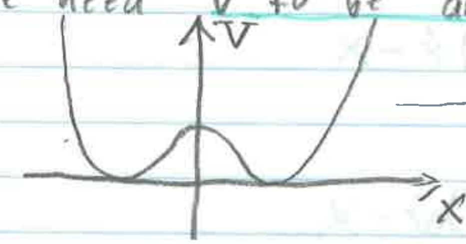
Lienard Systems

$$\ddot{x} + \underbrace{f(x)}_{\text{damping}}\dot{x} + \underbrace{g(x)}_{\text{force from external potential}} = 0.$$

$$V(x) = +\int_{x_0}^{x} g(x)\,dx.$$

We saw from the Van-der pol oscillator that we need several conditions for a limit cycle.

1. We need V to be an even function.



→ Oscillations for conservative systems

2. We need V to satisfy $\lim_{|x|\to\infty} V(x) = \infty$.

→ Trajectories are "trapped" for conservative system.

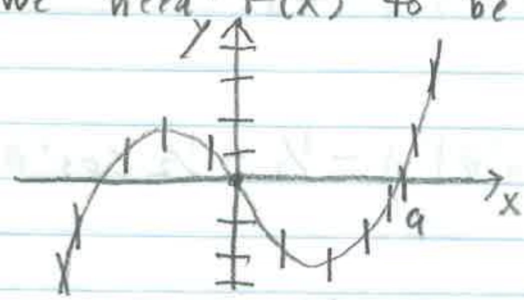
Rewrite

$$\frac{d}{dt}(\dot{x} + F(x)) = -g(x), \quad F(x) = \int_{x_0}^{x} f(x)\,dx$$

Let $y = \dot{x} + F(x)$

$$\Rightarrow \boxed{\begin{aligned}\dot{x} &= y - F(x)\\ \dot{y} &= -g(x)\end{aligned}}$$

3. We need $F(x)$ to be odd:



4. We need $F(x)$ to satisfy:

$F(x)$ is increasing on $\mathbb{R}^+$ and, $\exists a \in \mathbb{R}$ such that $F(x) > 0$ if $x > a$

*These four conditions guarantee the existence of a limit cycle. This is a sufficient condition.