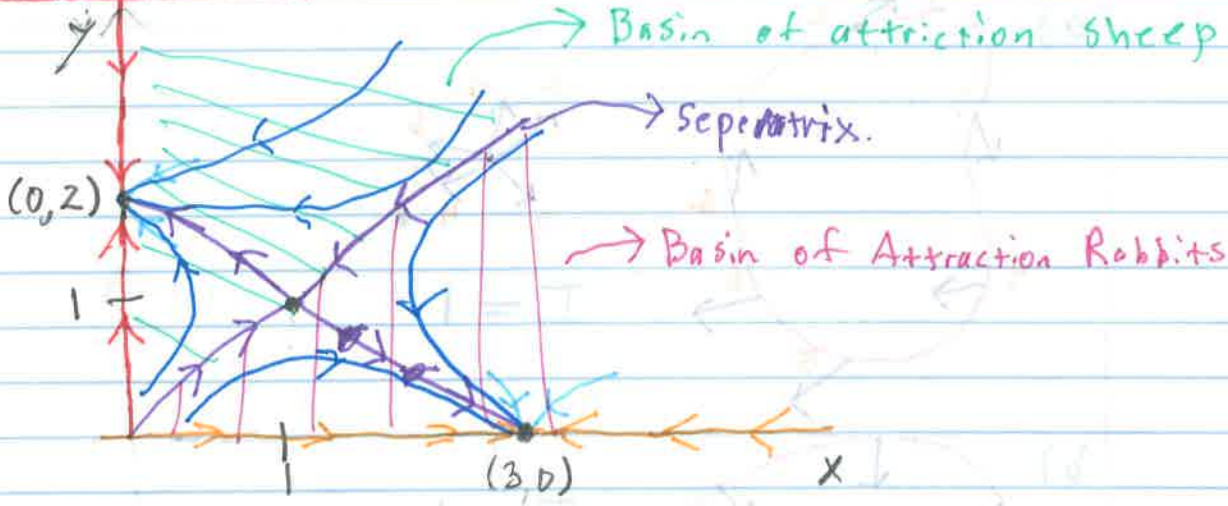
Phase Portrait



Index Theory

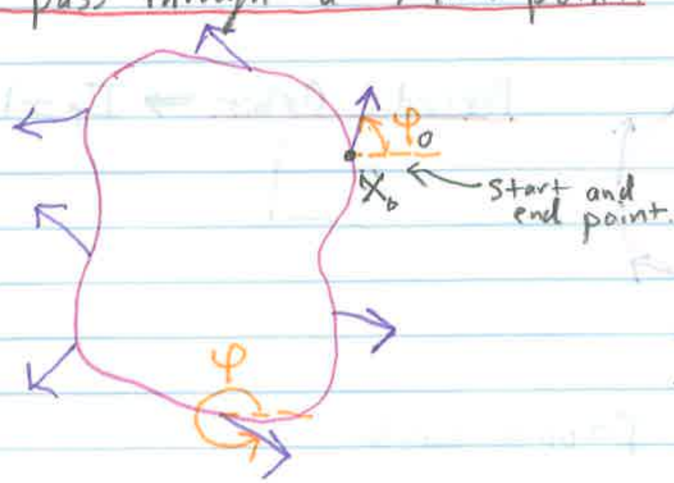
How can we be sure no periodic orbits exist?

Consider

$$\dot{\vec{x}} = F(\vec{x})$$

with $F: \mathbb{R}^2 \to \mathbb{R}^2$ continuously differentiable.

Take a closed curve Γ with no self intersections, that does not pass through a fixed point.

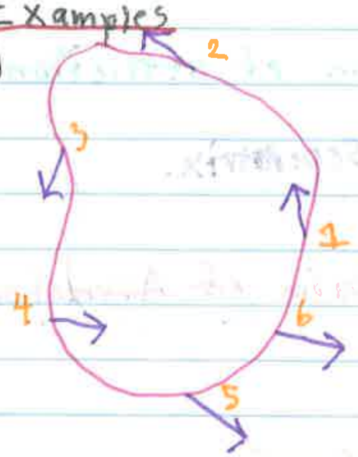


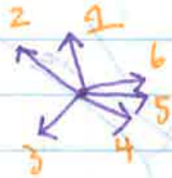
1. Start at U_0, traverse Γ counterclockwise and take angle φ of $F(\vec{x})$. $\to$ this angle changes continuously as Γ is traversed.
2. After one pass we again end up at U_0 with an angle $\varphi_1 = \varphi_0 + 2\pi n$; $n \in \mathbb{Z}$

$$\boxed{I_\Gamma = \frac{1}{2\pi}(\varphi_1 - \varphi_0)}$$

Examples

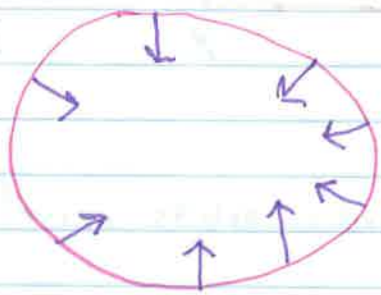
a.)





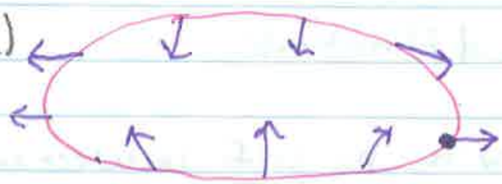
$I_\Gamma = 1$

b.)



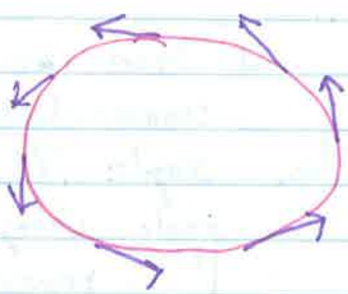
$I_\Gamma = 1$

c.)



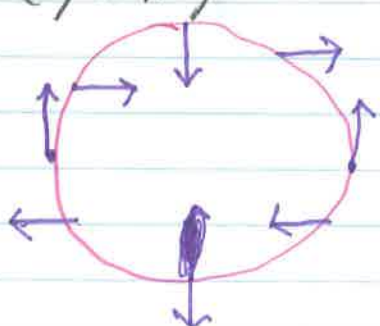
$I_\Gamma = -1$.

d.)



Periodic Orbit $\Rightarrow I_\Gamma = 1$

e.) $\begin{cases} \dot{x} = x^2 y \\ \dot{y} = x^2 - y^2 \end{cases}$; Γ = unit circle.



$I_\Gamma = 0$

Properties of the Index

1. If Γ can be deformed continuously into $\tilde{\Gamma}$ without passing through any equilibrium points then
$$I_\Gamma = I_{\tilde{\Gamma}}$$
proof:
I_Γ varies continuously as Γ is deformed, but I_Γ is integer valued.

2. If Γ does not contain any fixed points then $I_\Gamma = 0$
proof:
Property 1 implies we can shrink Γ to a point without changing the index.

3. If we replace $F(\vec{x})$ by $F(-\vec{x})$ the index is not changed.
proof:
Each angle is replaced by $\varphi + \pi$, hence $\varphi_1 - \varphi_0$ is the same.

4. The index of a periodic orbit is one.
5. If $F(\vec{x})$ is deformed continuously without creating any fixed points on Γ, I_Γ stays the same.

Theorem - Assume F is continuously differentiable. Inside each periodic orbit, there is at least one equilibrium.
proof:
Follows from items 2 and 4.

Index of isolated fixed point - Let $\vec{x}^*$ be an isolated fixed point of $\dot{\vec{x}} = F(\vec{x})$. Define
$$I(\vec{x}^*) = \text{index of simple closed curve that encloses } x^* \text{ and no other fixed points.}$$
$I(\vec{x}^*)$ is well defined by property 1.

Consequences:

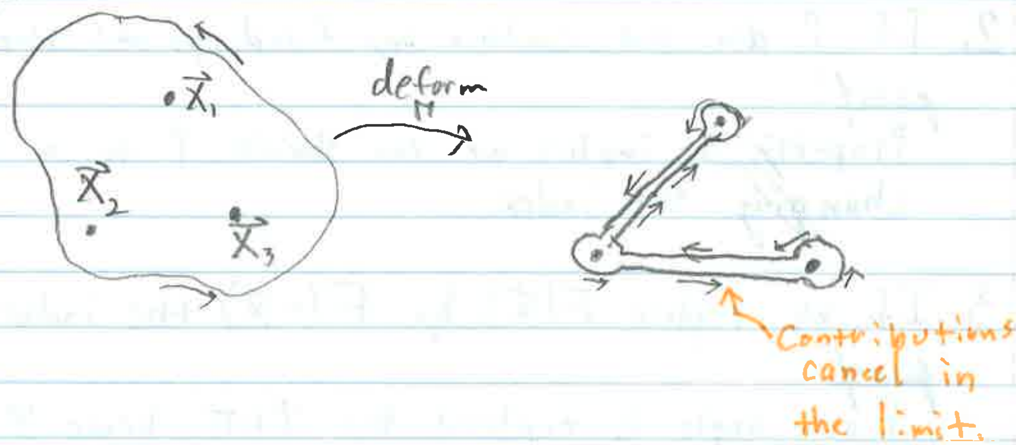
1. If $\vec{x}^*$ is an attractor or repellor then $I(\vec{x}^*) = 1$.
2. If $\vec{x}^*$ is a saddle point then $I(\vec{x}^*) = -1$.

~~proof~~:

Follows from examples b and c and properties 1, 3, 5.

Theorem - If Γ is a closed simple curve that contains n isolated fixed points $\vec{x}_1, \dots, \vec{x}_n$ then $I_\Gamma = I(\vec{x}_1) + \dots + I(\vec{x}_n)$.

~~proof~~:



Corollary: A periodic orbit must enclose fixed points whose indices sum to $+1$.

Omnivore example:

The index of all the fixed points is -2.

Sheep and Rabbits:

The index of all the fixed points is 0.

Conservative Systems:

Inertial Systems of the Form:

$$\ddot{x} = F(x)$$

A first integral can be found as follows:

$$\dot{x}\ddot{x} = \dot{x}F(x)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}(\dot{x}^2) = \frac{dx}{dt}\left(-\frac{dV}{dx}\right), \quad \text{(For any solution curve)}$$

where $V(x) = -\int_{x_0}^{x} F(x)\,dx$ (x_0 can be chosen arbitrarily)

$$\Rightarrow \frac{d}{dt}\left(\frac{\dot{x}(t)^2}{2} + V\right) = 0$$

For any solution curve there is a constant E such that

$$\boxed{\frac{\dot{x}(t)^2}{2} + V(x(t)) = E}$$

We can also write as a system:

$$\dot{x} = v \qquad \frac{v^2}{2} + V(x) = E$$
$$\dot{v} = F(x)$$

phase portrait $\Longleftrightarrow$ contour plot

Theorem - A conservative system cannot have any attractors or repellors.

~~proof~~:

Suppose there exists (x^*, v^*) that is an attracting point with a basin of attraction A. Then, for all $(x_1, v_1), (x_2, v_2) \in A$ it follows that $E(x_1, v_1) = E(x_2, v_2)$ since

$$E(x_1, v_1) = \lim_{t\to\infty} E(x_1(t), v_1(t))$$

$$= E(x^*, v^*(t))$$

$$= \lim_{t\to\infty} (x_2(t), V_2(t))$$

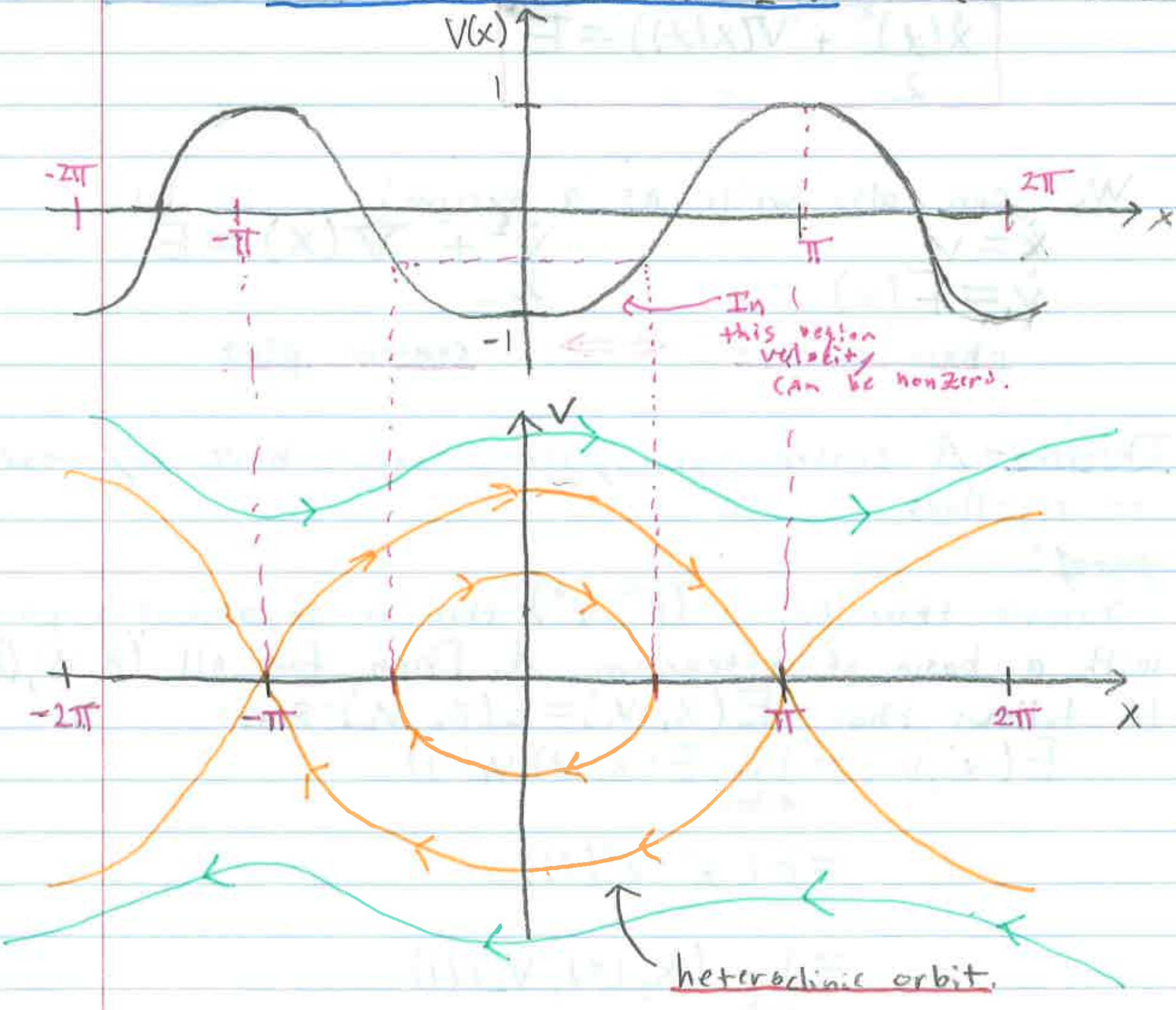
$$= E(x_2, v_2).$$

Therefore, E must be constant in entire basin of attraction which we preclude by definition. $\blacksquare$

Example.

$$\ddot{x} + \sin(x) = 0$$
$$V(x) = -\cos(x).$$

$$E = \frac{1}{2}v^2 - \cos(x)$$
$$\Rightarrow v = \pm\sqrt{2E + 2\cos(x)}$$
$$v = \pm\sqrt{2(\cos(x) - \cos(x_0))} \Leftrightarrow (\text{If } v(0) = 0)$$
$$v = \pm\sqrt{2(\cos(x) - \cos(x_0)) + \frac{1}{2}v_0^2} \Leftrightarrow (\text{If } v(0) \neq 0).$$



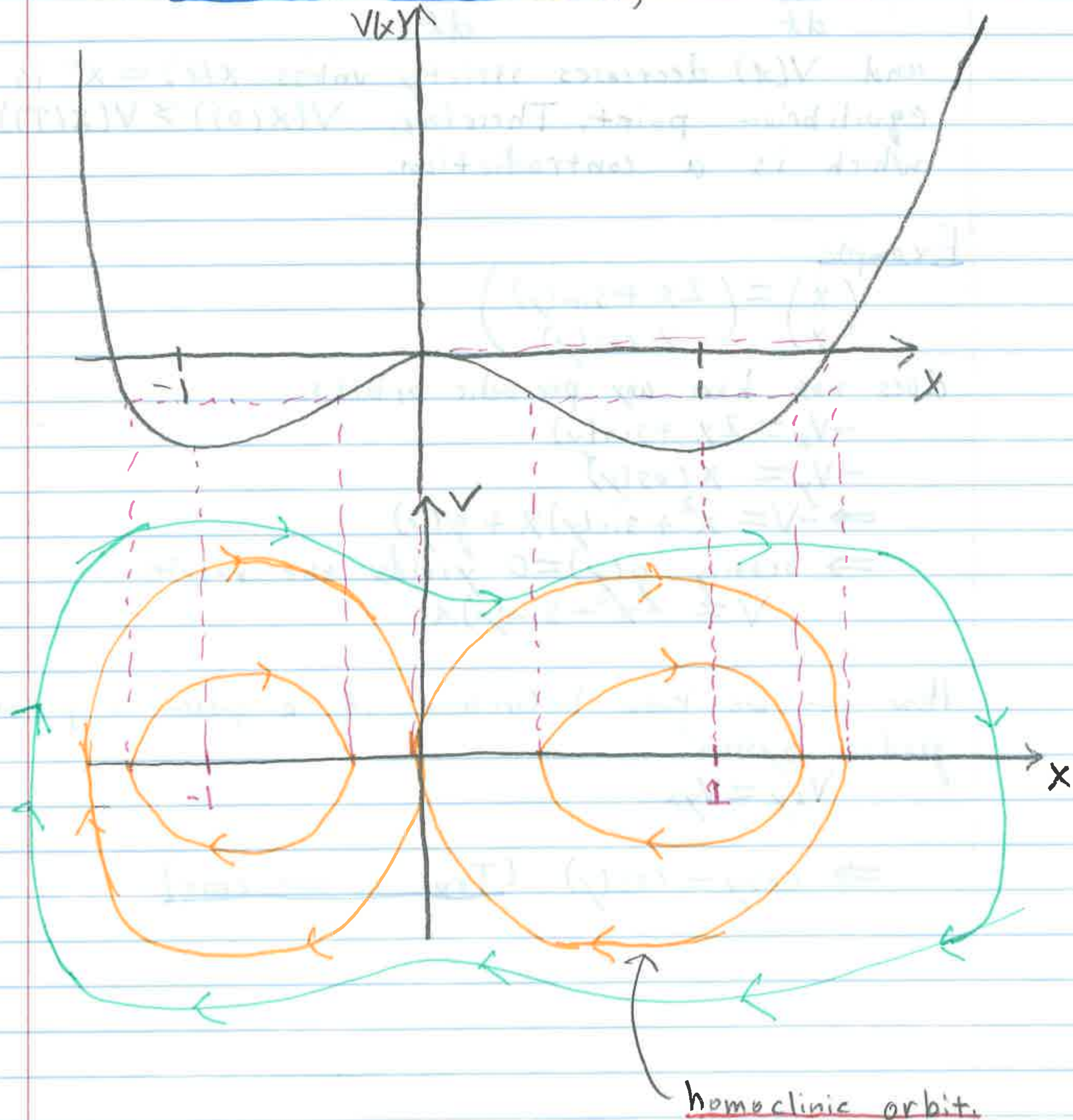
Example

$\ddot{x} = x - x^3$

$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$

$E = \frac{1}{2}v^2 - \frac{x^2}{2} + \frac{x^4}{4}$

$\Rightarrow v = \pm\sqrt{x^2 - \frac{x^4}{2} - x_0^2 + \frac{x_0^4}{4}}$, (If $v(0)=0$).



G Gradient Systems.

$$\dot{\vec{x}} = -\nabla V, \quad \text{where } V: \mathbb{R}^2 \to \mathbb{R}$$

Lemma - Gradient systems cannot have closed orbits.

proof:

Let $x(t)$ be a closed orbit with period T. Then,

$$\frac{d}{dt} V(x(t)) = \nabla V \cdot \frac{d\vec{x}}{dt} = -|\nabla V|^2 \leq 0.$$

and $V(t)$ decreases strictly unless $x(t) = x^*$ is an equilibrium point. Therefore, $V(x(0)) > V(x(T))$ which is a contradiction.

Example

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2x + \sin(y) \\ x\cos(y) \end{pmatrix}$$

does not have any periodic orbits.

$$-V_x = 2x + \sin(y)$$

$$-V_y = x\cos(y)$$

$$\Rightarrow -V = x^2 + \sin(y)x + g(y)$$

$\Rightarrow$ Setting $g(y) = 0$ yields the result.

$$V = -x^2 - \sin(y)x.$$

How can we know beforehand if a system is potentially a gradient system?

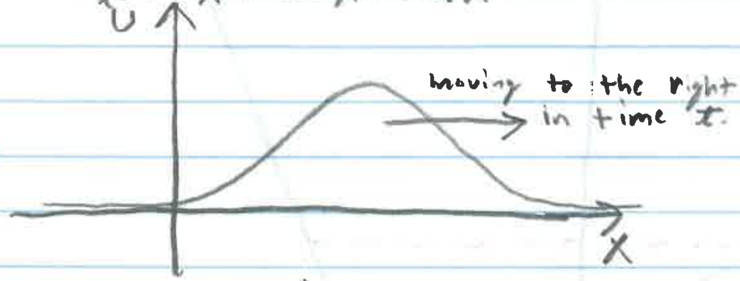
$$V_{xy} = V_{yx}$$

$\Rightarrow \cos y = \cos(y)$ (True in our case)

Solitons.

Shallow water waves in a narrow canal

$$u_t + u_x + u u_x + u_{xxx} = 0$$



Let $z = x - ct$:

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{d}{dz} = \frac{d}{dz}$$

$$\frac{\partial}{\partial t} = \frac{\partial z}{\partial t}\frac{d}{dz} = -c\frac{d}{dz}$$

$$\Rightarrow (1-c)\frac{du}{dz} + \frac{1}{2}\frac{d}{dz}(u^2) + u_{zzz} = 0$$

$$\Rightarrow (1-c)u + \frac{1}{2}u^2 + u_{zz} = \nu$$

$$\Rightarrow u_{zz} = -(1-c)u - \frac{1}{2}u^2 + \nu$$

This is a conservative system with potential:

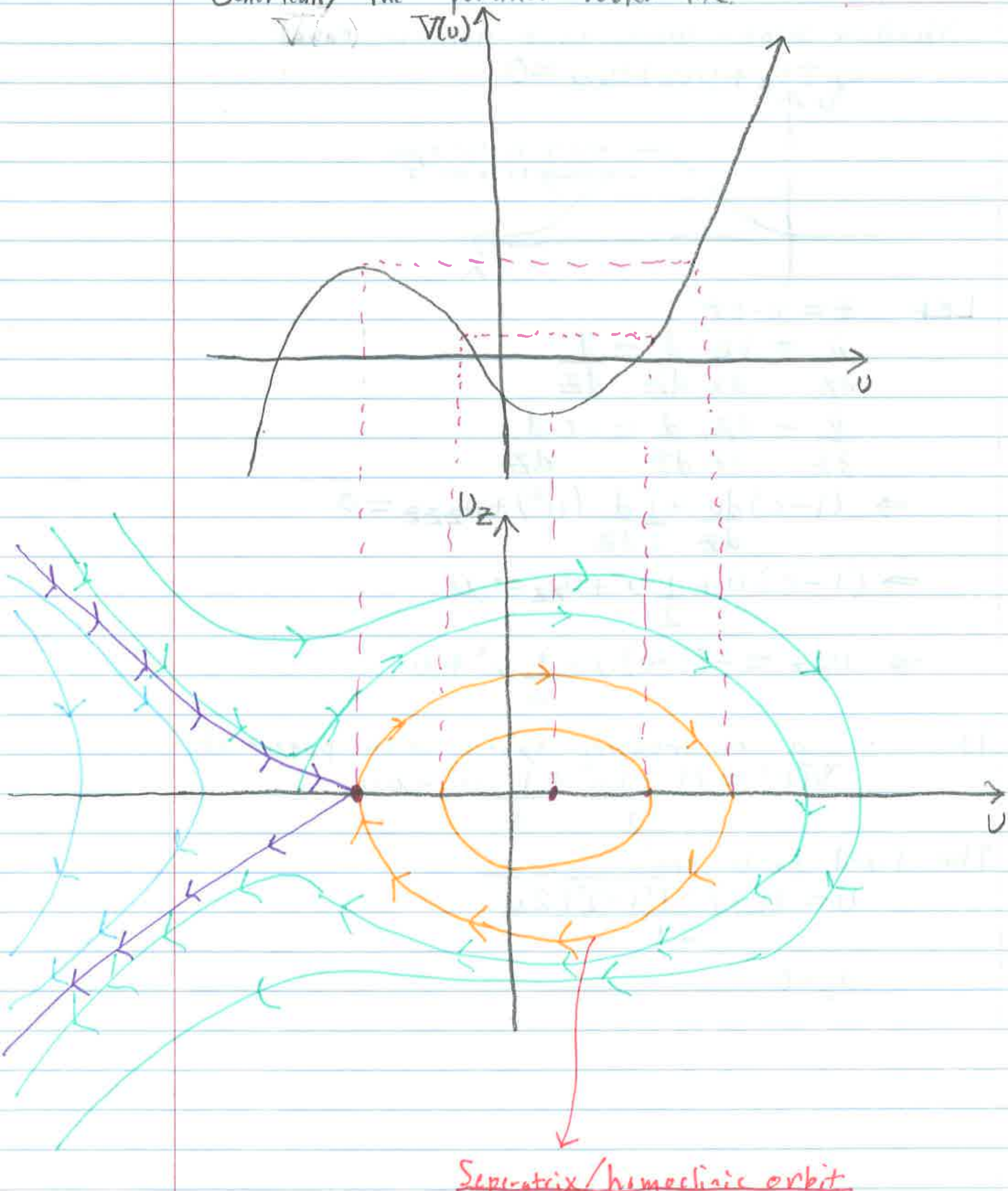
$$V(u) = \frac{(1-c)u^2}{2} + \frac{1}{6}u^3 - \nu u.$$

The fixed points are:

$$u = \frac{(1-c) \pm \sqrt{(1-c)^2 + 2\nu}}{-1}$$

$$u_z = 0$$

Generically the potential looks like:



Lyapunov Functions

$$\dot{\vec{x}} = F(\vec{x})$$

A continuously differentiable function $L: \mathbb{R}^2 \to \mathbb{R}$ is called a Lyapunov function if $L(x(t))$ strictly decreases along each solution of $\dot{\vec{x}} = F(\vec{x})$ that is not an equilibrium.

Lemma - If $\dot{\vec{x}} = F(\vec{x})$ admits a Lyapunov function, then it cannot have any periodic orbits.

Example

$$\ddot{x} + \alpha \dot{x} = g(x), \quad \alpha > 0.$$

Let $V(x) \triangleq -\int_{x_0}^{x} g(x)\,dx$. Then,

$$\frac{1}{2} \cdot \frac{d(\dot{x}^2)}{dt} + \alpha \dot{x}^2 = -\frac{d}{dt}(V(x(t)))$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + V(x(t))\right) = -\alpha \dot{x}^2 < 0.$$

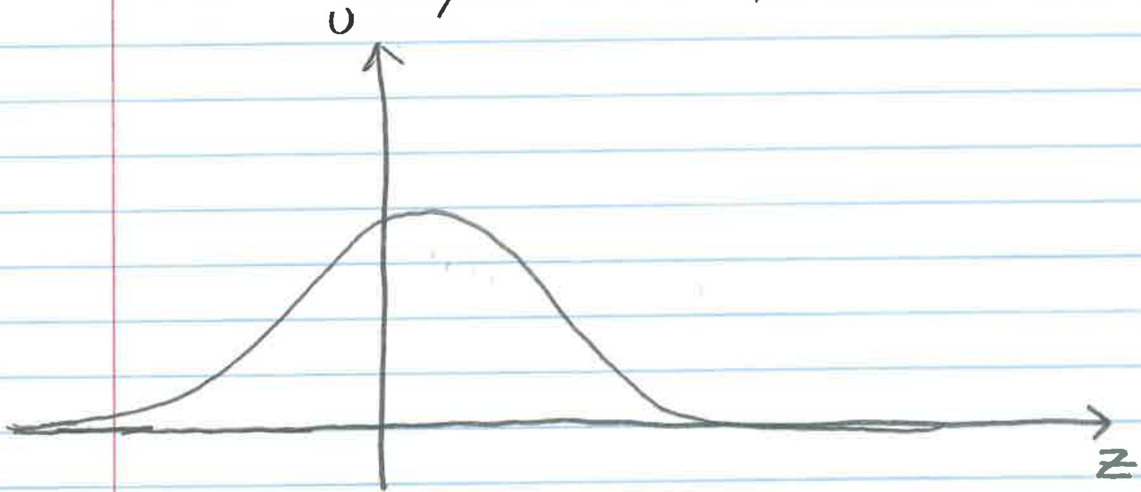
The function $L(v, x) = \frac{1}{2}v^2 + V$ is a Lyapunov function.

Summary:

1. $\ddot{x} = -\frac{dV}{dx} \to$ conservative $E(x, \dot{x})$ is conserved } Many periodic orbits.
2. $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\nabla V \to$ gradient system, V decreases along solutions
3. $\ddot{x} + \alpha\dot{x} = -\frac{dV}{dx} \to E(x, \dot{x})$ decreases

} No periodic solutions

The travelling wave corresponds to the seperatrix homoclinic o.



Sketch of the solution.