Chapter 5 and 6: Phase Plane

Differential Equations with two variables (x,y):

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

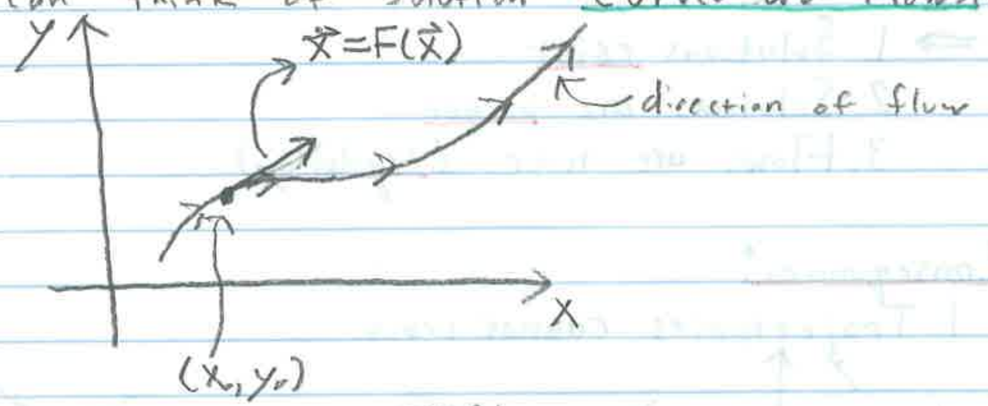
where $(x,y) \in \mathbb{R}^2$ and $f,g: \mathbb{R}^2 \to \mathbb{R}$ are continuously differentia

We can also write this system in the form:

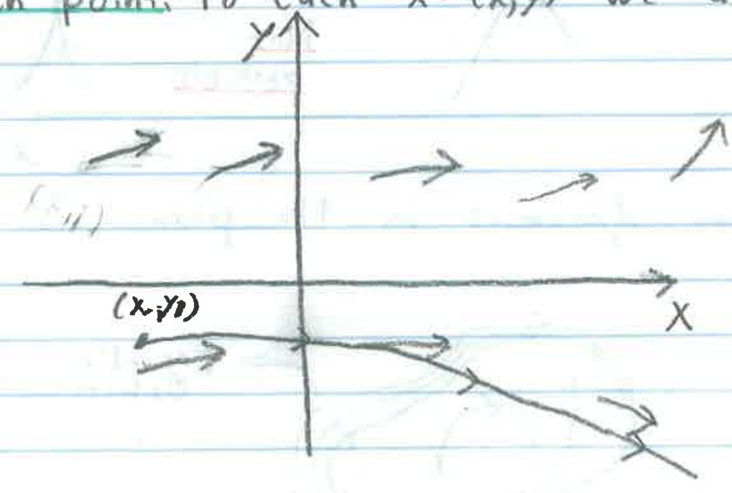
$$\dot{\vec{x}} = F(x,y),$$

where $F: \mathbb{R}^2 \to \mathbb{R}^2$.

We can think of solution curves as flows.



We can think of $F(\vec{x})$ as assigning a velocity vector to each point: to each $\vec{x} = (x,y)$ we assign the vector $F(\vec{x})$.



Particular Solutions of interest:

1. Fixed Points: Each $\vec{x}_0$ satisfy $F(\vec{x}_0) = 0$
2. Periodic Trajectories: $\vec{x}(t)$ is periodic if $\exists T > 0$

not a fixed point.

Theorem - Assume $F: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiabl for all $\vec{x} \in \mathbb{R}^n$, then for each $\vec{x}_0 \in \mathbb{R}^n$ the system

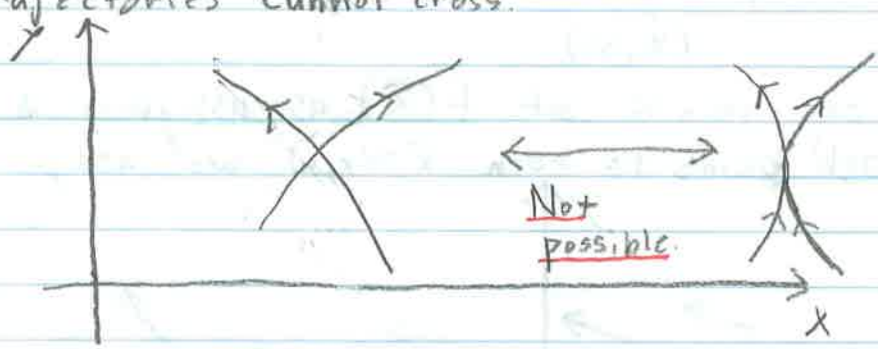
$$\dot{\vec{x}} = F(\vec{x})$$
$$\vec{x}(0) = \vec{x}_0$$

1 has a solution $\vec{x}(t)$ in the interval $(-\tau, \tau)$ for some $\tau > 0$ and the solution is 2 unique. The map $\vec{x}_0 \mapsto \vec{x}(t, \vec{x}_0)$ is 3 continuously differentiable for each fixed t.

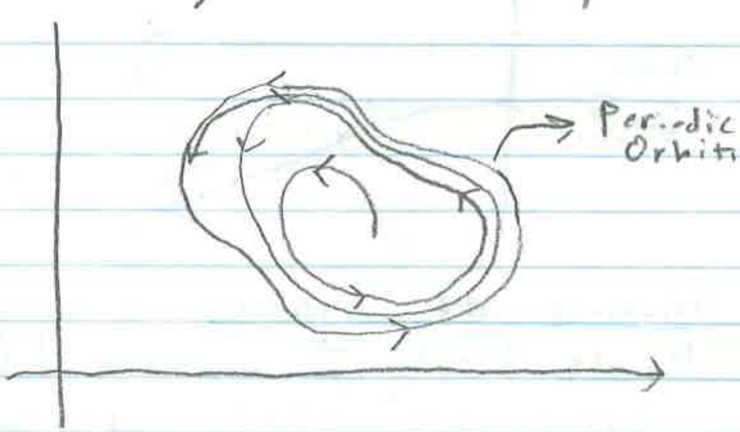
$\Rightarrow$ 1. Solutions exist.
2. Solutions are unique.
3. Flows are nice (regularity).

Consequences:

1. Trajectories cannot cross.



2. Constrains dynamics in the plane.



Linear Systems:

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

We can write

$$\dot{\vec{x}} = A\vec{x}$$

with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

1. If $\vec{x}_1, \vec{x}_2$ satisfy $\dot{\vec{x}} = A\vec{x}$ then so does any linear combination
$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2.$$
Therefore, it suffices to find two solutions $\vec{x}_1, \vec{x}_2$ with $\vec{x}_1(0)$ and $\vec{x}_2(0)$ linearly independant to generate all solutions.

2. If λ is an eignvalue of A with corresponding eigenvector $\vec{v}$ then $A\vec{v} = \lambda\vec{v}$. Therefore,
$$\vec{x} = e^{\lambda t}\vec{v}$$
is a solution:
proof:
$$\frac{d\vec{x}}{dt} = \lambda e^{\lambda t}\vec{v} = e^{\lambda t}Av = A\vec{x}.$$

3. All solutions can be written in the form:
$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2,$$
where λ_1, λ_2 and $\vec{v}_1, \vec{v}_2$ are the eigenvalues and corresponding eigenvectors, respectively.

4. The eigendirections tell us where the flow is invariant. (Invariant manifolds).

5. At $t=0$ we have
$$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2$$

Examples:

1. $\dot{\vec{x}} = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} \vec{x}$

$= A\vec{x}$

$\lambda_1 \cdot \lambda_2 = \det(A) = -2$

$\lambda_1 + \lambda_2 = tr(A) = -1$

$\Rightarrow \lambda_2 = -1 - \lambda_1$

$\Rightarrow \lambda_1 (1+\lambda_1) = 2$

$\lambda_1^2 + \lambda_1 - 2 = 0$

$$\boxed{\begin{array}{l} \lambda_1 = -2 \\ \lambda_2 = 1 \end{array}}$$

Clearly,

$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

To determine $\vec{v}_1$:

$(A + 2I)\vec{v}_2 = 0$

$\Rightarrow \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \vec{v}_2 = 0$

$\Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The general solution is then:

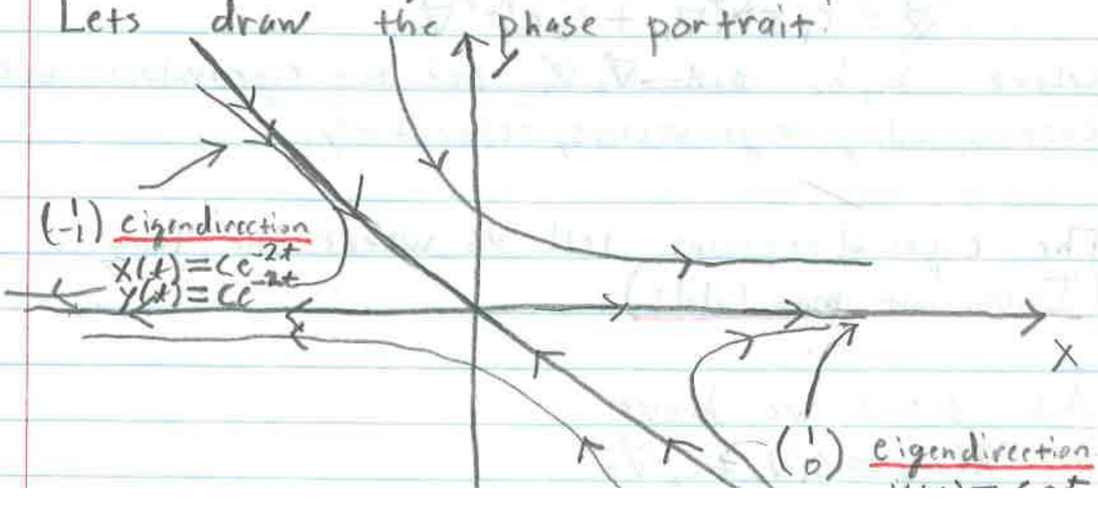
$\vec{x} = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

As $t \to \infty$ $x(t) \to \pm\infty$

$t \to \infty$ $y(t) \to 0$

Lets draw the phase portrait:



2. $\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}.$
$= A\vec{x}$

Eigenvalues satisfy

$$\lambda_1 \cdot \lambda_2 = 1$$
$$\lambda_1 + \lambda_2 = 0$$
$$\Rightarrow \lambda_1 = i$$
$$\lambda_2 = -i$$

The eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

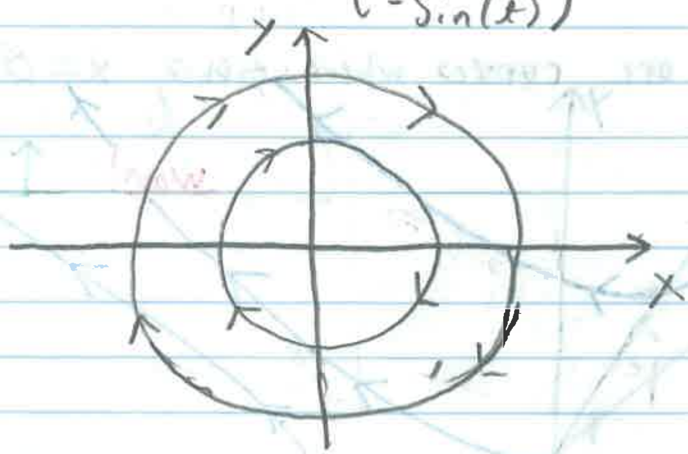
The general solution is:

$$\vec{x}(t) = c_1 e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

We of course want real valued solutions so we set

$$\vec{x}(t) = c_1 \operatorname{Re}\left[e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix}\right] + c_2 \operatorname{Im}\left[e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix}\right]$$

$$\Rightarrow \vec{x}(t) = c_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$



Periodic Orbit.
Period 2π

Example: (Richardson's Arm's Race)

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy.$$

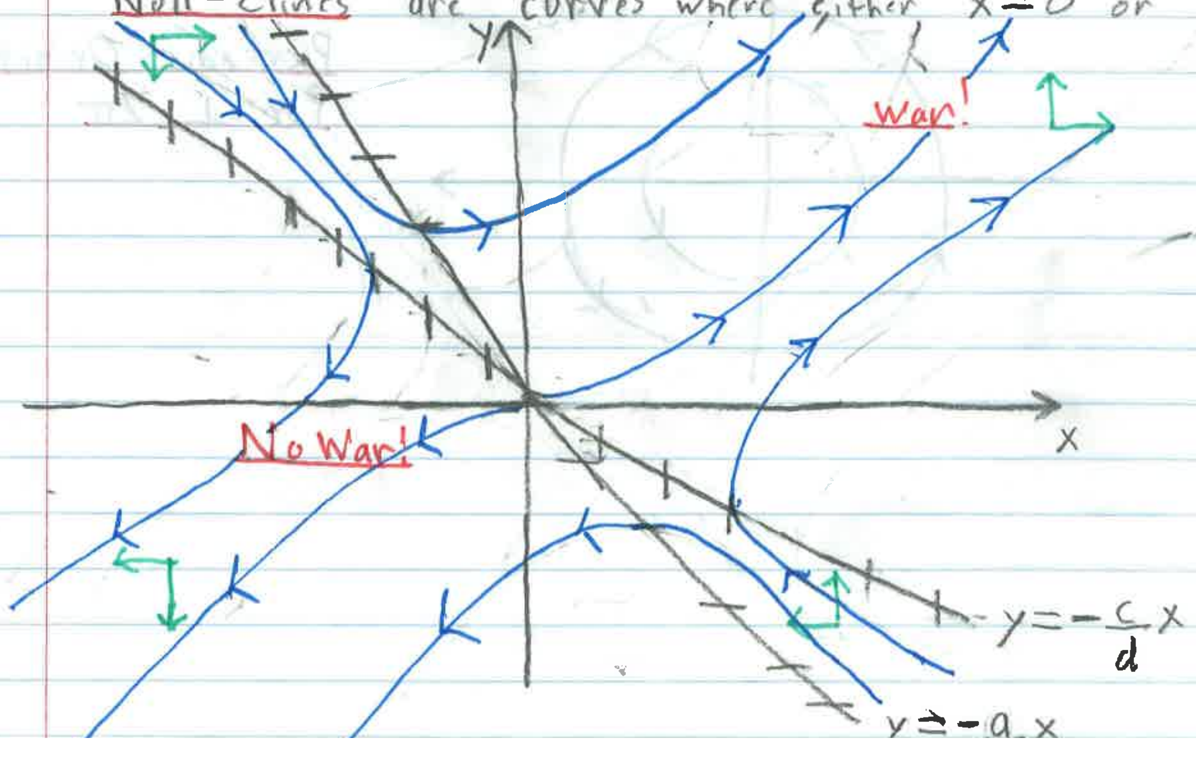
$x \rightarrow$ expendature on arms by nation x.
$y \rightarrow$ expendature on arms by nation y.

$a, d > 0$	nation x or y are arms dealers.
$a, d < 0$	nation x or y have limited resoourses
$b, d > 0$	nation x or y responds aggresively to other nation.
$b, d < 0$	nation x or y reduces spending depending on the other nations arms.

Case 1:

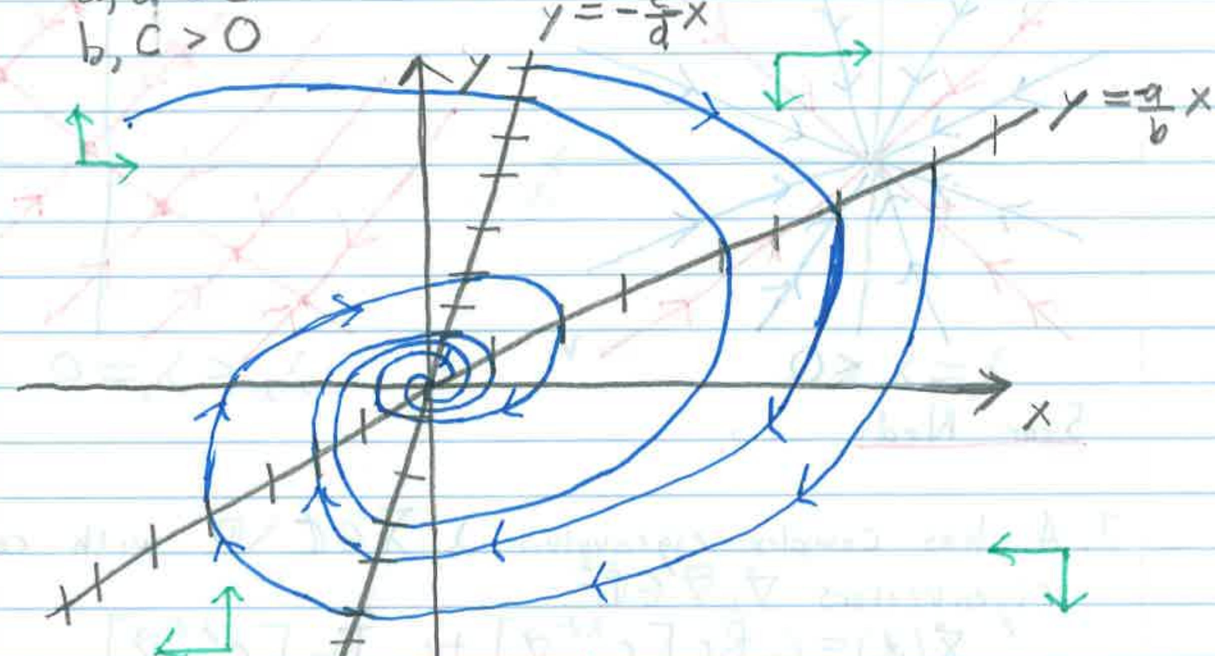
$a, d > 0$
$b, d > 0$

Null-clines are curves where either $\dot{x} = 0$ or $\dot{y} = 0$.



Case 2:

$a, d < 0$
$b, c > 0$

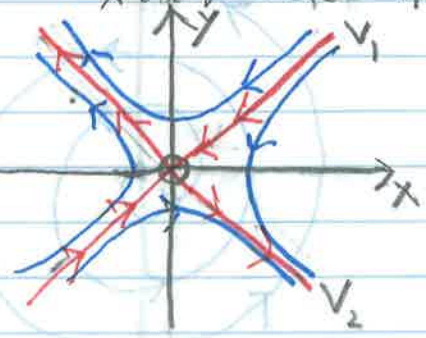


World Peace!!
Everybody focuses on Internal Nation Building.

Eigenvalue analysis:

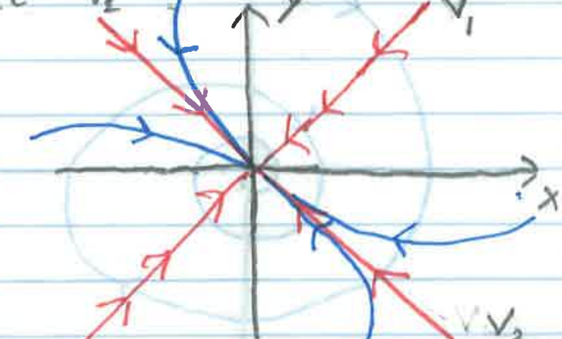
1. $\dot{\vec{x}} = A\vec{x}$, A has two real eigenvalues.

$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$



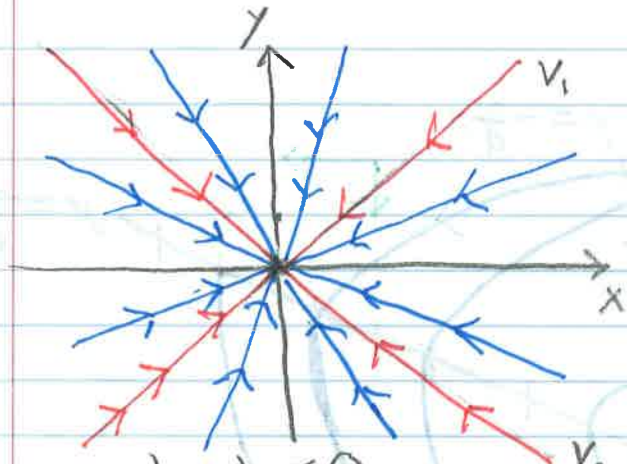
$\lambda_1 < 0 < \lambda_2$

Saddle node



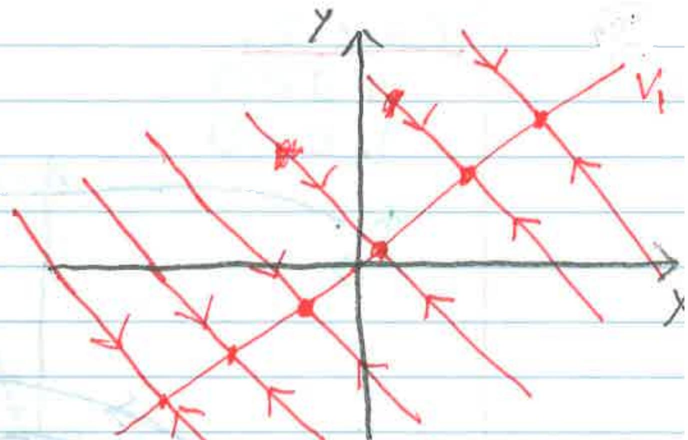
$\lambda_1 < \lambda_2 < 0$

Stable node
(V_1 is the fast direction)



$\lambda_1 = \lambda_2 < 0$

Star Node.



$\lambda_2 < \lambda_1 = 0$

2. A has complex eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C} \setminus \mathbb{R}$ with complex eigenvectors. $\vec{v}, \bar{\vec{v}} \in \mathbb{C}^2$.

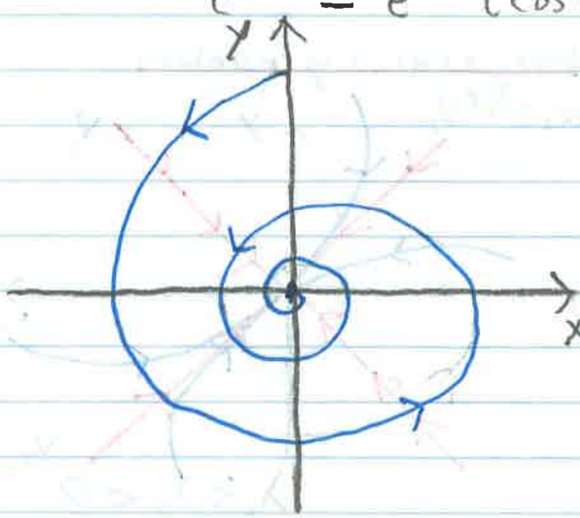
$$\vec{x}(t) = c_1 \,\mathrm{Re}[e^{\lambda t}\vec{v}] + c_2 \,\mathrm{Im}[e^{\lambda t}\vec{v}].$$

If we write

$$\lambda = \nu + i\mu$$

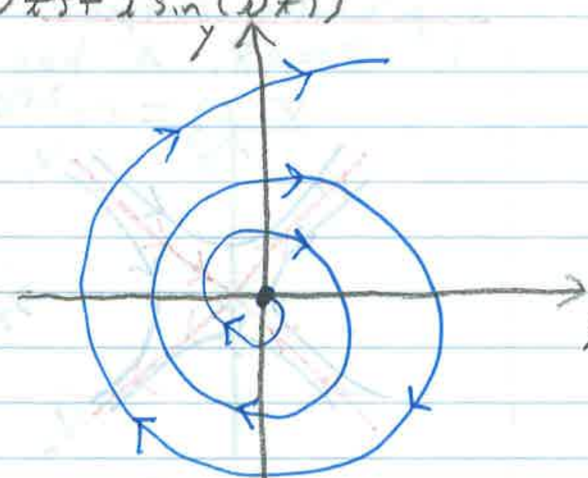
then

$$e^{\lambda t} = e^{\nu t}(\cos(\mu t) + i\sin(\mu t))$$



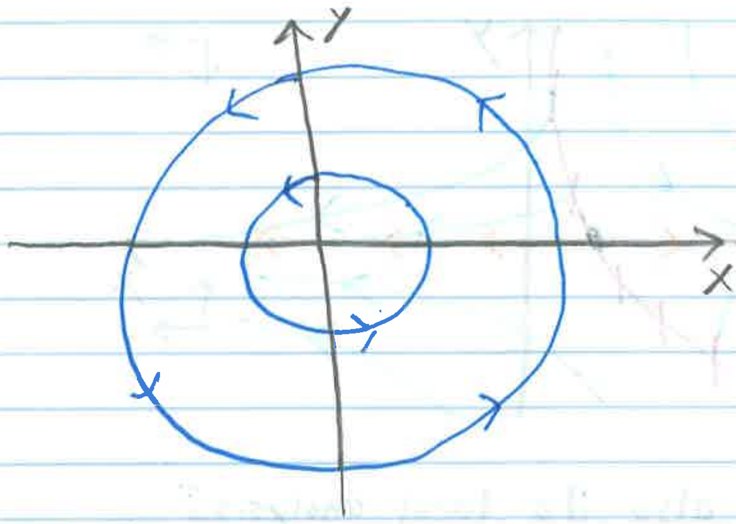
$\mathrm{Re}(\lambda) < 0$

(Stable Spiral).



$\mathrm{Re}(\lambda) > 0$

(Unstable Spiral).



$Re(\lambda)=0$
(center).

Notation.

1. A is called hyperbolic if $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
2. We say that the fixed point $x=0$ of $\dot{x}=Ax$ is
   a.) an attractor if $\vec{x}(t) \to 0$ as $t \to \infty \Rightarrow Re(\lambda_1, \lambda_2) < 0$.
   b.) a repellor if $\vec{x}(t) \to 0$ as $t \to -\infty \Rightarrow Re(\lambda_1, \lambda_2) > 0$.
   c.) a saddle if $\lambda_1 < 0 < \lambda_2$
   d.) nonhyperbolic if $Re(\lambda_1)=0$ or $Re(\lambda_2)=0$.

Example:

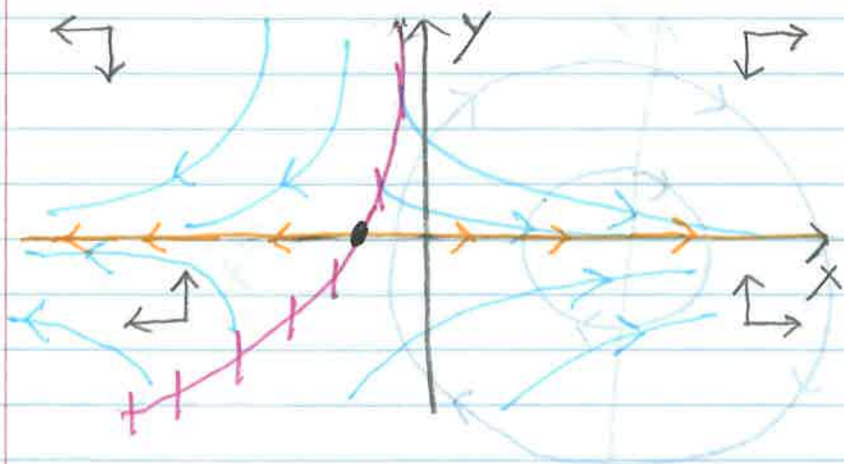
Sketch the phase portrait of
$$\begin{cases} \dot{x} = x + e^{-y} = f(x,y) \\ \dot{y} = -y = g(x,y). \end{cases}$$

Fixed points: $y=0$ and $x=-1$.

Nullclines:

a.) $-x = e^{-y} \Rightarrow y = -\ln(-x)$ N1: $\frac{dx}{dt}=0$

b.) $y=0$ N2: $\frac{dy}{dt}=0$.



We can also do local analysis:

Linearization about $(-1,0)$:

$$f(x,y) = f(\vec{x}^*) + \nabla f|_{\vec{x}^*} \cdot (\vec{x}-\vec{x}^*) + \frac{1}{2}(\vec{x}-\vec{x}^*)^T H(\nabla^2 f)(\vec{x}-\vec{x}^*)$$
$$g(x,y) = g(\vec{x}^*) + \nabla g|_{\vec{x}^*}(\vec{x}-\vec{x}^*) + \frac{1}{2}(\vec{x}-\vec{x}^*)^T H(\nabla^2 f)(\vec{x}-\vec{x}^*).$$

Near equilibrium:

$$F(\vec{x}-\vec{x}^*) \approx J(F)(\vec{x}-\vec{x}^*)$$
$$J(F) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

Changing variables $\tilde{x} = x - x^*$, $\tilde{y} = y - y^*$

$$F(\tilde{\vec{x}}) \approx J(F)|_{\vec{x}^*} \tilde{\vec{x}}$$

Locally, the solution is

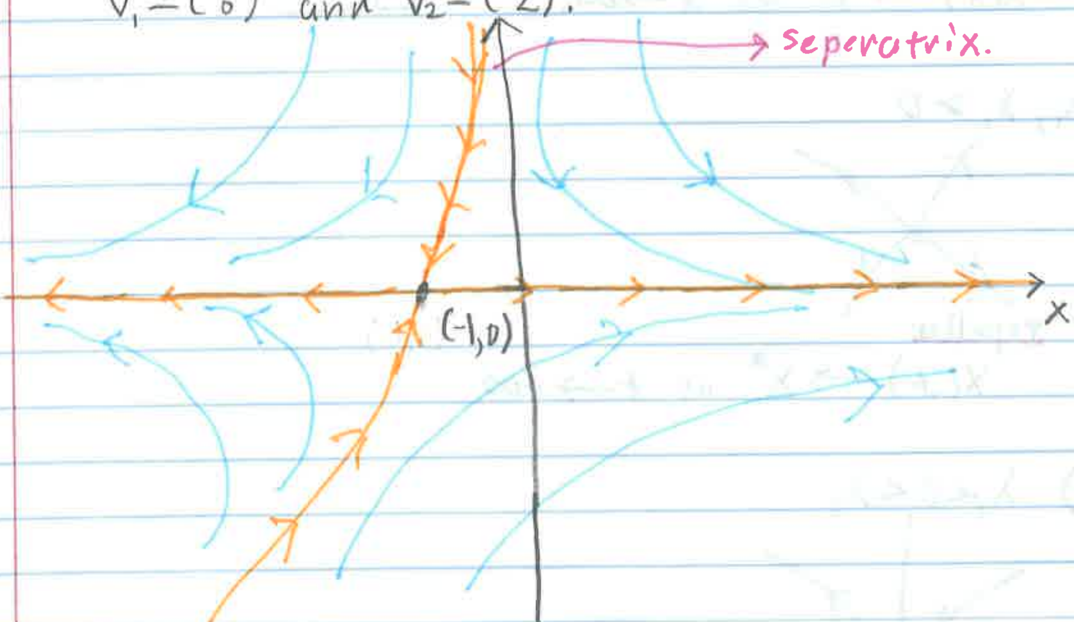
$$\tilde{\vec{x}} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2,$$

where λ_1, λ_2 are the eigenvalues of $J|_{\vec{x}^*}$.

For our problem:

$$J(F)\Big|_{(-1,0)} = \begin{pmatrix} 1 & -e^{-y} \\ 0 & -1 \end{pmatrix}\Bigg|_{(-1,0)} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$ with eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.



Phase Plane Summary:

1. Draw Nullclines
2. Draw direction arrows $\uparrow\!\!\rightarrow$
3. Find equilibrium
4. Calculate Jacobian
5. Determine eigenvalues and eigenvectors,

$\lambda_1, \lambda_2 \in \mathbb{R}$

$\Rightarrow$ hyperbolic points

* Determine fast direction

$\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$

$\Rightarrow$ spirals

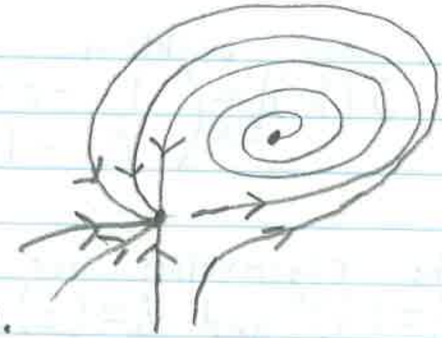
* Determine stable or unstable.

λ_1, λ_2 pure Imaginary.

$\Rightarrow$ Need more work.

λ_1 or $\lambda_2 = 0$

a.) $\lambda_1, \lambda_2 < 0$



attractor

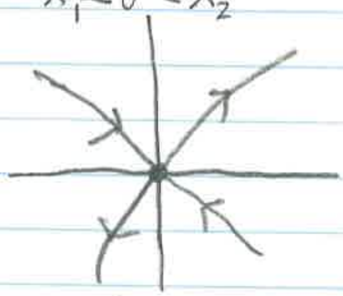
$X(t) \longrightarrow X^*$ as $t \to \infty$.

b.) $\lambda_1, \lambda_2 > 0$

repellor

$X(t) \longrightarrow X^*$ as $t \to -\infty$.

c.) $\lambda_1 < 0 < \lambda_2$



Saddle

Example

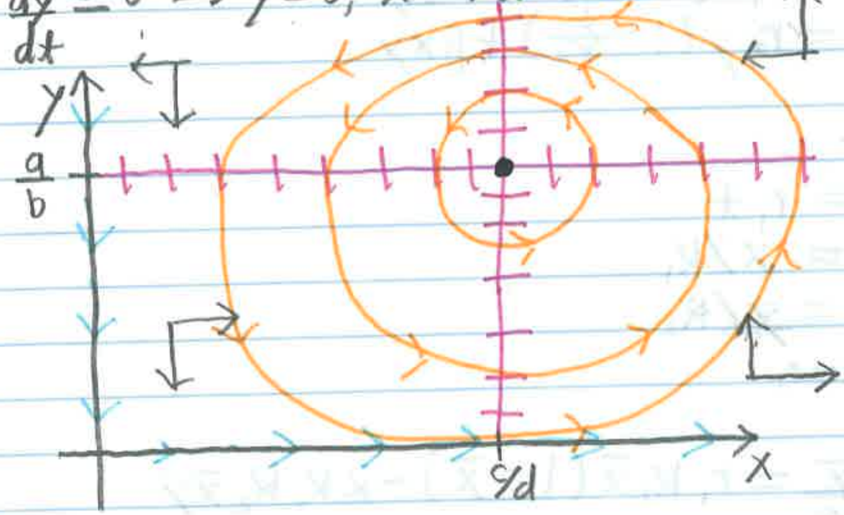
Predator - Prey

$$\dot{x} = ax - bxy$$
$$\dot{y} = -cy + dxy$$

Null Clines:

$$\frac{dx}{dt} = 0 \Rightarrow x=0, \ y = a/b$$

$$\frac{dy}{dt} = 0 \Rightarrow y=0, \ x = c/d$$



Eigenvalues analysis:

$$J = \begin{pmatrix} a-by & -bx \\ dy & -c+dx \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \Rightarrow \lambda_1 = a, \ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \lambda_2 = -c, \ \vec{v}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$J(c/d, a/b) = \begin{pmatrix} 0 & -bc/d \\ da/b & 0 \end{pmatrix}$$

The eigenvalues are pure imaginary $\Rightarrow$ cannot deduce anything.
However,

$$\frac{dy}{dx} = \frac{y}{x} \cdot \frac{(dx-c)}{(a-by)}$$

$$\Rightarrow \int \left(\frac{a}{y} - b\right) dy = \int \left(-\frac{c}{x} + d\right)$$

$$\Rightarrow a\ln(y) - by = -c\ln(x) + dx + C$$

$$a\ln(y) - by + c\ln(x) - dy = C$$

$$y^d e^{-by} x^c e^{-dy} = C$$

This describes a family of closed curves $\Rightarrow$ the solution

Example:

Logistic growth omnivores and prey.

$$\dot{x} = r_1 x\left(1-\frac{x}{K_1}\right) - \alpha x y$$
$$\dot{y} = r_2 y\left(1-\frac{y}{K_2}\right) + \beta x y$$

Rescale

$\tau = r_1 t$

$\bar{x} = x/K_1$

$\bar{y} = y/K_2$

Therefore,

$$r_1 K_1 \frac{d\bar{x}}{d\tau} = r_1 K_1 \bar{x}(1-\bar{x}) - \alpha K_1 K_2 \bar{x}\bar{y}$$
$$r_1 K_2 \frac{d\bar{y}}{d\tau} = r_2 K_2 \bar{y}(1-\bar{y}) + \beta K_1 K_2 \bar{x}\bar{y}$$

$$\Rightarrow \frac{d\bar{x}}{d\tau} = \bar{x}(1-\bar{x}) - \gamma \bar{x}\bar{y}$$
$$\frac{d\bar{y}}{d\tau} = \eta \bar{y}(1-\bar{y}) + \delta \bar{x}\bar{y}$$

Equilibrium points:

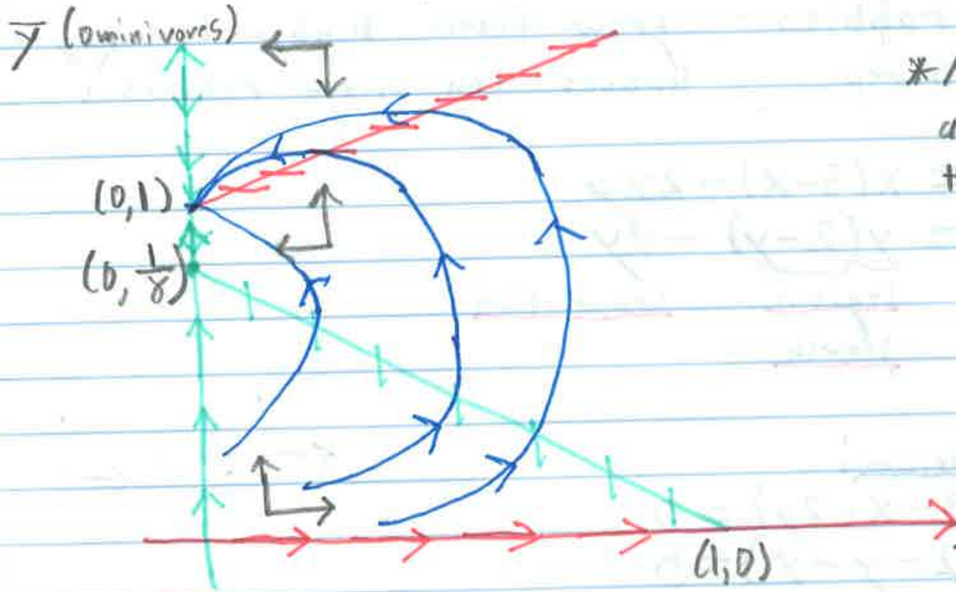
$$0 = \bar{x}(1-\bar{x}) - \gamma \bar{x}\bar{y}$$
$$0 = \eta \bar{y}(1-\bar{y}) + \delta \bar{x}\bar{y}$$

$\Rightarrow (\bar{x}, \bar{y}) = (0, 0)$, is one equilibrium. There are others.

Null clines:

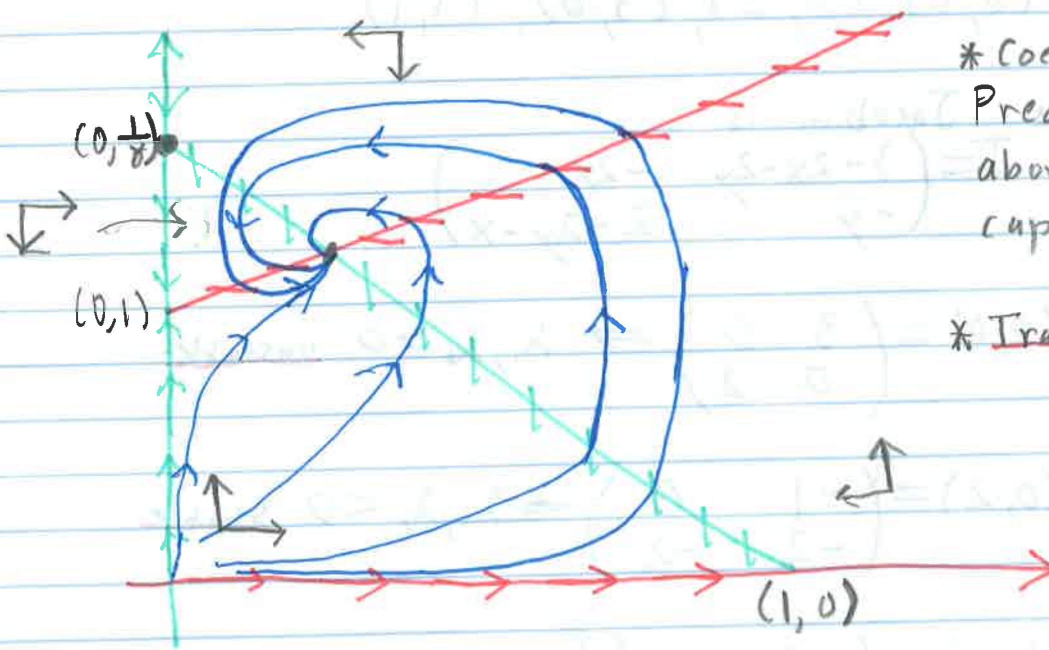
N1: $\bar{y} = \frac{1-\bar{x}}{\gamma}$, $\bar{x} = 0$: $\frac{dx}{d\tau} = 0$.

N2: $\bar{x} = \frac{\eta}{\delta}(\bar{y}-1)$, $\bar{y} = 0$: $\frac{dx}{d\tau} = 0$



*All prey die off and predators go to carrying capacity.

Omnivores take over



*Coexistence. Predators live above carrying capacity.

*Transcritical bifurca

Point of intersection:

$$\bar{x}^* = \frac{(1-\gamma)\eta}{\gamma\delta+\eta} \qquad \bar{y}^* = \frac{\delta+\eta}{\gamma\delta+\eta}$$

The Jacobian is a mess but it is complex valued at the critical point.

$\Longrightarrow$ Stable spiral

Example: (Competition)

$x \sim$ rabbits , grow faster higher carrying capacity

$y \sim$ sheep , hooves can smash rabbits.

$$\dot{x} = x(3-x) - 2xy$$
$$\dot{y} = \underbrace{y(2-y)}_{\text{logistic growth}} - \underbrace{xy}_{\text{Competition}}$$

Equilibrium:

$$x(3-x-2y)=0$$
$$y(2-y-x)=0$$

We get four equilibrium:

$(0,0)$, $(0,2)$, $(3,0)$, $(1,1)$

The Jacobian is

$$J = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-2y-x \end{pmatrix}$$

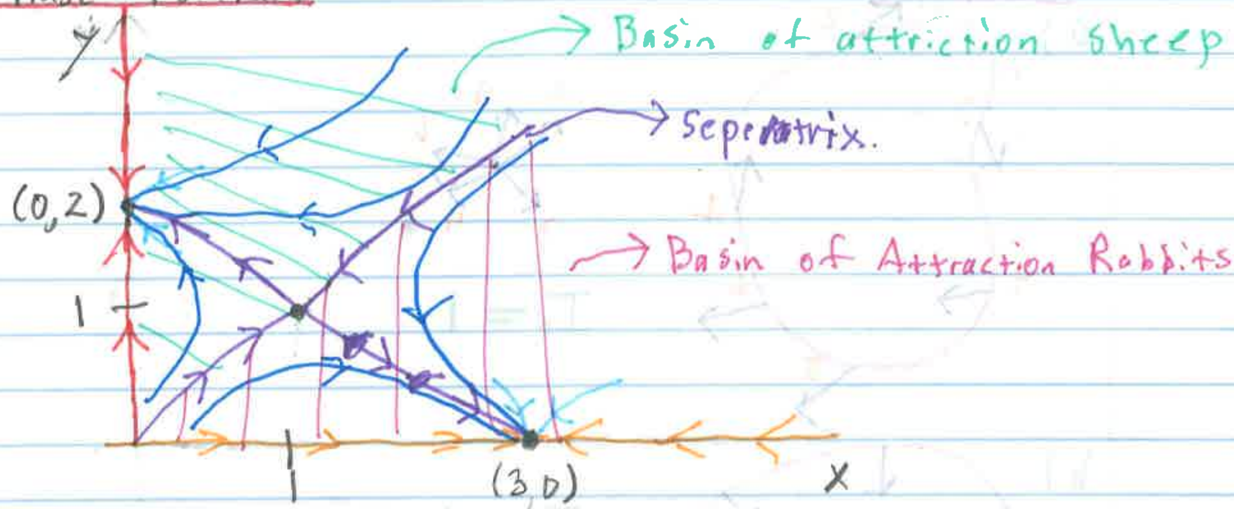
$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1, \lambda_2 > 0 \quad \text{unstable}$$

$$J(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \Rightarrow \lambda_1, \lambda_2 < 0 \quad \text{stable}$$

$$J(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \Rightarrow \lambda_1, \lambda_2 < 0 \quad \text{stable}$$

$$J(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \Rightarrow \lambda_1, \lambda_2 = -1 \pm \sqrt{2} \quad \text{saddle}$$

Phase Portrait



Index Theory

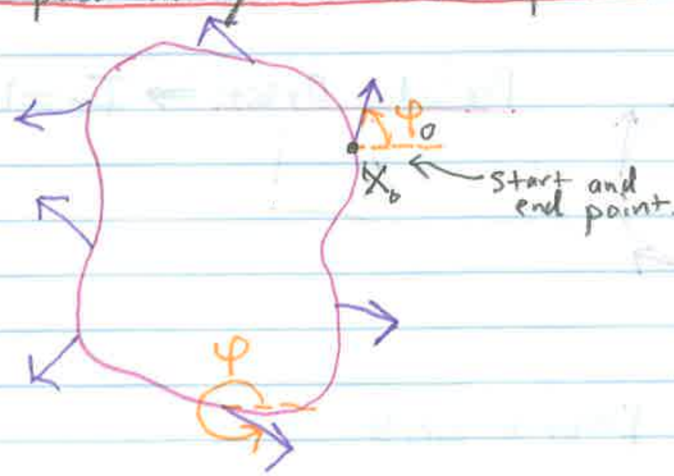
How can we be sure no periodic orbits exist?

Consider

$$\dot{\vec{x}} = F(\vec{x})$$

with $F: \mathbb{R}^2 \to \mathbb{R}^2$ continuously differentiable.

Take a closed curve Γ with no self intersections, that does not pass through a fixed point.



1. Start at v_0, traverse Γ counterclockwise and take angle φ of $F(\vec{x})$. $\to$ this angle changes continuously as Γ is traversed.
2. After one pass we again end up at v_0 with an angle $\varphi_1 = \varphi_0 + 2\pi n$; $n \in \mathbb{Z}$

$$\boxed{I_\Gamma = \frac{1}{2\pi}(\varphi_1 - \varphi_0)}$$