Chapter 4: Flows on the Circle

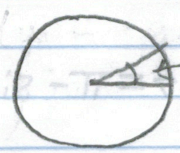
To fully describe a differential equation

$$\dot{x} = f(x)$$

one must also define the space the solution curves live on.

Examples:

1. $\mathbb{R}$ - position of a car on a straight track
2. $\mathbb{R}^+$ - population growth
3. S^1 (unit circle) - motion on circular track, angles.



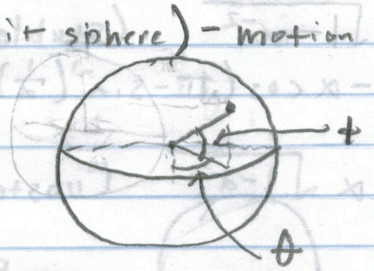
$\theta \rightarrow$ This labels a point

$\theta = 0, 2\pi$ are two labels for the same point.

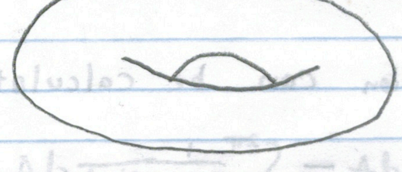
$\dot{\theta} = \theta^2$ cannot be vector field on S^1.

A vector field on S^1 must satisfy $\dot{\theta} = f(\theta + 2n\pi)$ for all $n \in \mathbb{Z}$.

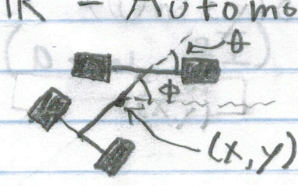
4. S^2 (unit sphere) - motion on earth.



5. $S^1 \times S^1$ (torus) - Two angles.



6. $S^1 \times S^1 \times \mathbb{R}^2$ - Automobile



We will concentrate on the circle.

Example:

$$\dot{\theta} = \omega - a\sin(\theta)$$

$\rightarrow$ Phase locking, θ is a phase difference.

Rescale:

$$\tau = \omega t$$

$$\Rightarrow \frac{d\theta}{d\tau} = 1 - \alpha\sin(\theta), \quad \alpha = a/\omega.$$

Fixed points:

$$\theta = \sin^{-1}\left(\frac{1}{\alpha}\right) \quad \text{and} \quad \theta = \pi - \sin^{-1}\left(\frac{1}{\alpha}\right)$$

if $|\alpha| \geq 1$. Stability analysis

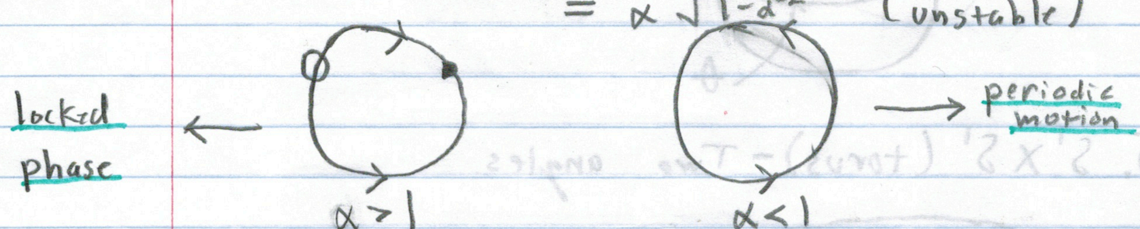
$$\frac{df}{d\theta} = -\alpha\cos(\theta)$$

$$\Rightarrow \left.\frac{df}{d\theta}\right|_{\sin^{-1}(\frac{1}{\alpha})} = -\alpha\cos\left(\sin^{-1}\left(\tfrac{1}{\alpha}\right)\right)$$

$$= -\alpha\sqrt{1-\alpha^{-2}} \quad \text{(stable)}$$

$$\left.\frac{df}{d\theta}\right|_{\pi-\sin^{-1}(\frac{1}{\alpha})} = -\alpha\cos\left(\pi - \sin^{-1}\left(\tfrac{1}{\alpha}\right)\right)$$

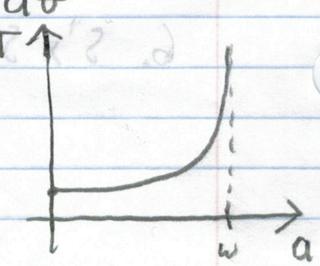
$$= \alpha\sqrt{1-\alpha^{-2}} \quad \text{(unstable)}$$



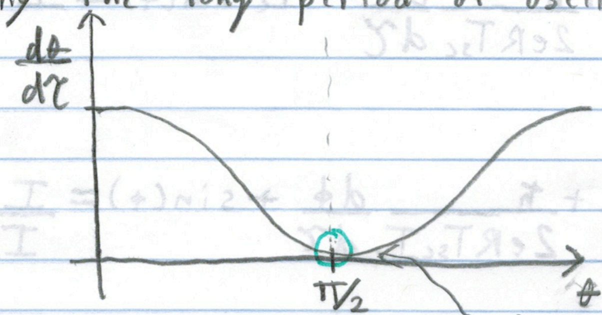
The period of oscillation can be calculated:

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{d\theta}\, d\theta = \int_0^{2\pi} \frac{1}{\omega - a\sin\theta}\, d\theta$$

$$\Rightarrow T = \frac{2\pi}{\sqrt{\omega^2 - a^2}} \quad (\text{If } \omega > a)$$



Why the long period of oscillation?



$a > 1$, and $a \approx 1$.

$\frac{d\theta}{d\tau} \approx 0 \Rightarrow$ very slow dynamics.

Near $\theta = \pi/2$, $1 - \alpha\sin\theta \approx 1 - \alpha + \alpha\frac{(\theta - \pi/2)^2}{2}$

$$\Rightarrow \frac{d\theta}{d\tau} \approx 1 - \alpha + \frac{\alpha\theta^2}{2}$$

The passage through the slow period can be estimated:

$$T_{passage} \approx \int_{-\infty}^{\infty} \frac{1}{1 - \alpha + \frac{\alpha}{2}x^2}\, dx = \frac{\sqrt{2}\pi}{\sqrt{\alpha}\sqrt{1-\alpha}} \approx \frac{\sqrt{2}\pi}{\sqrt{1-\alpha}}$$

This is known as a square root scaling law.

L8

Example:

$$\frac{\hbar C}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} + I_c\sin\phi = I$$ (superconducting Josephenson junction)

$[I] = I$ (current)

$\left[\frac{\hbar}{2eR}\right] = \frac{I}{T}$

$\left[\frac{\hbar C}{2e}\right] = \frac{I}{T^2}$

Rescale by $\tau = \frac{t}{T_{sc}}$

$$\Rightarrow \frac{\hbar C}{2eT_{sc}^2}\frac{d^2\phi}{d\tau^2} + \frac{\hbar}{2eRT_{sc}}\frac{d\phi}{d\tau} + I_c\sin(\phi) = I$$

Normalizing

$$\frac{\hbar C}{2eT_{sc}^2 I_c}\frac{d^2\phi}{d\tau^2} + \frac{\hbar}{2eRT_{sc}I_c}\frac{d\phi}{d\tau} + \sin(\phi) = \frac{I}{I_c}$$

We want

$$\frac{\hbar}{2eRT_{sc}I_c} = \mathcal{O}(1)$$

$$\Rightarrow \boxed{T_{sc} = \frac{\hbar}{2eRI_c}}$$

We obtain the system:

$$\frac{\cancel{\hbar} C}{\cancel{2e}\cdot\frac{\hbar^{\cancel{2}} I_c}{2^{\cancel{2}}e^{\cancel{2}}R^2I_c^2}}\frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} + \sin(\phi) = \gamma$$

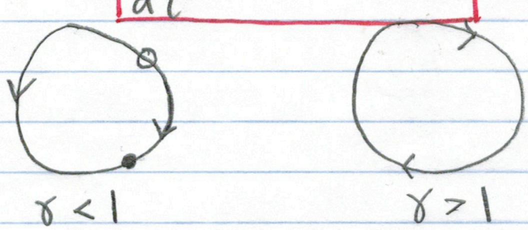
$$\Rightarrow \beta\frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} + \sin(\phi) = \gamma$$

$$\boxed{\beta = \frac{2eI_cR^2C}{\hbar}}$$

If $\beta << 1$ we have the system

$$\frac{d\phi}{d\tau} + \sin(\phi) = \gamma$$

$$\Rightarrow \boxed{\frac{d\phi}{d\tau} = \gamma - \sin(\phi)}$$



← periodic motion

$\gamma < 1$ $\gamma > 1$

What is the average velocity?

$$\left\langle \frac{d\phi}{dt} \right\rangle = \frac{1}{T_{per}}\int_0^{T_{per}} \frac{d\phi}{dt}\,dt = \frac{1}{T_{per}}\cdot 2\pi$$

where

$$T_{per} = \int_0^{T_{per}} dt$$

$$= \int_0^{2\pi} \frac{1}{\gamma - \sin(\phi)}\, dt$$

$$= \frac{2\pi}{\sqrt{\gamma^2 - 1}}$$

$$\Rightarrow \left\langle \frac{d\phi}{dt} \right\rangle = \sqrt{\gamma^2 - 1}, \text{ if } \gamma > 1.$$

Example:

$\dot{\phi} = \Omega$ entrainment frequency

$\dot{\theta} = \omega + A\sin(\phi - \theta)$

natural frequency — frequency update.

Let $\varphi = \phi - \theta$

$$\Rightarrow \dot{\varphi} = \Omega - \omega - A\sin(\varphi)$$

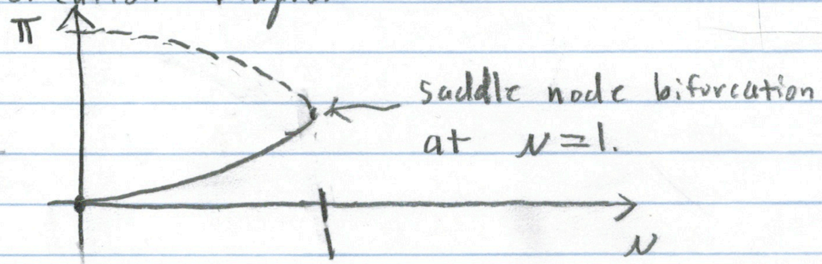
Rescale

$$\tau = At$$

and let $\mu = \frac{\Omega - \omega}{A}$. This gives the system

$$\frac{d\varphi}{d\tau} = \mu - \sin(\varphi)$$

Bifurcation Diagram



When $0 < \mu < 1$ we get entrainment. The range of entrainment is

$$\omega - A \leq \Omega \leq \omega + A.$$

This interval is the <u>range of entrainment</u>.

If $\mu > 1$, the <u>period of phase drift</u> is given by:

$$T_{drift} = \int_0^{T_{drift}} dt = \int_0^{2\pi} \frac{dt}{d\phi} d\phi = \int_0^{2\pi} \frac{d\phi}{\Omega - \omega - A\sin(\phi)}$$

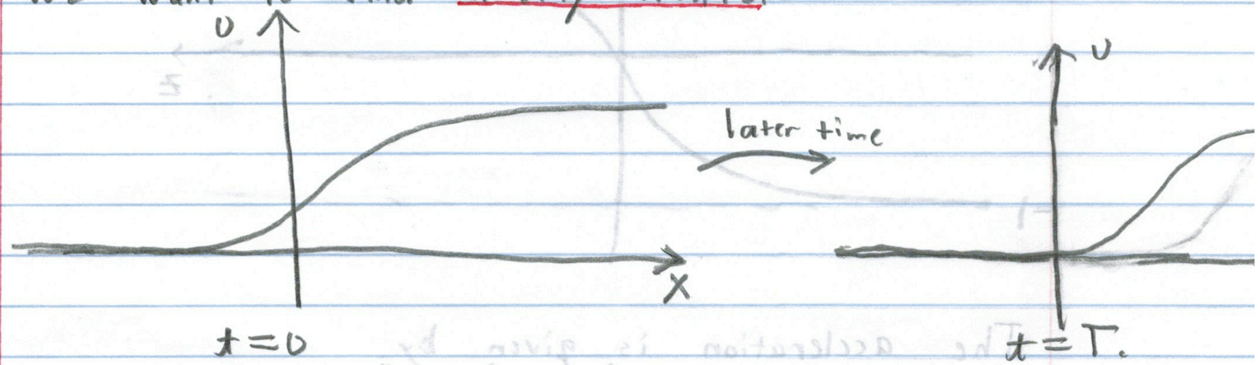
$$\Rightarrow \boxed{T_{drift} = \frac{2\pi}{\sqrt{(\Omega-\omega)^2 - A^2}}}$$

Moving Front

We want to study solutions to the P.D.E.

$$U_t = K U_x U^3 + 1 - U^2$$

We want to find moving fronts.



Move to a coordinate system $Z = x - ct$ that moves with the front and to the right.

$$Z = x - ct$$

$$\Rightarrow -cU_Z = K U_Z U^3 + U^2 - 1$$

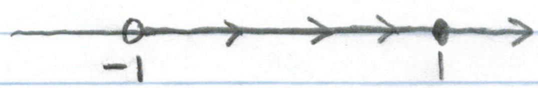
$$\Rightarrow U_Z = \frac{1-U^2}{c+KU^3}$$

Fixed points occur at $U = \pm 1$. Lets analyze stability:

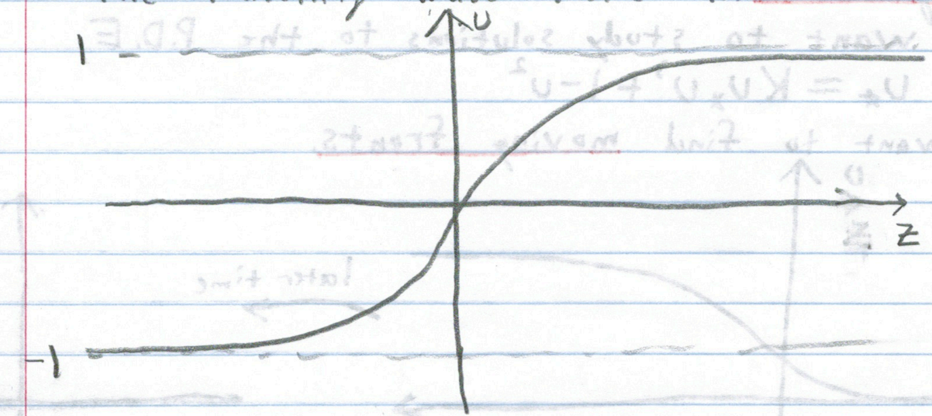
$$\frac{d}{dZ} U_Z = \frac{-2U}{c+KU^3} - \frac{(1-U^2)\cdot 3U^2}{(c+KU^3)^2}$$

$$\Rightarrow \frac{d}{dZ} U_Z \Big|_{\pm} = \frac{\mp 2}{c \pm K}$$

If $c > K$ then for $-1 < U < 1$ we have the following phase portrait



The travelling wave looks like:



The accleration is given by

$$u_{tt} = \frac{d}{dt}\left(-c u_z\right) = c^2 u_{zz}$$

Now,

$$c^2 u_{zz} = c^2 \frac{d}{dz} u_z = c^2 \frac{du}{dz} \frac{d}{du} u_z$$

$$\Rightarrow u_{tt} = c^2 \left(\frac{1-u^2}{Ku^3+c}\right)\left(\frac{-2u}{(c+Ku^3)^2}\right)$$