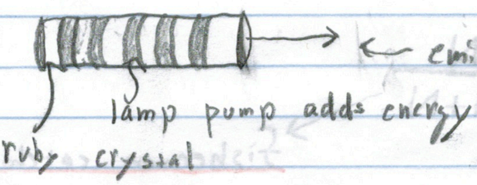
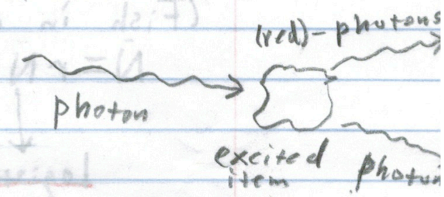
Example:





$$\dot{n} = GnN - Kn$$
$$\dot{N} = -GnN - fN + p$$

$n \sim$ # of photons
$N \sim$ # of excited atoms
$K \sim$ decay rate due to transmission
$f \sim$ decay rate due to spontaneous emission
$p \sim$ pump strength.

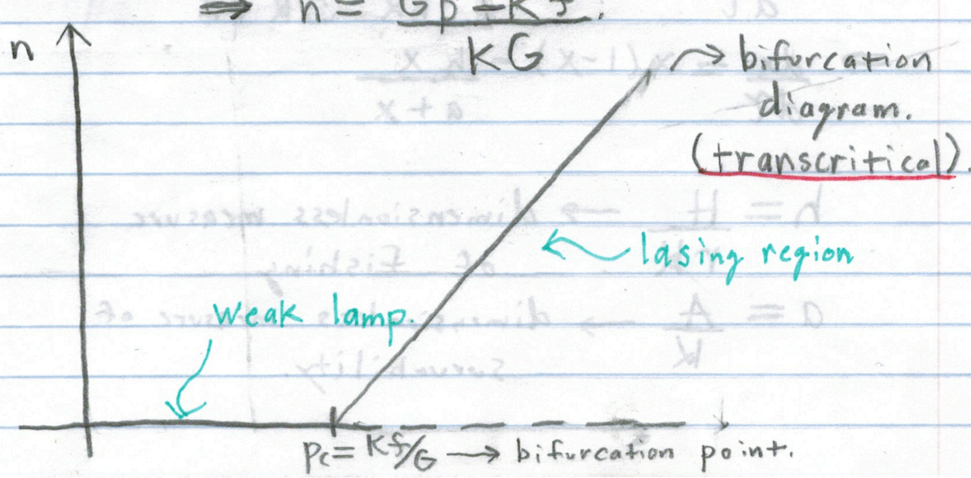
Assumption $- \dot{N} \approx 0$ (quasistatic approximation)

$$p = N(Gn + f)$$
$$\Rightarrow N = \frac{p}{Gn+f}$$
$$\dot{n} = \frac{Gpn}{Gn+f} - Kn$$

Fixed points:

$n = 0$, $\quad Gp - K(Gn+f) = 0$

$$\Rightarrow n = \frac{Gp - Kf}{KG}.$$



Example:

(Fish in a pond)

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\frac{N}{A+N}$$

$rN(1-\frac{N}{K})$ → logistic growth

$H\frac{N}{A+N}$ → fisherman catching fish.

$N \sim$ # of fish , $[N] \sim N$

$r \sim$ growth rate , $[r] \sim T^{-1}$

$K \sim$ carraying capacity , $[K] \sim N$,

$H \sim$ rate of fishing , $[H] \sim T^{-1}N$

$A \sim$ survivability of fish , $[A] \sim N$

There are a lot of parameters! Dimension reduction can simplify analysis:

$\tau = rt$ → dimensionless time

$x = K^{-1}N$ → dimensionless population

$$\frac{d}{dt}N = \frac{d}{dt}(Kx) = K\frac{d}{dt}x = K\frac{d\tau}{dt}\frac{d}{d\tau}x = Kr\frac{dx}{d\tau}$$

$$\Rightarrow Kr\frac{dx}{d\tau} = rKx(1-x) - \frac{HKx}{A+Kx}$$

$$\Rightarrow \frac{dx}{d\tau} = x(1-x) - \frac{H}{rK}\frac{x}{A/K + x}$$

$$\Rightarrow \frac{dx}{d\tau} = x(1-x) - \frac{hx}{a+x}$$

$h = \frac{H}{rK}$ → dimensionless measure of fishing

$a = \frac{A}{K}$ → dimensionless measure of survability.

Fixed Points:

$x^*=0$, $(a+x)(1-x)-h=0$

$$\Rightarrow -x^2+(1-a)x+(a-h)=0$$

$$x^*=\frac{(1-a)\pm\sqrt{(1-a)^2+4(a-h)}}{2}=\frac{(1-a)\pm\sqrt{(1+a)^2-4h}}{2}$$

Stability of $x^*=0$:

$$\frac{df}{dx}=1-2x-\frac{h}{a+x}+\frac{x}{(a+x)^2}$$

$$\Rightarrow \left.\frac{df}{dx}\right|_{0}=1-\frac{h}{a}$$

a.) If $h>a$, stable $\Rightarrow$ fish can go extinct.

b.) If $h<a$, unstable $\Rightarrow$ fish can survive.

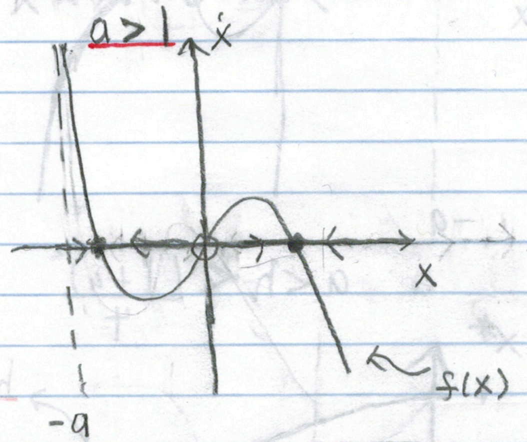
Other Fixed Points:

Exist if

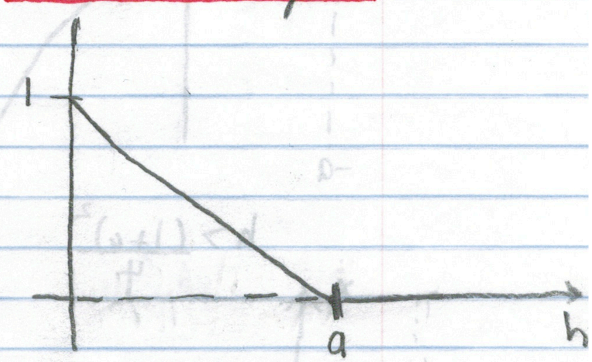
$$h<\frac{(1+a)^2}{4}$$

Case 1:

$a>1$



Bifurcation diagram:



$a>1$

Transcritical bifurcation.

$(1-a)+\sqrt{(1-a)^2+4(a-h)}>0$

$\Rightarrow \sqrt{(1-a)^2+4(a-h)}>-(1-a)$

True if and only if $h<a$.

This condition implies exactly that there is one root to the left and one to the right of 0.

Case 2:

$a < 1$

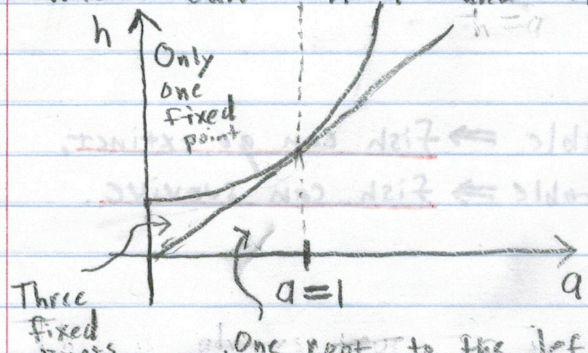
$(1-a) \pm \sqrt{(1-a)^2 + 4(a-h)} > 0$

This is the condition that both roots are to the right of zero.

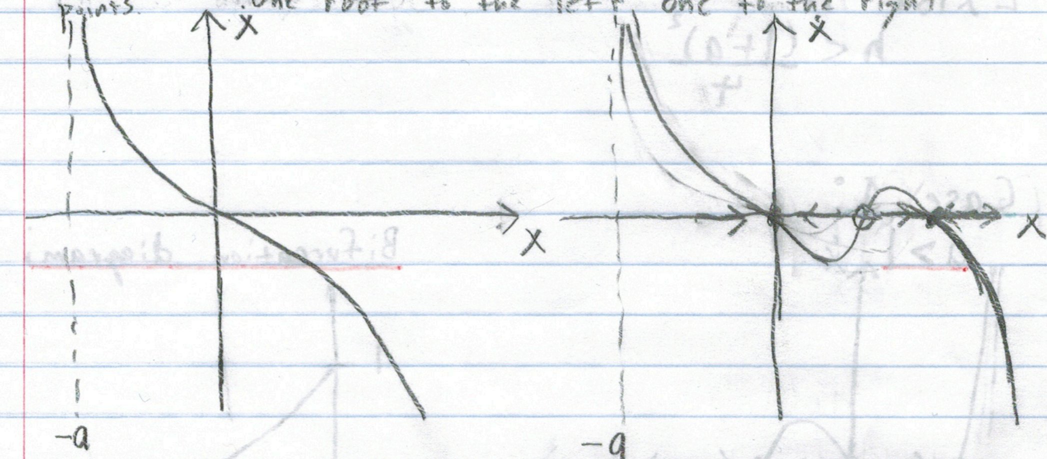
$\Rightarrow (1-a) > \sqrt{(1-a)^2 + 4(a-h)}$

$\Rightarrow \quad h > a$

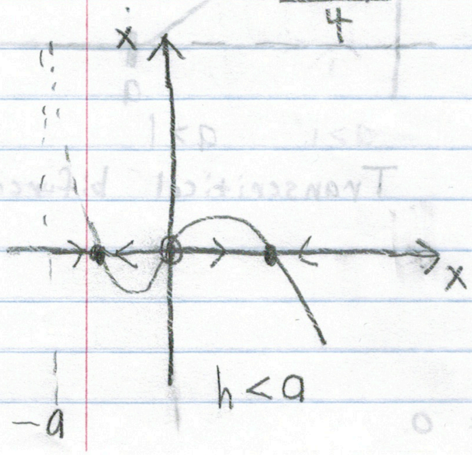
When can $h > a$ and $h < \frac{(1+a)^2}{4}$

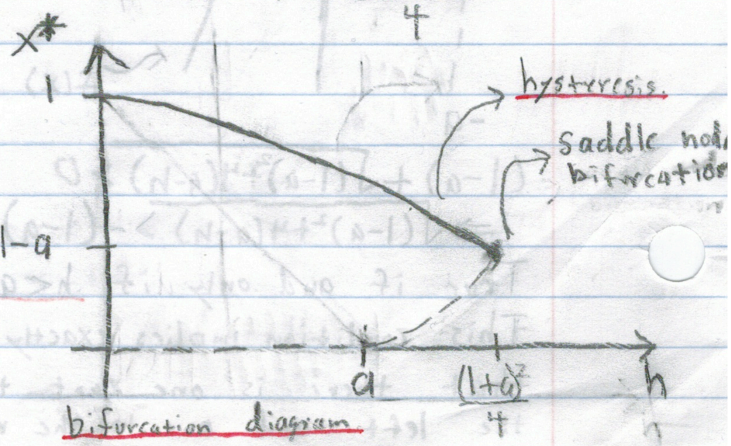


One root to the left, one to the right



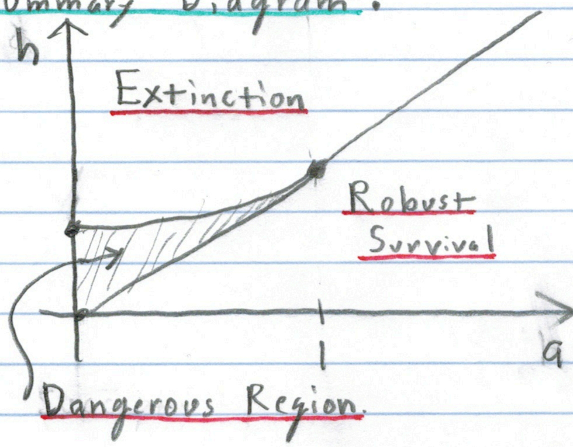
$h > \frac{(1+a)^2}{4}$ $\qquad$ $a < h < \frac{(1+a)^2}{4}$





bifurcation diagram

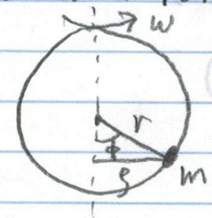
Summary Diagram:



L6

Example:

Bead on a rotating hoop.



$$m r \ddot{\phi} = -b\dot{\phi} - mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi) = 0$$

net force   friction   gravity   centrifugal

$[m] = M$

$[r] = L$

$[b] = T^{-1} L M$

$[g] = L T^{-2}$

$[\omega] = T^{-1}$

Rescale:

$$\tau = T_{sc}\, t$$

We will choose T_{sc} later.

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau}$$

$$\Rightarrow m r T_{sc}^2 \frac{d^2\phi}{d\tau^2} = -b\, T_{sc}\frac{d\phi}{d\tau} - mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi) = 0.$$

Divide by mg to nondimensionalize:

$$\frac{r}{g} T_{sc}^2 \frac{d^2\phi}{d\tau^2} = -\frac{b}{mg} T_{sc}\frac{d\phi}{d\tau} - \sin(\phi) + \frac{r\omega^2}{g}\sin(\phi)\cos(\phi) = 0.$$

In order to reduce to a first order system we need

$$\frac{b}{mg} T_{sc} = O(1) \quad \text{and} \quad \frac{r}{g} T_{sc}^2 \ll 1$$

$$\Rightarrow \boxed{T_{sc} = \frac{mg}{b} \Rightarrow \varepsilon = \frac{rm^2g}{b} \ll 1}$$

$$\Rightarrow \varepsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin(\phi) + \gamma\sin(\phi)\cos(\phi)$$

$$\boxed{\gamma = \frac{r\omega^2}{g}}$$

The 1-D system is then

$$\frac{d\phi}{d\tau} = -\sin(\phi) + \gamma \sin(\phi)\cos(\phi) = -\sin(\phi) + \frac{\gamma}{2}\sin(2\phi).$$

$$\frac{d\phi}{d\tau} = \sin(\phi)\gamma\left(\cos(\phi) - \frac{1}{\gamma}\right)$$

Fixed points:

$\phi = 0, \pi$, $\cos(\phi) = \frac{1}{\gamma}$

$$\left.\frac{df}{d\phi}\right|_0 = -1 + \gamma$$

$\Rightarrow$ $\gamma > 1$, $\phi = 0$ is unstable

$\gamma < 1$, $\phi = 0$ is stable

$$\left.\frac{df}{d\phi}\right|_\pi = 1 + \gamma$$

$\Rightarrow$ $\phi = \pi$ is unstable.

$\cos(\phi) = \frac{1}{\gamma}$