Chapter 3: Bifurcations

Systems with parameters can have drastic qualitative changes as a parameter is varied.

Examples:

* Euler Buckling, * Turbulence * outbreaks of epidemics.
* catastrophic environmental collapse.

Framework:

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}, \ \mu \in \mathbb{R} \text{ parameter}$$

Study behaviour of fixed points under parameter changes.
Suppose $x^*(\mu)$ is a fixed point.

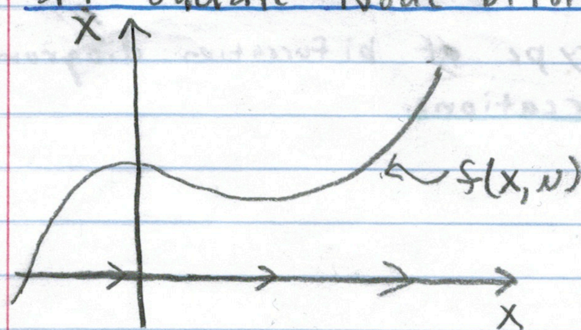
1. $\left.\dfrac{\partial f}{\partial x}\right|_{(x^*(\mu), \mu)} > 0 \rightarrow$ unstable fixed point.

2. $\left.\dfrac{\partial f}{\partial x}\right|_{(x^*(\mu), \mu)} < 0 \rightarrow$ stable fixed point.

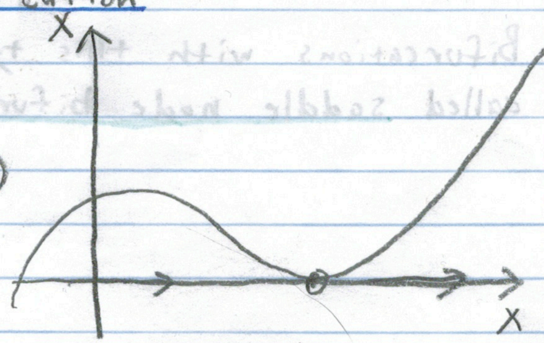
3. $\left.\dfrac{\partial f}{\partial x}\right|_{(x^*(\mu), \mu)} = 0 \rightarrow$ possibly semistable.

We can think of x^* as a (multivalued) function of μ.

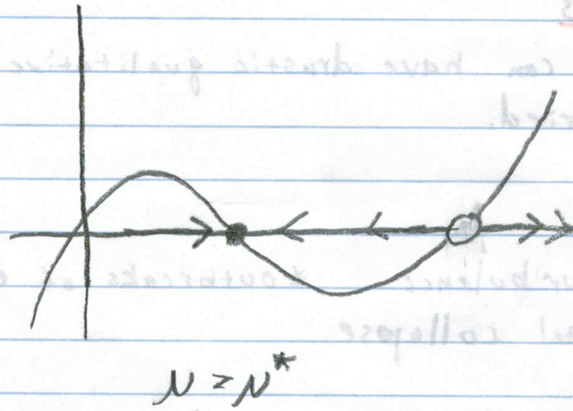
3.1 Saddle-Node Bifurcation



$\mu < \mu^*$



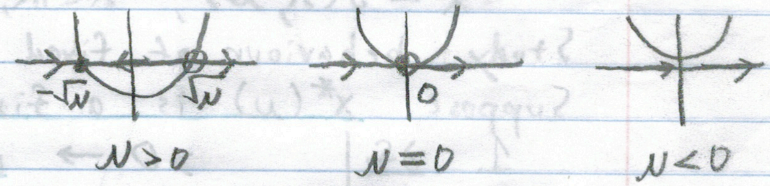
$\mu = \mu^*$



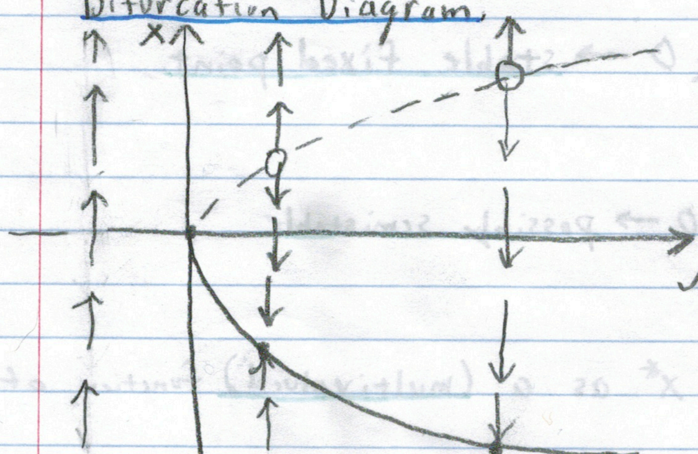
The bifurcation point is the value of μ where the fixed point changes stability.

Protypical Example:

$\dot{x} = -\mu + x^2$



Bifurcation Diagram:



location of fixed points as a function of μ. The phase portraits can be reconstructed

* The bifurcation diagram does not typically contain the arrows

Bifurcations with this type of bifurcation diagram are called saddle node bifurcations.

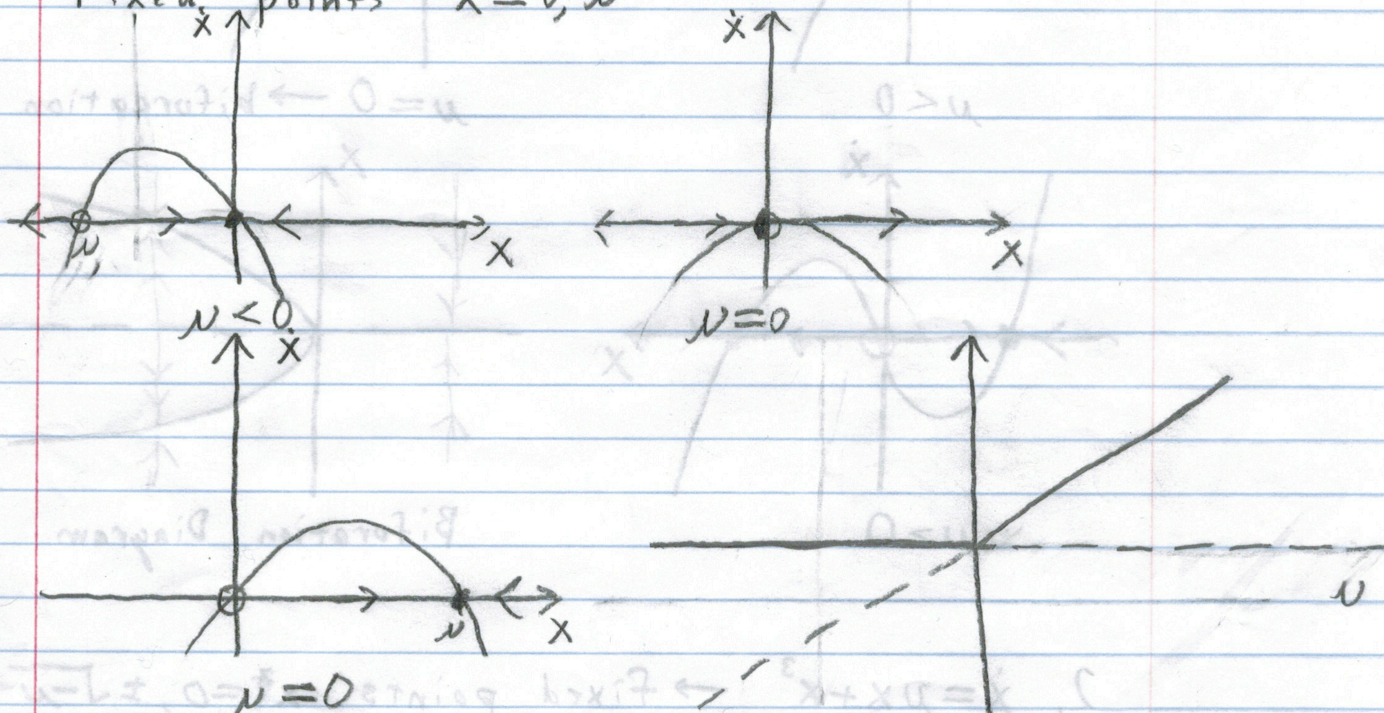
3.2 Transcritical Bifurcation

Many systems have a fixed point that cannot change position (population $P=0$, for example). However, the stability of the fixed point can change.

Typical Example

$$\dot{x} = \nu x - x^2$$

Fixed points $x = 0, \nu$

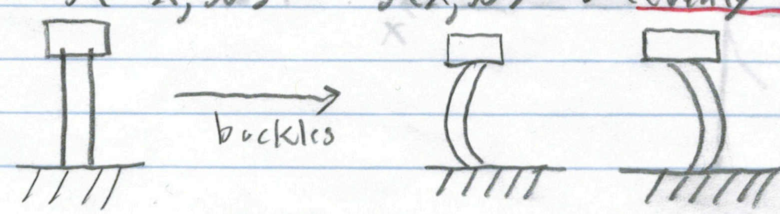


Bifurcation diagram.

3.4 Pitchfork Bifurcation

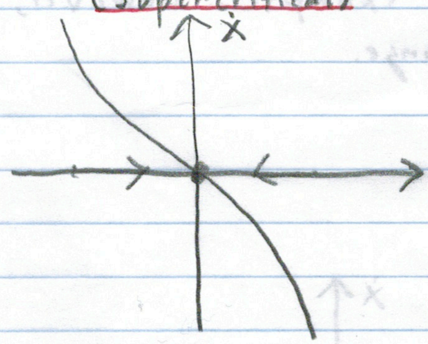
Many physical systems have odd symmetry:

$$f(-x,\nu) = -f(x,\nu) \Rightarrow \text{(evenly symetric potential energy)}$$

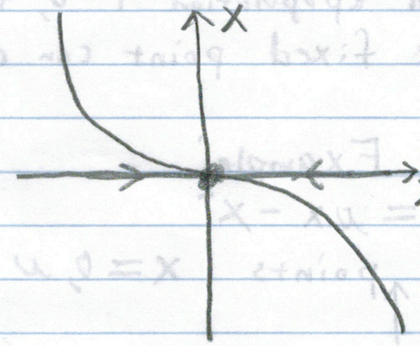


Prototypical Examples:

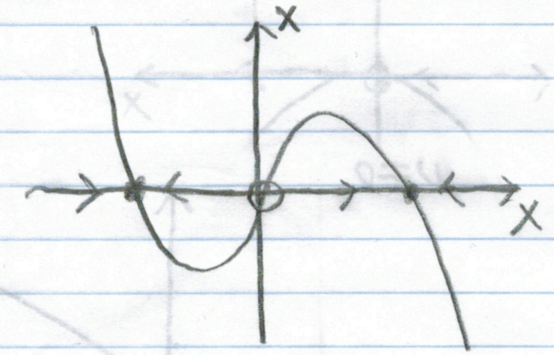
1. $\dot{x} = \mu x - x^3 \rightarrow$ fixed points: $x^* = 0, \pm\sqrt{\mu}$
(supercritical)



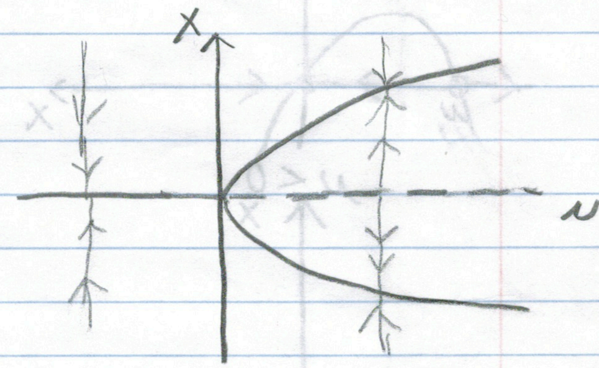
$\mu < 0$



$\mu = 0 \rightarrow$ bifurcation point

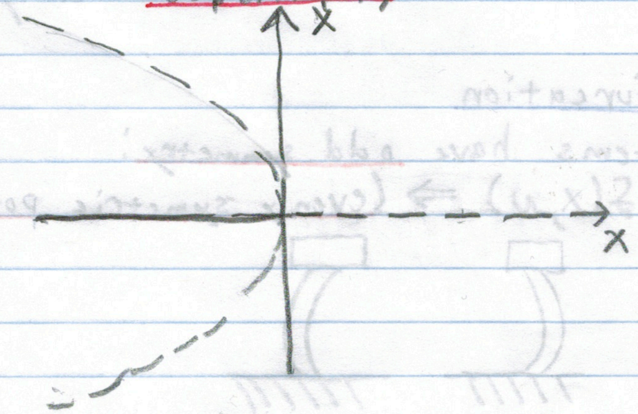


$\mu > 0$



Biforation Diagram

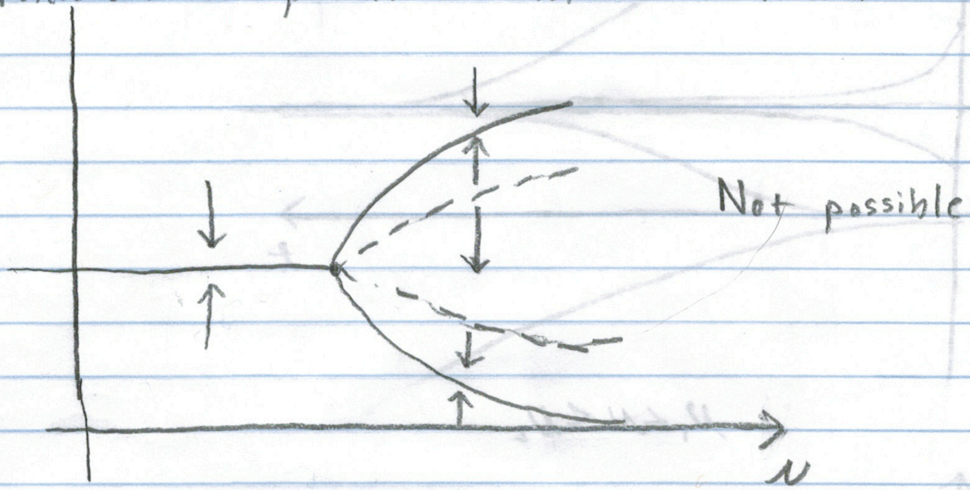
2. $\dot{x} = \mu x + x^3 \rightarrow$ fixed points: $x^* = 0, \pm\sqrt{-\mu}$
(subcritical)

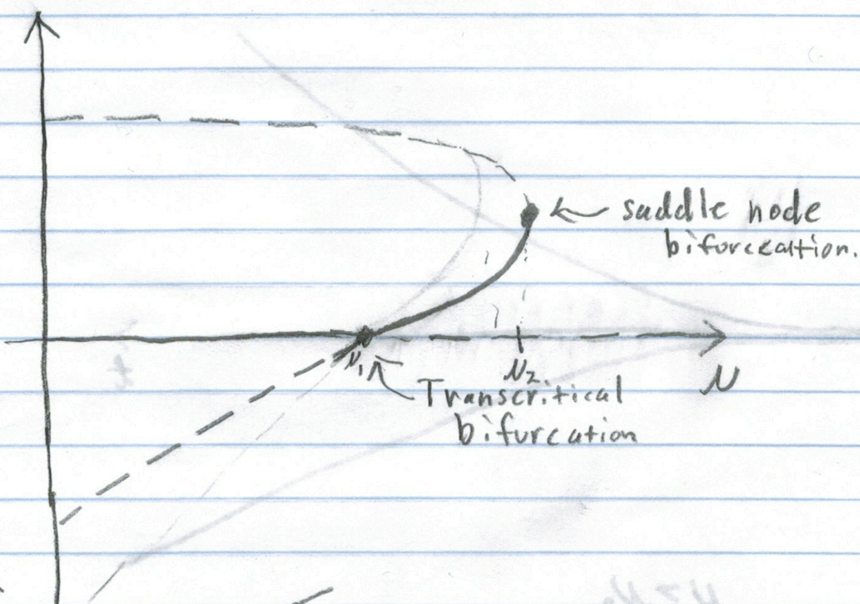


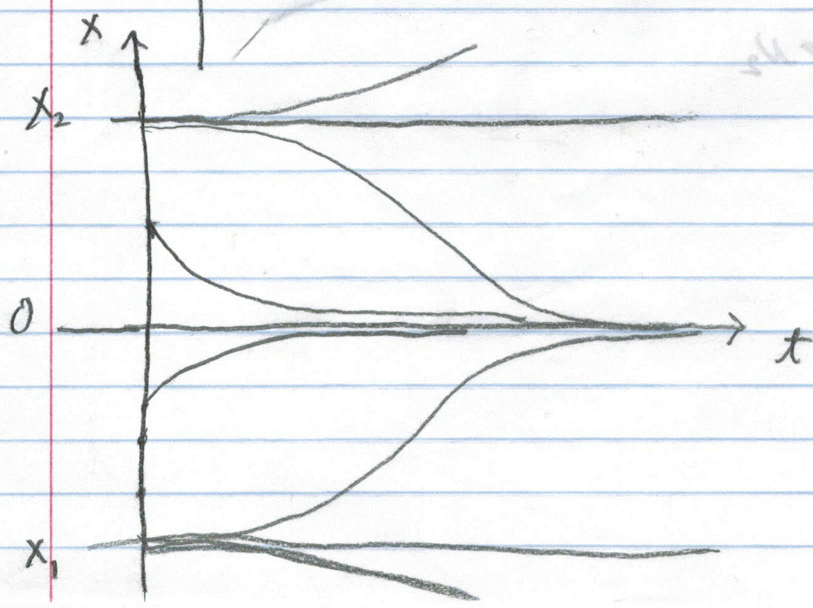
The fixed point switches to unstable in fact, the solution goes to ∞ in finite time.

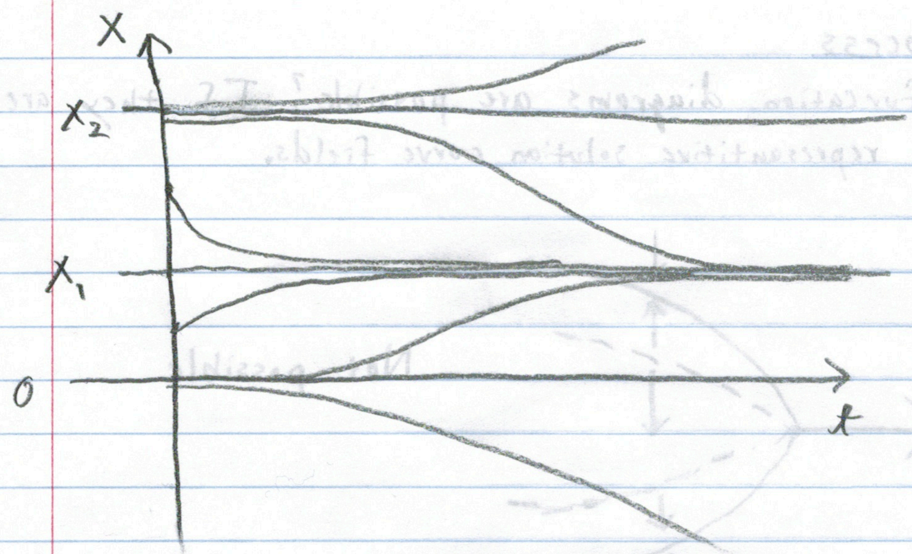
Reverse Process

Which bifurcation diagrams are possible? If they are possible sketch several represantitive solution curve fields.

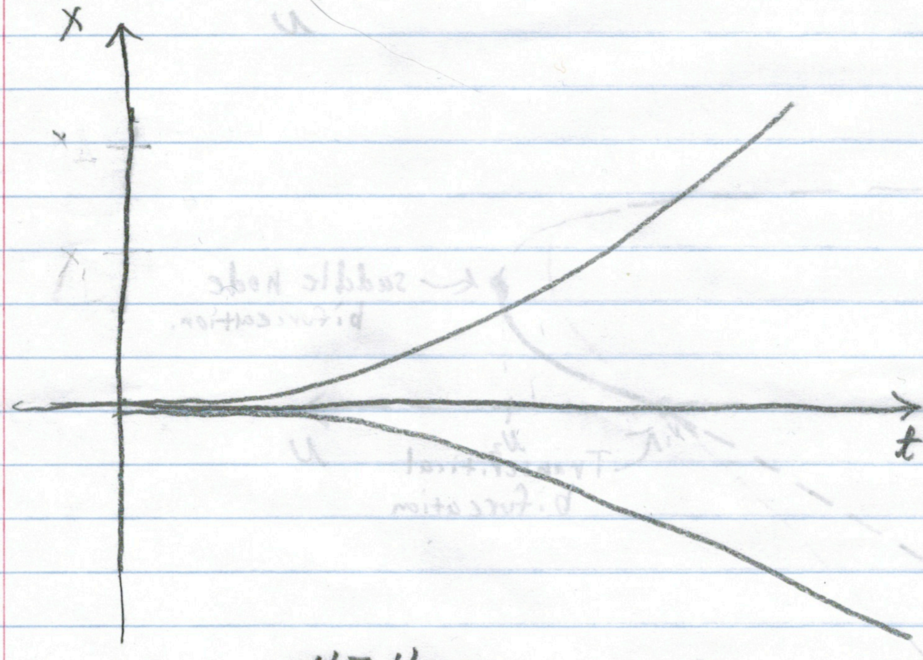








$\mu_1 < \mu < \mu_2$



$\mu > \mu_2$