Chapter 2: Flows on the Line

General Framework:

$\dot{x} = f(x)$, $x \in \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$ differentiable.

x - position

$f(x)$ - velocity (really f is a vector field)

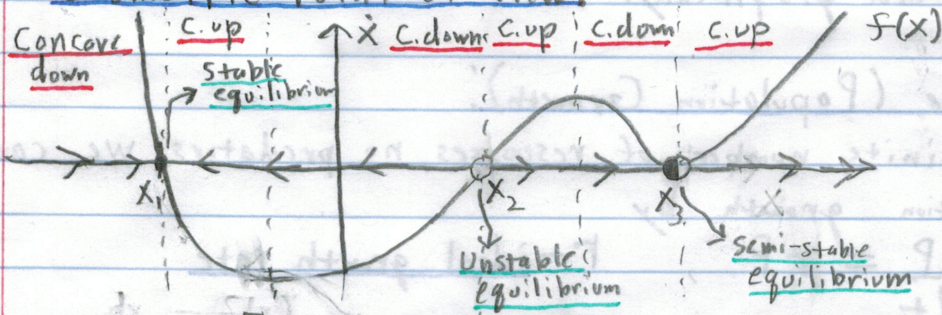
We can try to solve

$$t = \int_{x_0}^{x} \frac{1}{f(s)}\,ds$$

i.) We may not be able to integrate explicitly.

ii) It may not be possible to solve for $x(t)$.

Geometric Point of View:



$$\ddot{x} = \frac{df}{dx}\frac{dx}{dt} = \frac{df}{dx} f.$$

Fig. 2.1

i) $f(x) > 0$, $\frac{dx}{dt} > 0 \Rightarrow x(t)$ increases particle moves right

ii) $f(x) < 0$, $\frac{dx}{dt} < 0 \Rightarrow x(t)$ decreases particle moves left.

iii) $f(x) = 0$, $\frac{dx}{dt} = 0$, $x(t) = x_0$ is a solution of $\frac{dx}{dt} = f(x)$ with $x(0) = x_0$.

Solutions to $f(x) = 0$ are called fixed points, equilibrium, steady states, rest states.

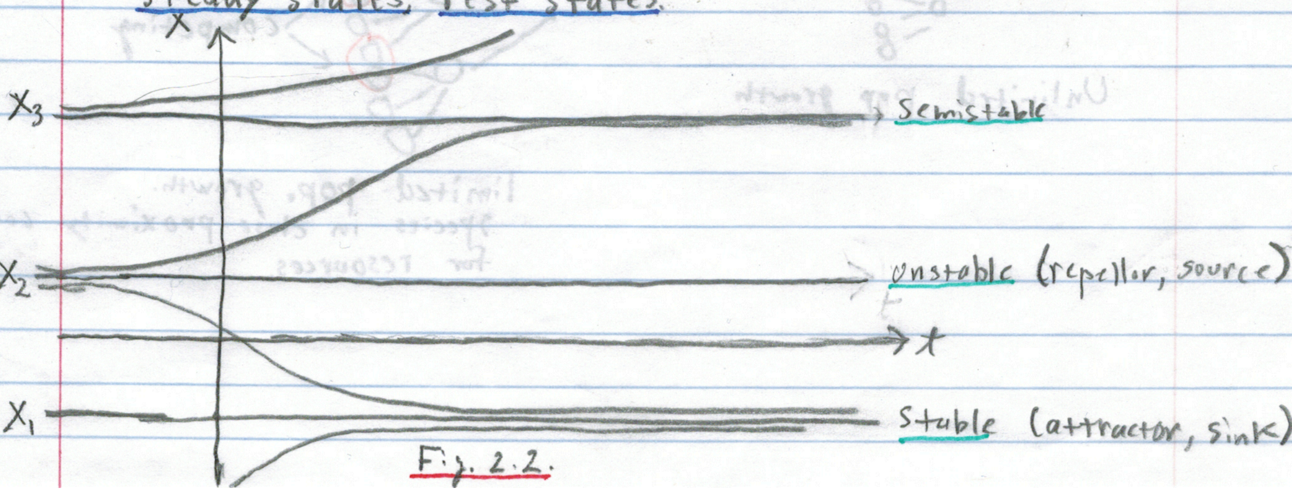


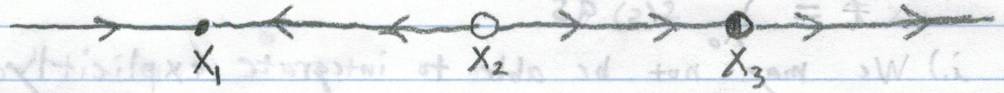
Fig. 2.2.

Note:

$$\ddot{x} = \frac{df}{dx}\frac{dx}{dt} = \frac{df}{dx}\cdot f$$

This allows us to identify changes in concavity.

The phase portrait captures all of this behavior



At each point in $\mathbb{R}$ the function f assigns a tangent vector pointing left or right. Local stability can be determined graphically.

Example (Population Growth):

Infinite number of resourses, no predators we can model population growth by

$$\frac{dP}{dt} = rP, \quad r - \text{ideal growth rate}$$

$$P(0) = P_0$$

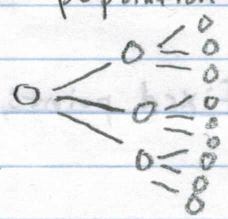
Derivation:

$$\underset{\text{new pop}}{P(t+\Delta t)} = \underset{\text{old pop}}{P(t)} + \underset{\text{children}}{r\Delta t P(t)}$$

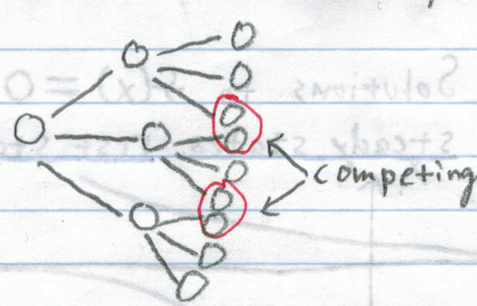
Easy solution

$$P = P_0 \exp(rt).$$

A more realistic model must account for finite resources. As population increases it becomes harder to reproduce.



Unlimited pop growth



limited pop. growth.
Species in close proximity compete for resources.

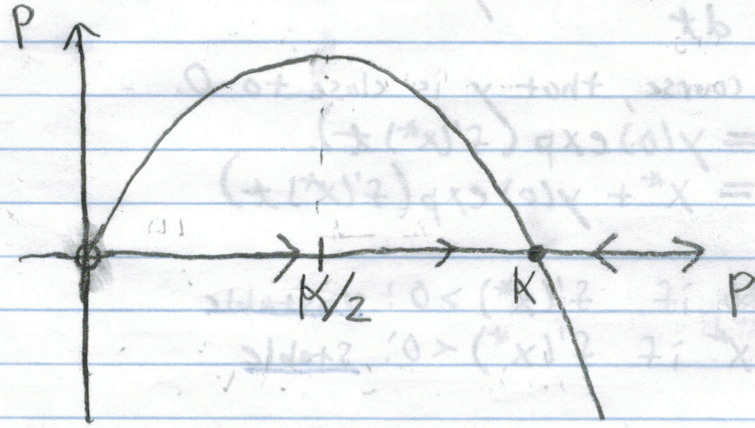
We can modify the rate of reproduction

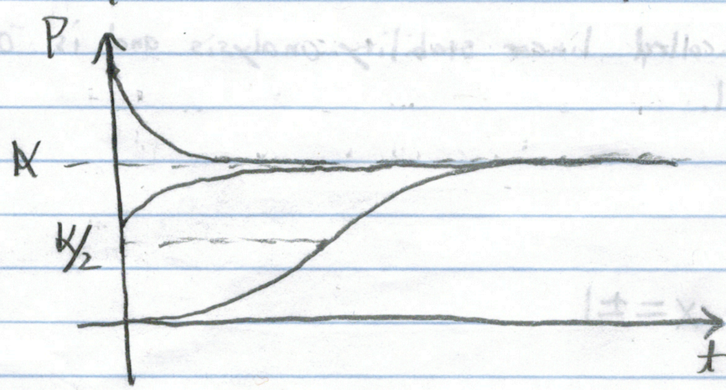
$$P(t+\Delta t) = P(t) + r\left(1-\frac{P(t)}{K}\right)\Delta t\, P(t) = P(t) + r\left(\frac{K-P(t)}{K}\right)\Delta t \cdot P(t)$$

$P(t+\Delta t)$: New pop; $P(t)$: old pop; $r\left(1-\frac{P(t)}{K}\right)$: fraction that get to duplicate

$$\frac{dP}{dt} = r\left(1-\frac{P(t)}{K}\right)P(t)$$

$$P(0) = P_0$$

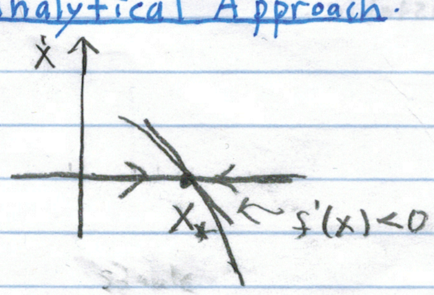


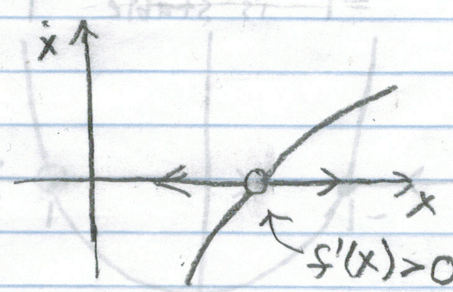


→ Population simply goes to carrying capacity.

L2

Analytical Approach:





Take a Taylor expansion near x^*

$$f(x) = \underbrace{f(x^*)}_{=0} + f'(x^*)(x-x^*) + \underbrace{O((x-x^*)^2)}_{\text{terms that are at least quadratic.}}$$

Set $y(t) = x(t) - x_*$ and omit higher order terms.

$$\Rightarrow \frac{dy}{dt} = \frac{dx}{dt} \approx f'(x^*)y$$

Provided, of course, that y is close to 0.

$$\Rightarrow y(t) = y(0)\exp(f'(x^*)t).$$

$$\Rightarrow x(t) = x^* + y(0)\exp(f'(x^*)t)$$

Conclusion:

1. $x(t) \to \infty$ if $f'(x^*) > 0$: unstable
2. $x(t) \to x^*$ if $f'(x^*) < 0$: stable

This process is called linear stability analysis and is a fundamental tool.

Example:

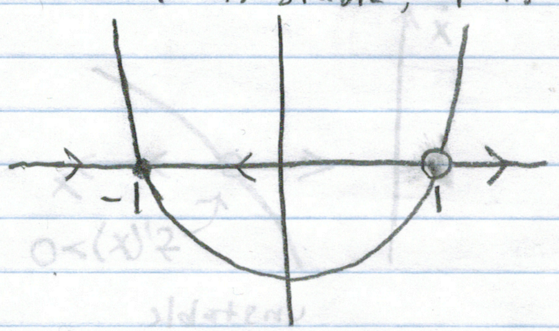
$$\dot{x} = x^2 - 1$$

Fixed points $x = \pm 1$.

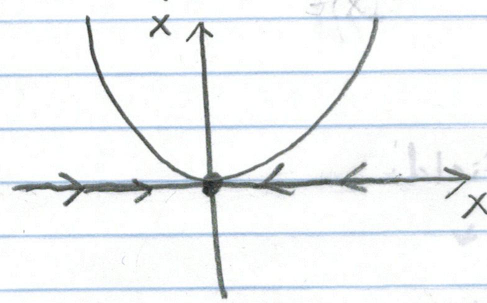
$$f'(x) = 2x$$

Hence,

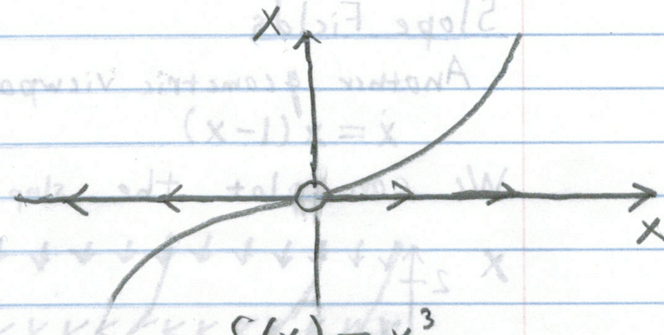
-1 is stable, 1 is unstable.



What happens when $f'(x) = 0$ at an equilibrium?



$f(x) = x^2$



$f(x) = x^3$

Potentials (Energy or Lyapunov functions)

Overdamped motion

$$\ddot{x} + \alpha\dot{x} = -\frac{dV}{dx}$$

acceleration | friction   potential energy gradient.

If α is really large we can look at equivalent system

$$\dot{x} = -\frac{dV}{dx} = f(x)$$

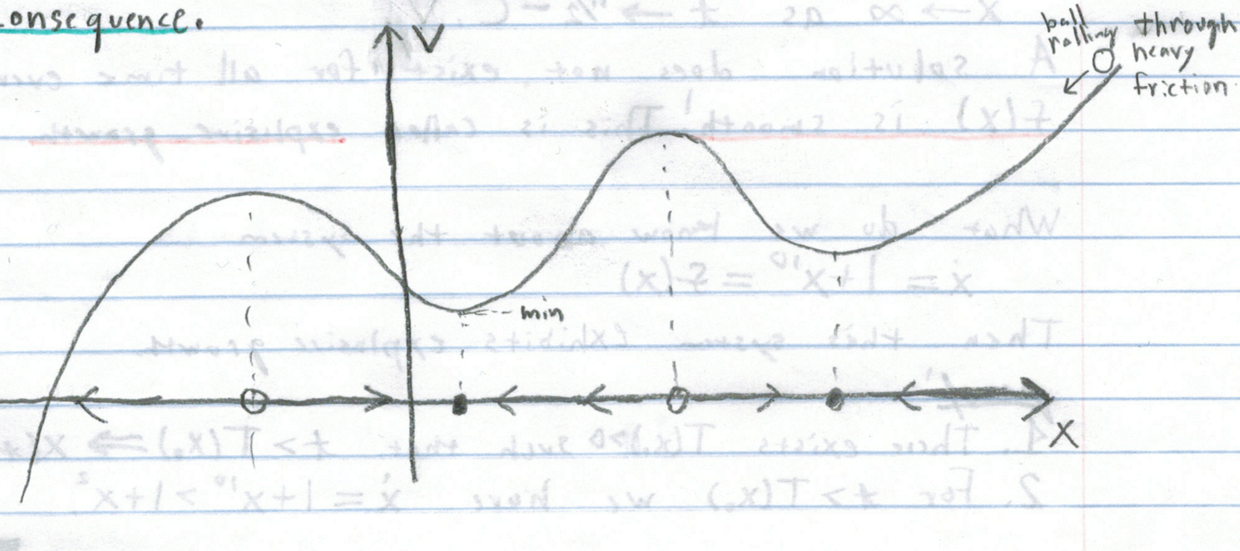
Since there is no inertia the potential energy is always decreasing.

proof

We need to prove that $\frac{dV}{dt} < 0$.

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = -\left(\frac{dV}{dx}\right)^2 < 0.$$

Consequence:

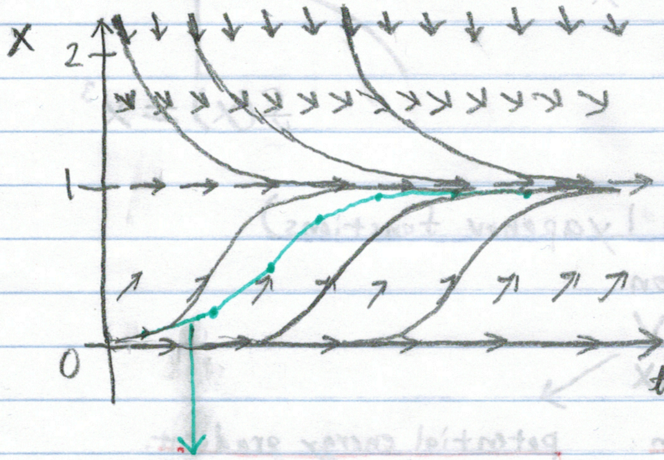


Slope Fields

Another geometric viewpoint.

$$\dot{x} = x(1-x)$$

We can plot the slope field:



* Euler's method uses slopes to construct piecewise linear approximations. Smarter methods use more information to gain better guesses on slopes.

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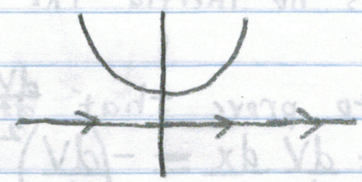
Explosions

$$\dot{x} = 1 + x^2 = f(x)$$



Solution:

$$x = \tan(t + C)$$

$$x \to \infty \text{ as } t \to \pi/2 - C.$$

A solution does not exist for all time even though f(x) is smooth! This is called explosive growth.

What do we know about the system

$$\dot{x} = 1 + x^{10} = f(x)$$

Then this system exhibits explosive growth

~~proof~~:

1. There exists $T(x_0) > 0$ such that $t > T(x_0) \Rightarrow x(t) > 1$.
2. For $t > T(x_0)$ we have $\dot{x} = 1 + x^{10} > 1 + x^2$.

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