Chapter 9: Lorenz Equations.

3-D dynamics.

Old tools:

1. local linearization
2. global analysis

* Trajectories can be attracted to lower dimensional subsets:

Surfaces:

a.) Spheres

b.) Torus.

Example.

$$\dot{x} = -y + \frac{x(1-x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}$$

$$\dot{y} = x + \frac{y(1-x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}$$

$$\dot{z} = \frac{z(1-x^2-y^2-z^2)}{\sqrt{x^2+y^2+z^2}}$$

Convert to spherical polar:

$$\dot{\rho} = 1-\rho^2$$
$$\dot{\theta} = 1$$
$$\dot{\phi} = 0$$

* All trajectories go to sphere of radius 1 and form limit cycles of constant angle ϕ.

* Approach sphere along a cone.

Example

$$\dot{x} = \frac{xz}{\sqrt{x^2+y^2}} + \frac{x(1-(x^2+y^2+z^2))}{\sqrt{x^2+y^2+z^2}}$$

$$\dot{y} = \frac{yz}{\sqrt{x^2+y^2}} + \frac{y(1-(x^2+y^2+z^2))}{\sqrt{x^2+y^2+z^2}}$$

$$\dot{z} = \frac{z(1-(x^2+y^2+z^2))}{\sqrt{x^2+y^2+z^2}}$$

Convert to spherical Polar

$$\dot{\rho} = 1-\rho^2$$
$$\dot{\theta} = 0$$
$$\dot{\phi} = 1$$

Converge to latitudes → This violates existence and uniqueness
This occurs because x^2+y^2 can equal 0.

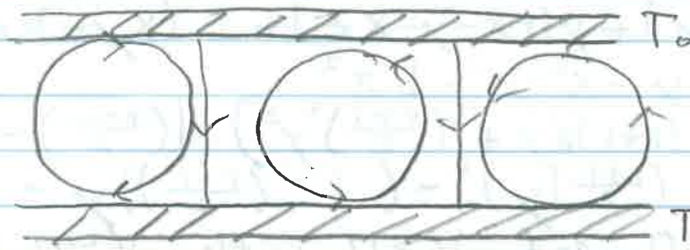
Example:

$$\dot{\rho} = (1-r^2)$$
$$\dot{\theta} = 1$$
$$\dot{\phi} = 1$$

This forms Lissajous cycles.

* In these examples the long term behaviour is attracted to surfaces which we can analyze dynamics on.

Lorenz Equations:



$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

σ – Prandtl number → ratio of $\frac{\text{viscosity}}{\text{thermal diffusivity}}$
r – Rayleigh number: ($r \ll 1$ conduction, $r \gg 1$ convection)
b – diminsionless aspect ratio.

Neet stuff happens as we play with r.

Fixed Points:

1. $(0,0,0)$ → Pure conduction

2. $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ → left and right moving rolls.

$$J(0,0,0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$$2\lambda_{1,2} = -\sigma - 1 \pm \sqrt{(\sigma+1)^2 - 4(\sigma - \sigma r)}$$
$$= -\sigma - 1 \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}$$
$$= -\sigma - 1 \pm \sqrt{(\sigma-1)^2 + 4\sigma r}$$

Case 1:

$r<1$, only one fixed point and it is a stable node.

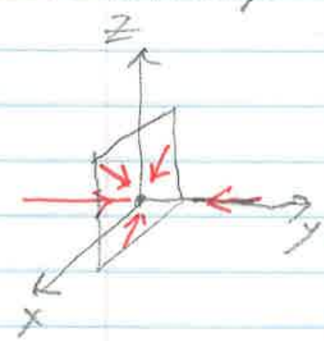
How can we eliminate closed orbits? Construct a Lyapunov function.

Let $V = \frac{1}{2}\left(\frac{1}{\sigma}x^2 + y^2 + z^2\right)$

$$\Rightarrow \dot{V} = \frac{1}{\sigma}x\dot{x} + y\dot{y} + z\dot{z}$$
$$= x(y-x) + y(rx - y - xz) + z\cdot(xy - bz)$$
$$= (r+1)xy - x^2 - y^2 - bz^2$$
$$= -\left[x^2 - (r+1)xy + \left(\frac{r+1}{2}\right)^2 y^2\right] + \left[\left(\frac{r+1}{2}\right) - 1\right]y^2 - bz^2$$
$$= -\left(x - \left(\frac{r+1}{2}\right)y\right)^2 - \left(1 - \left(\frac{r+1}{2}\right)\right)y^2 - bz^2$$

If $r<1$ $\dot{V}<0$, and $V \geq 0$ with $V=0$ if and only if $x=y=z=0$.

$$\Longrightarrow \lim_{t\to\infty} V(t) = 0.$$



If $r<1$, all trajectories go to the origin

Case 2:

$r>1$

$(0,0,0)$ has eigenvalues $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$.

$\longrightarrow$ This is like a saddle point.

$(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ now exists. $\Rightarrow$ pitchfork bifurcation

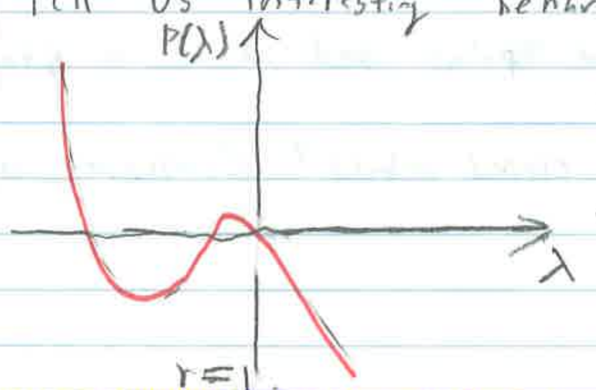
$\downarrow$

We should do local analysis,

From now on fix $b = 8/3$, $\sigma = 10$. We denote new fixed points by C^+ and C^-.

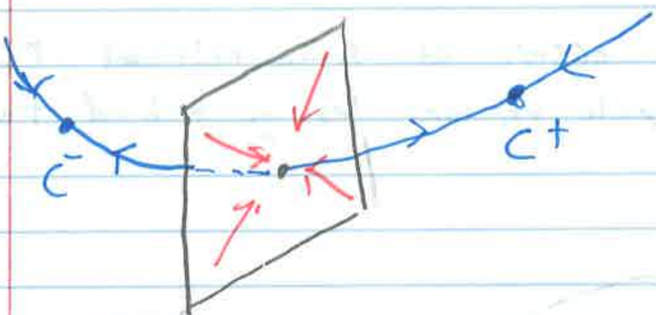
The characteristic polynomial is given by:

$3P(\lambda) = -\lambda^3 - 41\lambda^2 - 88\lambda - 80(r-1)$. Plotting this can tell us interesting behavior



$\Rightarrow$ Changing r shifts this graph down

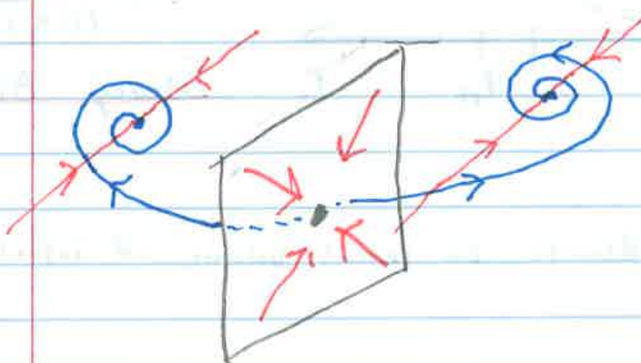
$\Rightarrow$ There exists r_s such that if $1<r<r_s$ C_+, C_- are stable



$\Rightarrow$ System has two steady state convection rolls.

<u>Case 3:</u>

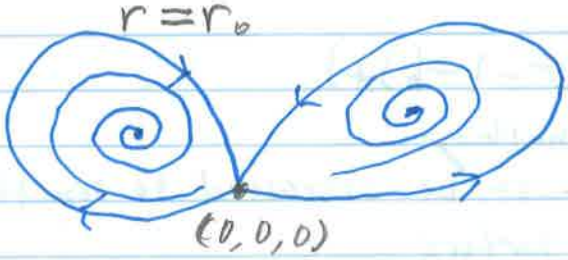
$r_s < r < r_0$



, stability analysis tells us that they are stable spirals around the fixed points.

<u>Case 4:</u>

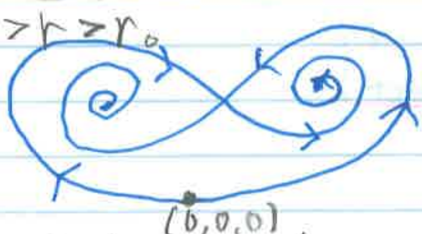
$r = r_0$



$\rightarrow$ Unstable limit cycle born in a homoclinic bifurcation.

<u>Case 5:</u>

$r_H > r > r_0$

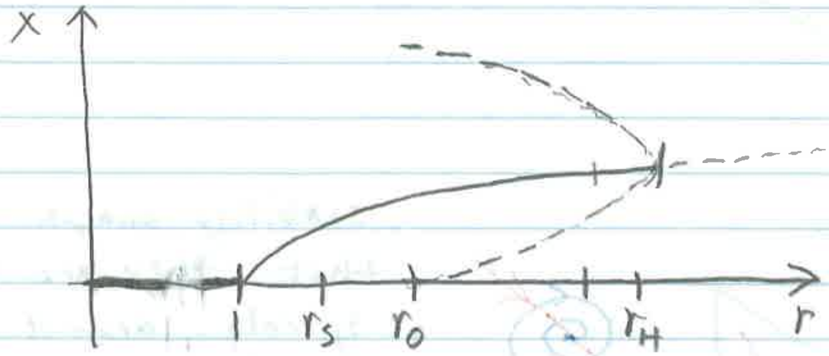


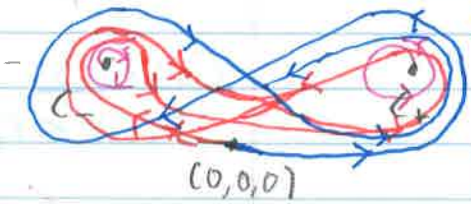
$\longrightarrow$ This creates a strange invariant set

spirals may loop around a lot. Impossible to predict which fixed point we go to

Case 6:

A Hopf bifurcation occurs at some critical r_H. The unstable limit cycle vanishes leaving behind two unstable spirals.





Strange Attractor.

Volume Contraction:

$\dot{\vec{x}} = F(\vec{x})$, let V_0 denote an intial volume of intial conditions

$$\Rightarrow \dot{V} = \int_{\partial S} \vec{F} \cdot \vec{n} \, dA$$

$$= \int_V \nabla \cdot \vec{F} \, dV$$

$$= \int_V (-\sigma - 1 - b) dV$$

$$= (-\sigma - 1 - b) V$$

$$\Rightarrow V(t) = V_0 \exp((-\sigma - 1 - b)t).$$

All volumes shrink to nothing.

$\Rightarrow$ The trajectories must be attracted to something...

$\rightarrow$ 1. Can't be a surface

2. can't be a fixed point (all unstable if $r > r_H$).

3. possible to eliminate limit cycles.

4. Can't be quasiperiodic.

$\rightarrow$ Must be a new object!

$\downarrow$

Goals:

1. Find out what new object is.

2. Try to study chaos.

What is Chaos?

* Chaos - Aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on intial conditions.

1. Aperiodic - There exist trajectories which do not go to fixed points, periodic orbits, or quasi-periodic orbits.
2. Sensitive Dependence - We say $\dot{\vec{x}} = F(\vec{x})$ has sensitive dependence on intial data on a set A if $\forall \vec{x}_0 \in A$, $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists \tilde{\vec{x}}_0$ and $T > 0$ with $|x_0 - \tilde{x}| < \delta$ and $|x(T) - \tilde{x}(T)| > \varepsilon$.

Trajectories remain bounded

Consider a spherical surface $S_R = x^2 + y^2 + (z - r - \sigma)^2 = R^2$.
For a trajectory starting on this surface we have that

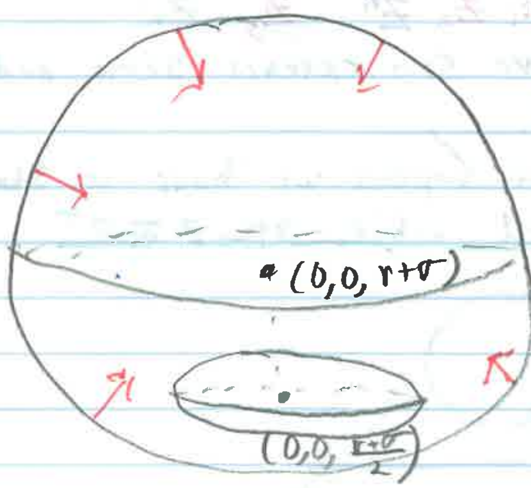
$$\frac{d}{dt}\left[x^2 + y^2 + (z - r - \sigma)^2\right] = 2x\dot{x} + 2y\dot{y} + 2(z - r - \sigma)\dot{z}$$

$$= -2\left[\sigma x^2 + y^2 + b\left(z - \frac{r+\sigma}{2}\right)^2 - \frac{b(r+\sigma)^2}{4}\right]$$

Pick R large enough that the ellipse

$$\sigma x^2 + y^2 + b\left(z - \frac{r+\sigma}{2}\right)^2 = \frac{b(r+\sigma)^2}{4}$$

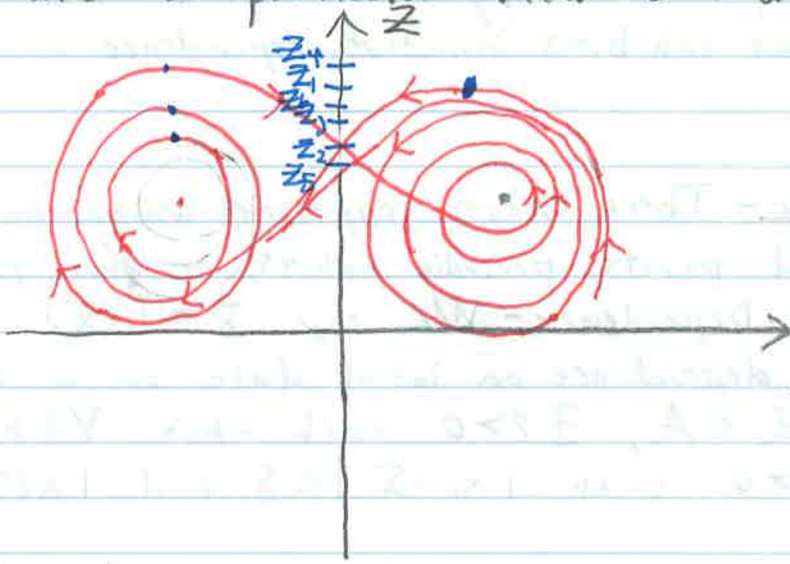
is enclosed in S_R. This guarantees S_R is a trapping region.



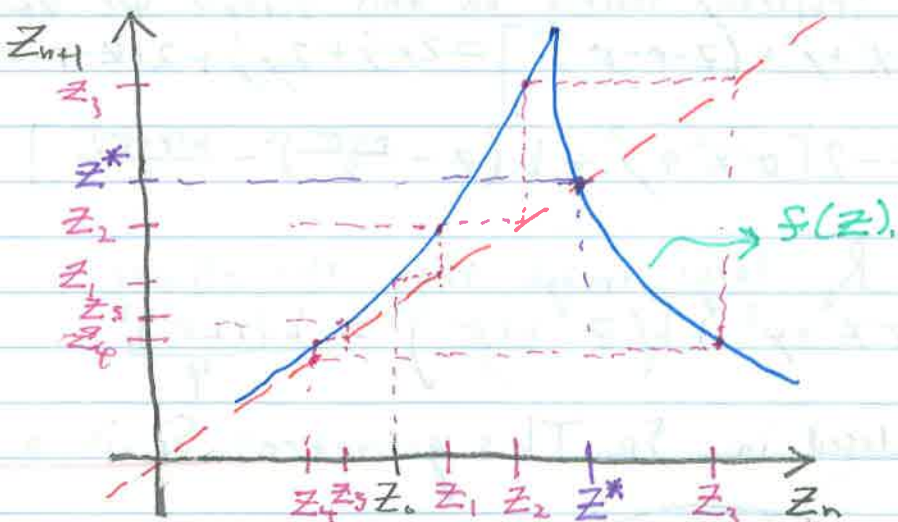
$\Rightarrow$ Trajectories remain bounded and must go to something.

Lorenz Map

Take a particular view of a trajectory:



The Lorenz map looks at the relationship between z_n and z_{n+1}

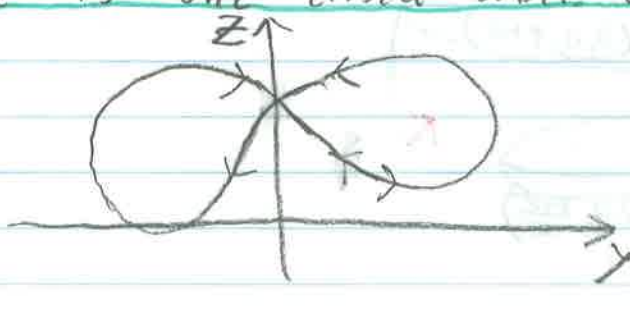


This is awesome! We can extract some order from Choas

$$z_{n+1} = f(z_n).$$

Note $|f'(z)| > 1$. Now suppose we have a stable limit cycle.

There is one closed orbit at $z = z^*$



Consider

$$z_0 = z^* + \delta_0$$

δ_0 = intial peturbation.

$$\Rightarrow z_1 = f(z^* + \delta_0)$$

$$\Rightarrow f(z) \approx f(z^*) + f'(z^*)(z - z^*)$$

$$\Rightarrow z_1 = f(z_0) \approx z^* + f'(z^*)\delta_0$$

$$\Rightarrow \delta_1 = z_1 - z^* \approx f'(z^*)\delta_0 \quad , \quad \delta_1 = z_1 - z^*.$$

$$\Rightarrow |\delta_1| > |\delta_0|$$

The trajectory grows away from the set. This fixed point is unstable.

$\Rightarrow$ No stable limit cycles.

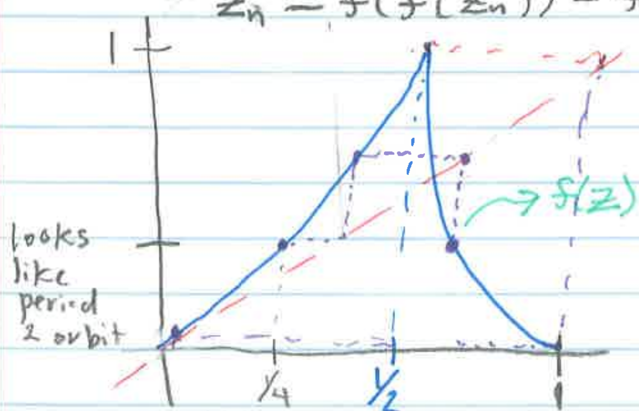
Attractor

An attractor is a closed set A with the following properties.

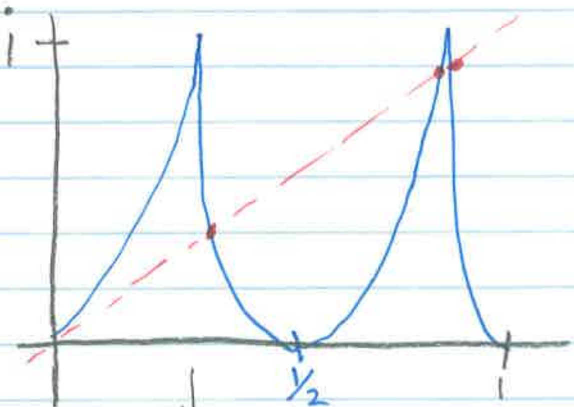
1. A is invariant: Any $\vec{x}(t)$ that starts in A remains in A.
2. A attracts an open set of initial conditions: $\exists$ exists an open set U containing A such that if $\vec{x}(0) \in A$, then $d(\vec{x}(t), A) \to 0$ as $t \to \infty$. The largest U is called the basin of attraction.
3. A is minimal: There is no proper subset of A.

What about Double Periodic Iterations?

$$z_n = f(f(z_n)) = f^2(z_n)?$$

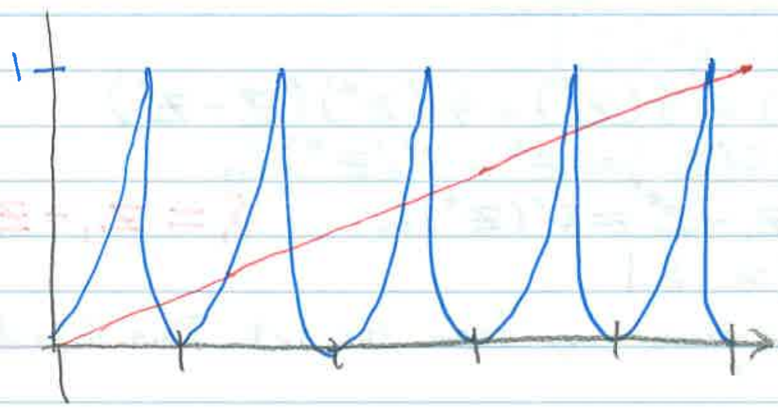


$f^2(1/2) \approx 0, \; f^2(1/4) \approx 1/4.$



3 double period orbits all unstable.

General Case



We can continue forever

→ There exists an infinite number of unstable limit cycles!