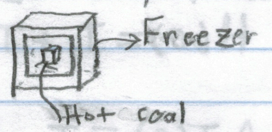
Chapter 1: Introduction

Classic physics - Derive a system of differential equations and solve them. The entire evolution of the system is determined from initial conditions.

Example - An object is placed in a cold (hot) environment at a fixed temperature T_E.



If the temperature difference between the object and its environment changes at rate proportional to the temperature difference, what is a function for the temperature of the object.

$$\frac{dT}{dt} = K(T_E - T) \quad \text{(Newton's Law of Cooling)}$$

$T(0) = T_0$

$K \sim$ units of inverse time.

Solution:

Let $\Delta T = T - T_E$ and $\Delta T_0 = T_0 - T_E$. Then,

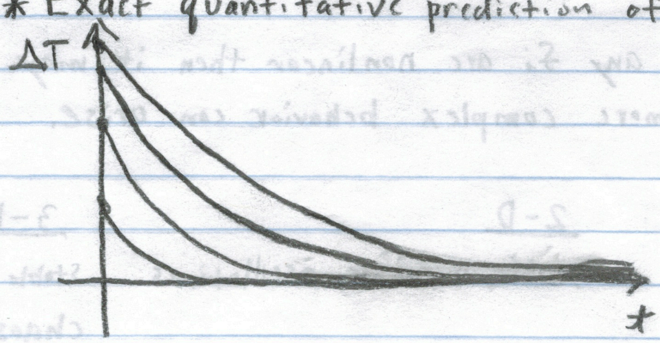
$$\Rightarrow \frac{dT}{dt} = \frac{d\Delta T}{dt}$$

and

$$\frac{d\Delta T}{dt} = -K\Delta T, \quad \Delta T(0) = \Delta T_0$$

$$\Rightarrow \Delta T = \Delta T_0 e^{-Kt}$$

* Exact quantitative prediction of temperature.



Example: Now suppose the object and its environment influence each other:

$$\frac{dT}{dt} = -K_1(T - T_E), \quad T(0) = T_0$$

$$\frac{dT_E}{dt} = K_2(T - T_E) \quad T_E(0) = T_{E,0}$$

Again let $\Delta T = T - T_E$ and $H = T + T_E$. Then,

$$\frac{dH}{dt} = (K_2 - K_1)\Delta T, \quad H(0) = T_0 + T_{E,0} = H_0$$

$$\frac{d\Delta T}{dt} = -(K_1 + K_2)\Delta T, \quad \Delta T(0) = T_0 - T_{E,0} = \Delta T_0$$

$$\Rightarrow \Delta T = \Delta T_0 e^{-(K_1+K_2)t}$$

$$\frac{dH}{dt} = (K_2 - K_1)\Delta T_0 e^{-(K_1+K_2)t}$$

$$\Rightarrow H(t) = \frac{(K_1 - K_2)}{(K_1 + K_2)}\Delta T_0 \left(e^{-(K_1+K_2)t} - 1\right) + H_0$$

Notation - A general framework for (continuous) dynamical systems is the system of equations

$$\dot{x}_1 = f_1(x_1, \ldots, x_n)$$
$$\vdots \qquad\qquad \vdots$$
$$\dot{x}_n = f_n(x_1, \ldots, x_n)$$

(overdot denotes differentation in time)

Such a system is called an n-dimensional dynamical system.

Linearity - If the functions f_i are linear then the system can be solved exacty using linear algebra and complex solutions can be built up as a superposition.

Nonlinear - If any f_i are nonlinear then it may not be solvable exactly and more complex behavior can arise.

1-D	2-D	3-D and beyond
stable, unstable	stable, unstable, oscillations	stable, unstable, oscillations, chaos!

Example:

Nonlinear version of Newton's law of cooling

$$\dot{\Delta T} = -\kappa \Delta T - \alpha \Delta T^3$$
$$\Delta T(0) = T_0$$

Note: If we think of $\dot{\Delta T} = f(\Delta T)$ then this is the first two nonzero terms in a Taylor expansion.

We can try seperating variables again.

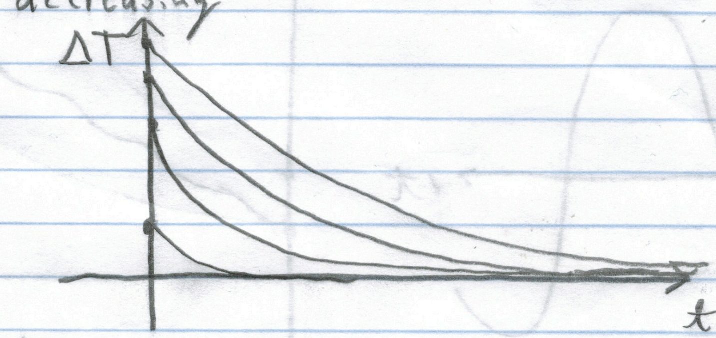
$$-t = \int_{T_0}^{\Delta T} \frac{1}{\kappa s + \alpha s^3}\, ds.$$
$$= \int_{T_0}^{\Delta T} \left(\frac{1}{\kappa s} + \frac{\kappa - \alpha}{\alpha} \frac{1}{\kappa + \alpha s^2} \right) ds$$
$$= \frac{1}{\kappa} \ln\left(\frac{\Delta T}{T_0}\right) + \frac{\kappa - \alpha}{\kappa \alpha} \sqrt{\frac{\kappa}{\alpha}} \left(\tan^{-1}\left(\sqrt{\tfrac{\alpha}{\kappa}} \Delta T\right) - \tan^{-1}\left(\sqrt{\tfrac{\alpha}{\kappa}} T_0\right) \right)$$

To get a complete solution we would need to invert.

Poincare took a different approach

$$\dot{\Delta T} = 0 \quad \text{when} \quad \Delta T = 0,$$

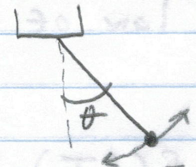
i.e. the object stops cooling when $\Delta T = 0$. Before that $\dot{\Delta T}$ is always negative implying ΔT is monotone decreasing



All initial conditions end up at the endpoint.

Poincare had the insight to study flows instead of solutions.

Example



Pendulum

Classic Form:

$$\ddot{\theta} + \sin(\theta) = 0$$

$$\theta(0) = \theta_0$$

$$\dot{\theta}(0) = V_0$$

Equivalent Form:

$$\dot{V} = -\sin(\theta)$$

$$\dot{\theta} = V$$

$$\theta(0) = \theta_0, \quad V(0) = V_0$$

The equation is nonlinear, so it is difficult to solve.

Can we answer some questions about this system.

$$\frac{dv}{d\theta} = \frac{-\sin\theta}{v}$$

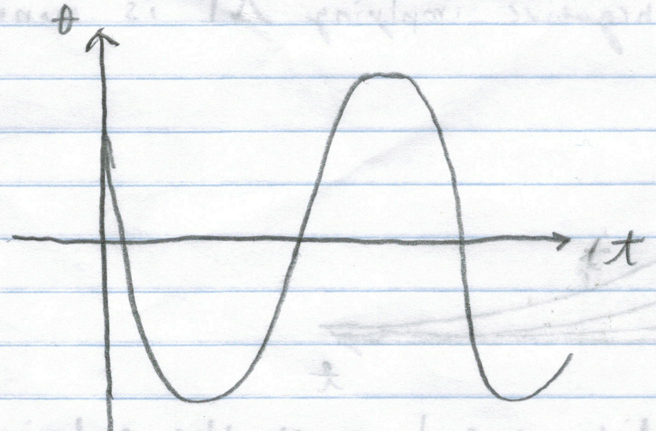
$$\Rightarrow \int_{V_0}^{V} v\,dv = \int_{\theta_0}^{\theta} -\sin\theta\,d\theta$$

$$\Rightarrow \frac{1}{2}V^2 - \frac{1}{2}V_0^2 = \cos(\theta) - \cos(\theta_0)$$

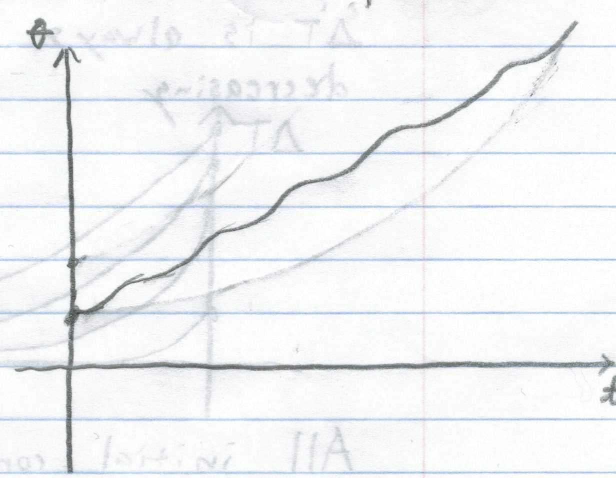
$$V = \pm\sqrt{V_0^2 + 2\cos(\theta) - 2\cos(\theta_0)}$$

$$\frac{d\theta}{dt} = \pm\sqrt{V_0^2 + 2\cos(\theta) - 2\cos(\theta_0)}$$

If $V_0^2 - 2\cos(\theta_0) > -2$ then no oscillations possible.



Oscillations



No oscillations.