Lecture 5: Linear Systems of Differential Equations

A two dimensional linear system is in the form

$$\begin{cases} \dfrac{dx}{dt} = ax+by \\[2mm] \dfrac{dy}{dt} = cx+dy \end{cases}$$

This system is linear since it satisfies the following:

Solutions can be scaled ←

1. If $(x_1(t), y_1(t))$ is a solution then so is $(\alpha x_1(t), \alpha y_1(t)) = (x_2(t), y_2(t))$

proof:

$$\frac{d}{dt}(\alpha x_1(t)) = \alpha \frac{dx_1}{dt} = \alpha a \frac{dx_1}{dt} + \alpha b \frac{dy_1}{dt} = a\frac{d}{dt}(\alpha x_1) + b\frac{d}{dt}(\alpha y_1)$$

$$\Rightarrow \frac{dx_2}{dt} = a\frac{dx_2}{dt} + b\frac{dy_2}{dt}.$$

Solutions can be joined together by addition. ←

2. If $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$ are solutions then so is

$$(x_3(t), y_3(t)) = (x_1(t)+x_2(t), y_1(t)+y_2(t))$$

proof:

$$\frac{d}{dt}(x_3(t)) = \frac{d}{dt}(x_1(t)+x_2(t)) = \frac{dx_1}{dt} + \frac{dx_2}{dt} = ax_1 + by_1 + ax_2 + by_2$$

$$\Rightarrow \frac{d}{dt}(x_3(t)) = a(x_1+x_2) + b(y_1+y_2) = ax_3 + bx_3.$$

Example:

$$\frac{dx}{dt} = ax \quad , \quad x(0)=x_0$$

$$\frac{dy}{dt} = -y \quad\quad y(0)=y_0$$

We can solve exactly:

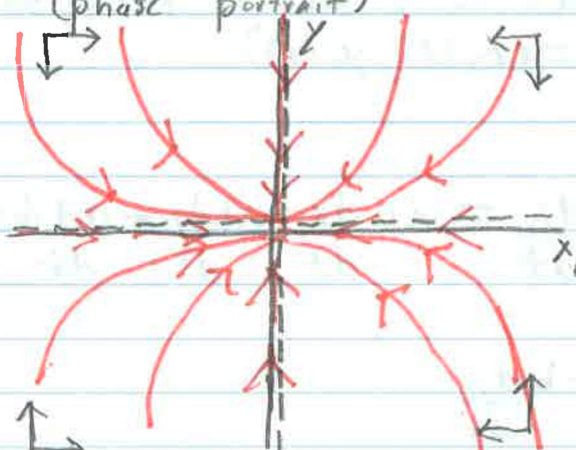
$$x(t) = x_0 e^{at}$$

$$y(t) = y_0 e^{-t} \Rightarrow \lim_{t\to\infty} y(t) = 0$$

Case 1 ($-1<a<0$):

(phase portrait)

stable node ←



$$x(t) = x_0 e^{at}$$

$$y(t) = y_0 e^{-t}$$

$$\Downarrow$$

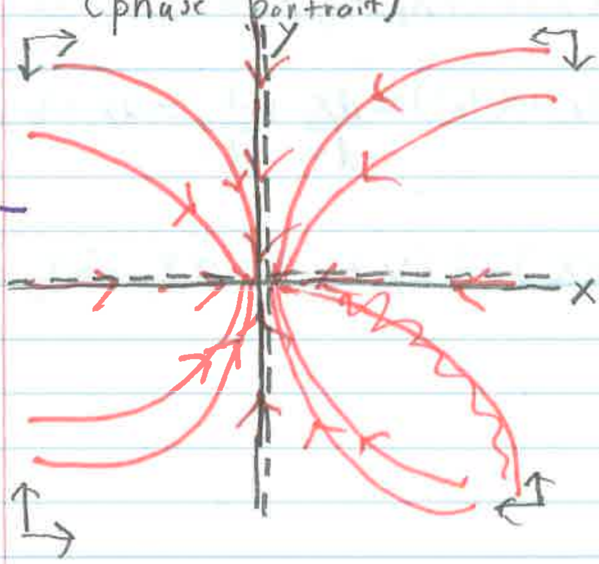
$$\frac{dy}{dx} = \frac{y(t)}{x(t)} = -\frac{y_0}{ax_0} e^{-(1+a)t}$$

$$\Rightarrow \lim_{t\to\infty} \frac{dy}{dx} = 0.$$

Case 2 ($a<-1$):

(phase portrait)

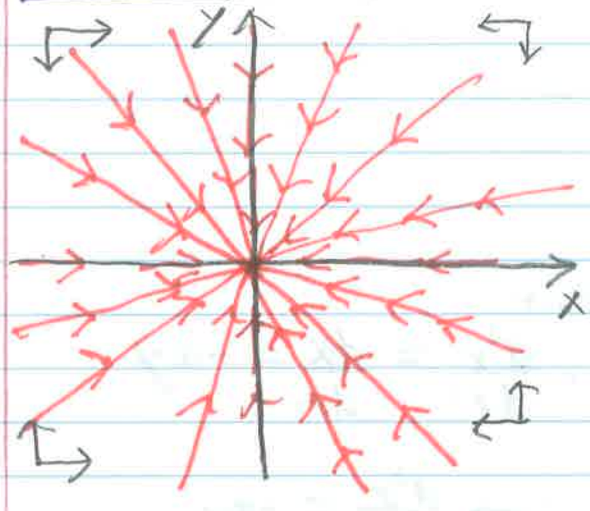
stable node ←



$$\frac{dy}{dx} = -\frac{y_0}{ax_0} e^{-(1+a)t}$$

$$\Rightarrow \lim_{t\to\infty} \frac{dy}{dx} = \infty.$$

Case 3 ($a=-1$):

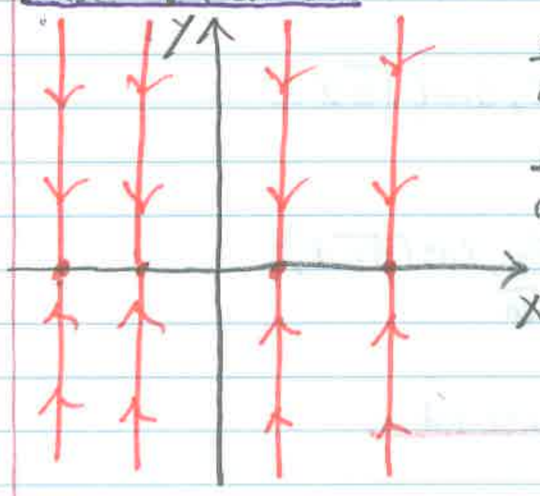


$$\frac{dy}{dx} = \frac{y_0}{x_0}$$

$$\Rightarrow \lim_{t\to\infty} \frac{dy}{dx} = \frac{y_0}{x_0}$$

(Note: $a<0 \Rightarrow$ origin is stable fixed point)

Case 4 ($a=0$):

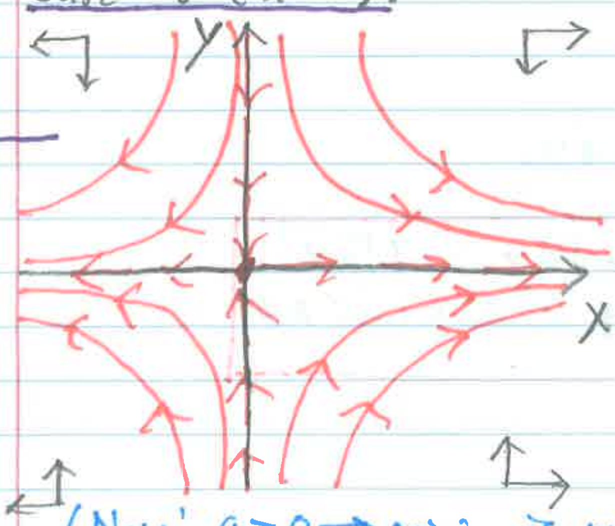


$$\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = -1$$

Case 5 ($a>0$):

Actually called a saddle node



$$\frac{dy}{dx} = -\frac{y}{ax_0} e^{-(1+a)t}$$

$$\rightarrow \lim_{t\to\infty} \frac{dy}{dx} = 0$$

(Note: $a>0 \Rightarrow$ origin is unstable fixed point).

Example:

$$\frac{dx}{dt} = -ay$$

$$\frac{dy}{dt} = x$$

Differentiating we have that

$$\frac{d^2x}{dt^2} = -a\frac{dy}{dt} = -ax \quad , \quad \frac{d^2y}{dt^2} = \frac{dx}{dt} = -ay$$

$$\Rightarrow \frac{d^2x}{dt^2} = -ax \qquad \Rightarrow \frac{d^2y}{dt^2} = -ay$$

$$\Rightarrow x(t) = c_1\cos(\sqrt{a}\,t) + c_2\sin(\sqrt{a}\,t)$$

$$\Rightarrow \frac{dy}{dt} = c_1\cos(\sqrt{a}\,t) + c_2\sin(\sqrt{a}\,t))$$

$$\Rightarrow y(t) = \frac{c_1}{\sqrt{a}}\sin(\sqrt{a}\,t) - \frac{c_2}{\sqrt{a}}\cos(\sqrt{a}\,t)$$

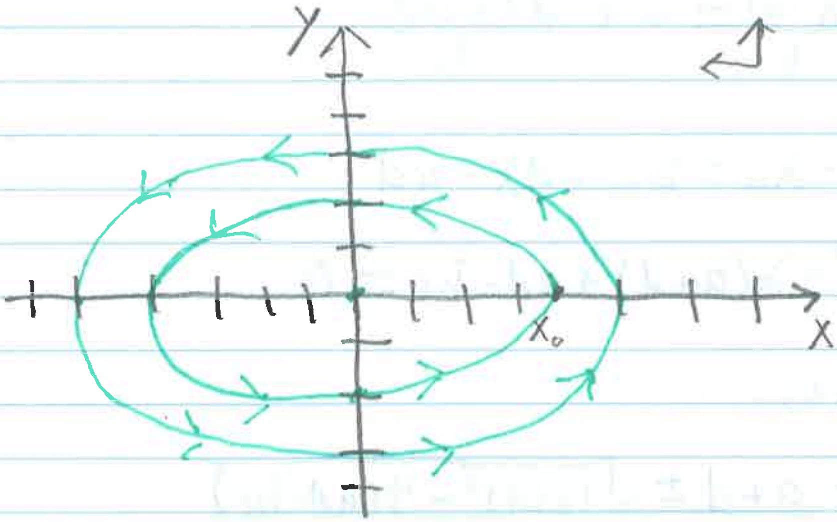
These solutions are periodic.

$$x(0) = c_1 = x_0$$

$$y(0) = -\frac{c_2}{\sqrt{a}} = y_0$$

$$\Rightarrow c_1 = x_0, \; c_2 = \sqrt{a}\,y_0$$

$$\boxed{\begin{aligned} x(t) &= x_0\cos(\sqrt{a}\,t) - \sqrt{a}\,y_0\sin(\sqrt{a}\,t) \\ y(t) &= \frac{x_0}{\sqrt{a}}\sin(\sqrt{a}\,t) + y_0\cos(\sqrt{a}\,t) \end{aligned}}$$



Solving Generic Systems (eigenvalues analysis)

Lets solve the generic system:

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

Guess

$$x(t) = c_1 e^{\lambda t}$$
$$y(t) = c_2 e^{\lambda t}$$

$$\Rightarrow c_1 \lambda e^{\lambda t} = a c_1 e^{\lambda t} + b c_2 e^{\lambda t}$$
$$c_2 \lambda e^{\lambda t} = c c_1 e^{\lambda t} + d c_2 e^{\lambda t}$$

$$\Rightarrow c_1 \lambda = a c_1 + b c_2$$
$$c_2 \lambda = c c_1 + d c_2$$

$$\Rightarrow c_2 = \frac{c_1(\lambda - a)}{b}$$

$$\Rightarrow \frac{c_1(\lambda - a)}{b}\lambda = c c_1 + \frac{d c_1 (\lambda - a)}{b}$$

$$\Rightarrow \lambda \frac{(\lambda - a)}{b} = c + \frac{d(\lambda - a)}{b}$$

$$\lambda^2 - \lambda a = bc + d\lambda - ad$$

$$\Rightarrow \lambda^2 - \lambda(a+d) + ad - bc = 0$$

Therefore,

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

$\tau = a + d$ (trace)
$\Delta = ad - bc$ (determinant)

Example:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -2x + 3y$$

$a = 0,\ b = 1$
$c = -2,\ b = 3$

$\tau = 3,\ \Delta = 2$

$$\lambda = \frac{3 \pm \sqrt{9-8}}{2} \Rightarrow \lambda_1 = 2,\ \lambda_2 = 1$$

The solutions must be of the form:

$$x(t) = c_1 e^{2t} + c_3 e^{t}$$
$$y(t) = c_2 e^{2t} + c_4 e^{t}$$

$$\Rightarrow 2c_1 e^{2t} + c_3 e^{t} = c_2 e^{2t} + c_4 e^{t}$$
$$2c_2 e^{2t} + c_4 e^{t} = -2c_1 e^{2t} + 3c_2 e^{2t} + 3c_4 e^{t} - 2c_3 e^{t}$$

Matching coefficient we have:

$$2c_1 = c_2$$
$$c_3 = c_4$$
$$2c_2 = -2c_1 + 3c_2$$
$$c_4 = 3c_4 - 2c_3$$

$$\Rightarrow 4c_1 = -2c_1 + 6c_1$$
$$\Rightarrow 4c_1 = 4c_1$$

Set $c_1 = 1$, we get $c_2 = 2$.

$$c_4 = 3c_4 - 2c_4$$
$$\Rightarrow c_4 = c_4$$

Set $c_4 = 1$, we get $c_3 = 1$.

Solution:

$$x(t) = (e^{2t} + e^{t})c_5$$
$$y(t) = (2e^{2t} + e^{t})c_6$$

$$\Rightarrow 2c_5 = x_0 \Rightarrow c_5 = x_0/2$$
$$3c_6 = y_0 \Rightarrow c_6 = y_0/3$$

$$\boxed{\begin{aligned} x(t) &= x_0/2\,(e^{2t} + e^{t}) \\ y(t) &= y_0/3\,(2e^{2t} + e^{t}) \end{aligned}}$$

Summary:

For a linear system

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

Solution are of the form

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y(t) = c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t}$$

with the eigenvalues satisfying

$$\lambda^2 - \tau\lambda + \Delta = 0$$

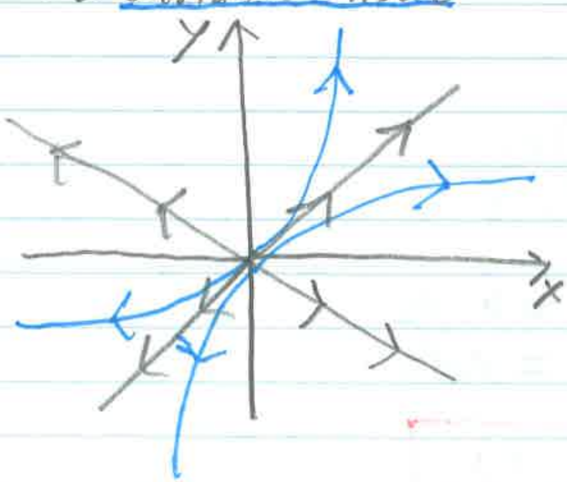
where

$$\tau = a + d \quad \text{(trace)}$$
$$\Delta = ad - bc \quad \text{(determinant)}.$$

Case 1:

$\lambda_1, \lambda_2 > 0$

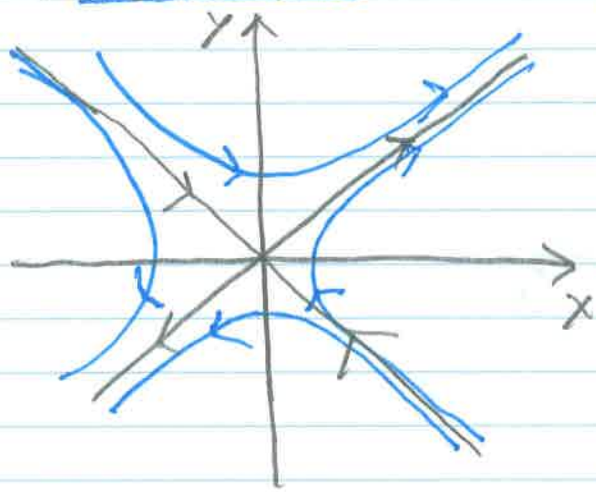
$\Rightarrow$ unstable node



Case 2:

$\lambda_1 < 0, \ \lambda_2 > 0$

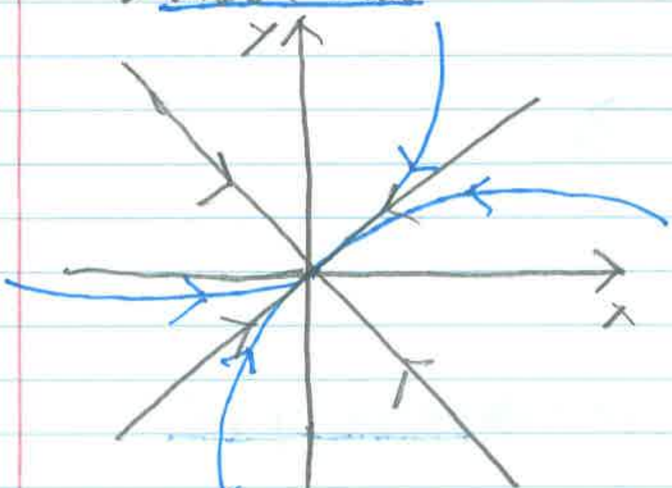
$\Rightarrow$ Saddle node



Case 3:

$\lambda_1 < 0, \ \lambda_2 < 0$

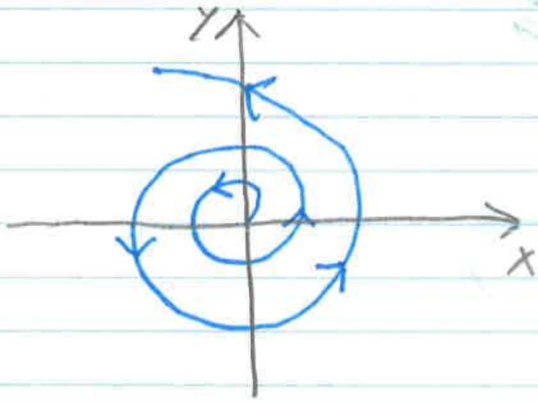
$\Rightarrow$ Stable node



Case 4:

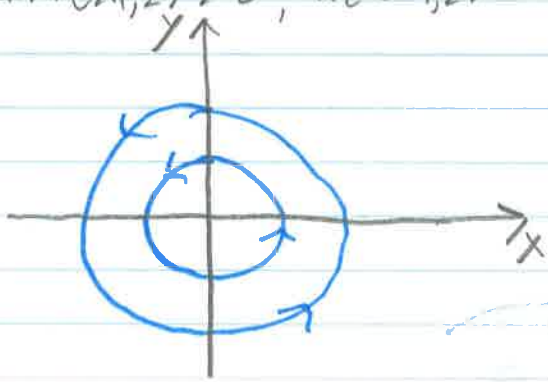
$Im(\lambda_{1,2}) \neq 0$, $Re(\lambda_{1,2}) > 0$

$\Rightarrow$ unstable spiral



Case 5:

$Im(\lambda_{1,2}) \neq 0$, $Re(\lambda_{1,2}) = 0 \Rightarrow$ linear center



Case 6:

$Im(\lambda_{1,2}) \neq 0$, $Re(\lambda_{1,2}) < 0 \Rightarrow$ stable spiral